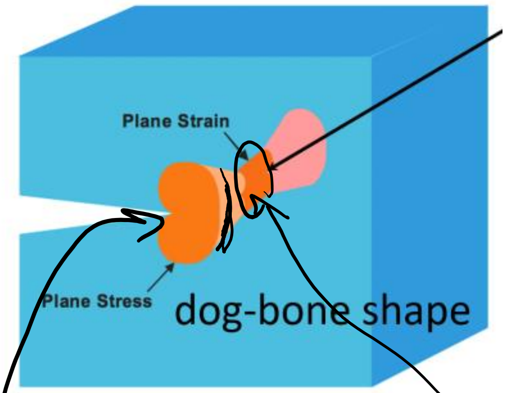
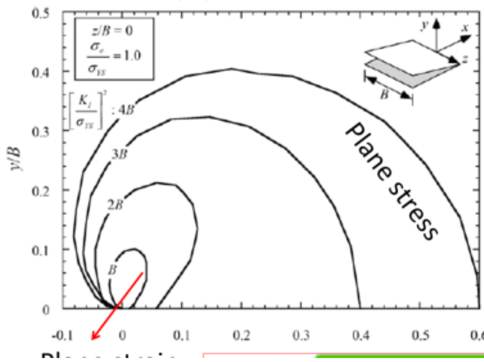
2020/10/05

Monday, October 5, 2020 2:51 PM

From the last time



- Note that $r_p \propto \left(\frac{K}{\sigma_{ys}}\right)^2$



$$\frac{\left(\frac{K}{\sigma_{ys}}\right)^2}{B} \propto \frac{r_p}{B} : \begin{cases} \text{low (high } B) & \text{plane strain} \\ \text{high (low } B) & \text{plane stress} \end{cases}$$

Change of plastic loci to plane stress mode as "relative B decreases". Nakamura & Park, ASME 1988

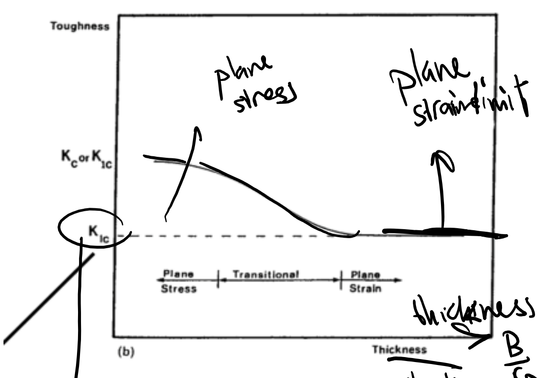
$\frac{r_p}{B} \to 0$ p. strain approx.

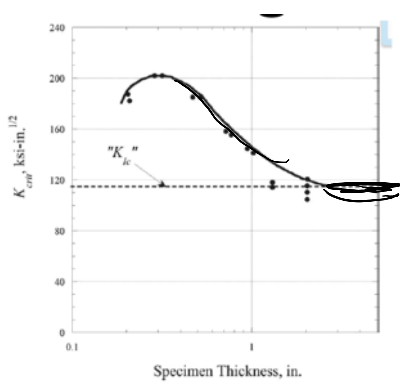
$\frac{r_p}{B} \sim O(1) \to$ p. stress

Bigger fracture process zone -> Larger energy dissipation per unit area of crack advance -> Higher resistance

=> Plane stress -> Should have a larger resistance => Larger fracture roughness $G = \frac{K^2}{E'} \to$

$K_{Ic} = \sqrt{RE'}$





we report p. strain K_{Ic} as fracture toughness

1. it reaches a plateau
2. it's conservative

$K_c(B) = f(K_{Ic}, \frac{r_p}{B})$ $\quad r_p \propto \left(\frac{K_I}{\sigma_Y}\right)^2$

$\downarrow$ p. strain

(safe)

(Irwin) $K_c = K_{Ic}\left(1+\frac{1.4}{B^2}\left[\frac{K_{Ic}}{\sigma_{ys}}\right]^4\right)^{1/2}$ Note that $\frac{1}{B^2}\left[\frac{K}{\sigma_{ys}}\right]^4 \propto \left(\frac{r_p}{B}\right)^2$

$\propto \left(\frac{r_p}{B}\right)^2$

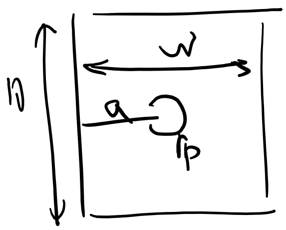
$B \to \infty$ $K_c \to K_{Ic}$

Experiments: -> we want to be in **plane-strain** mode -> B/r_p >> 1

$B \gg 2.5\, r_p$

SSY criterion: FPZ must be much smaller than all relevant length scales of the problem

$W, W-a, H, r_s, \ldots \gg r_p$

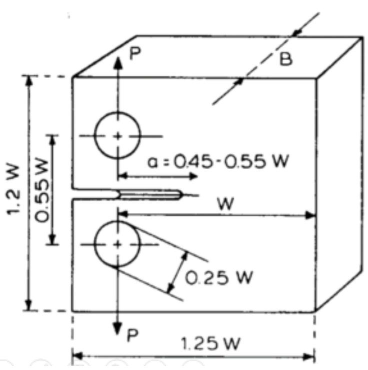


- Prediction of failure in real-world applications: need the value of fracture toughness
- Tests on cracked samples: PLANE STRAIN condition!!!

Compact Tension Test

$$K_I = \frac{P}{B\sqrt{W}}\frac{\left(2+\frac{a}{W}\right)\left[0.886+4.64\frac{a}{W}-13.32\left(\frac{a}{W}\right)^2+14.72\left(\left(\frac{a}{W}\right)^3-5.6\left(\frac{a}{W}\right.\right.\right.}{\left(1-\frac{a}{W}\right)^{3/2}}$$

$a, B, (W-a) \geq 2.5\left(\frac{K_{Ic}}{\sigma_{YS}}\right)^2$



ASTM (based on Irwin's model) fo plane strain condition:

$$\boxed{a, B, (W-a) \geq 2.5\left(\frac{K_{Ic}}{\sigma_{Y}}\right)^2}$$ $\propto r_p$

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Plastic Fracture Mechanics (PFM)

5.3. J Integral (Rice 1958)

LEFM	PFM
G — $d\Pi$ energy	J

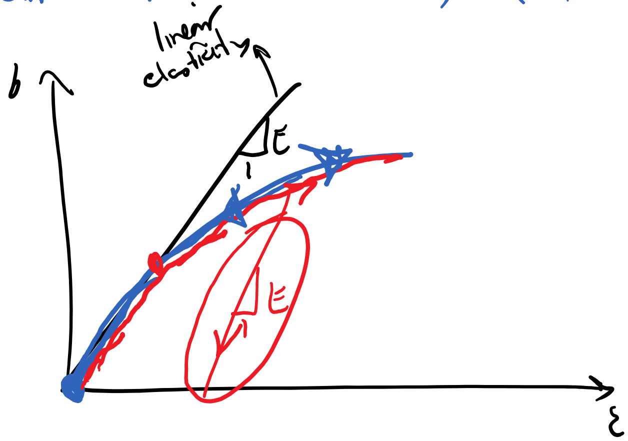
	LEFM	NLFM
Global perspective "energy"	$G = \frac{-d\Pi}{dA}$ energy release rate (E.R.R) $G_c R$ crack can grow	$G = J$ ERR even when LEFM is not applicable
Local perspective "stresses"	$\sigma_{ij}^{I} = \frac{K_I}{\sqrt{2\pi r}} f_{ij}^{I}(\theta)$ $G \leftrightarrow K_I$ for pure mode I $G = \frac{K_I^2 + K_{II}^2}{E'}$ mode I / mode II $\sigma = \frac{\sqrt{GE'}}{\sqrt{2\pi r}} f_{ij}^{I}(\theta)$	$\sigma_{yy} = J^{\alpha} r^{\beta}$

Basically for nonlinear elasticity, J can both provide energy release rate and local stress field.

Difference between Nonlinear elasticity & Plasticity

1D

$\varepsilon \leftrightarrow \sigma$ relation



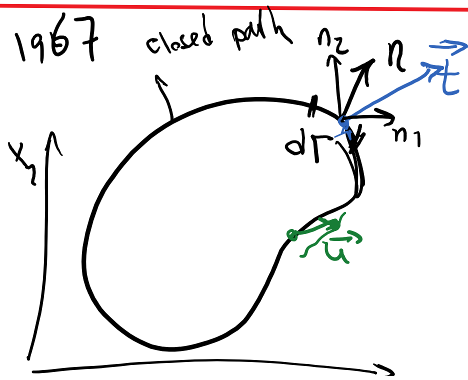
When can we use nonlinear elastic constitutive equation as an approximation for plastic constitutive equation?

As long as we don't have significant unloading, nonlinear elasticity is a very good substitute for plasticity.

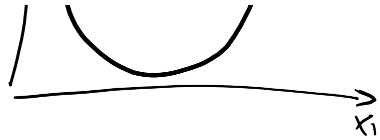
Eshelby, Cherepanov 1967

$$J_k = \int \Big(w n_k - t_i \cdot \underbrace{\frac{\partial u_i}{\partial x_k}}_{\vec{t} \cdot \frac{\partial \vec{u}}{\partial x_k}} \Big) d\Gamma$$

$k = 1$ or 2

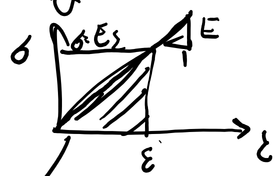


$k = 1$ or 2 $\qquad \vec{t} \cdot \dfrac{\partial \vec{u}}{\partial x_k}$



strain energy density $W(\varepsilon)$

1D linear elasticity



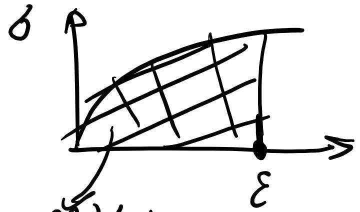
$$\boxed{\sigma = \frac{\partial W}{\partial \varepsilon}}$$

$$W = \frac{1}{2}\sigma\varepsilon = \frac{1}{2}E\varepsilon^2$$

linear elasticity in general $W(\varepsilon) = \frac{1}{2}\sigma : \varepsilon$

$$= \frac{1}{2}\varepsilon : \underbrace{C\varepsilon}_{\sigma}$$

Nonlinear elasticity



$\rightarrow$ can be generalized to 2D / 3D

Eshelby and Cherepanov showed that this integral is zero over closed path for smooth solutions for **elastostatic** problem with no **body force**.

Equation of motion $\quad \nabla \cdot \sigma + \rho b = \rho \ddot{u}$

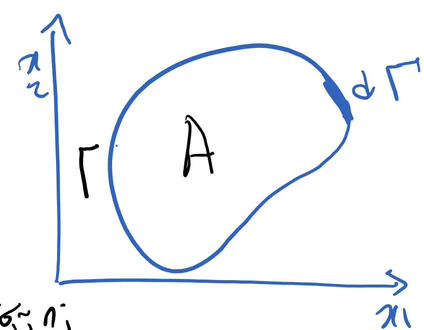
body force $\rightarrow 0$ $\qquad$ acceleration

Proof of why J = 0 over closed paths

$$J_k = \int \left(W(\varepsilon) n_k - t_i \frac{\partial u_i}{\partial x_k} \right) d\Gamma$$

$$t = \sigma n \Rightarrow t_i = \sigma_{ij} n_j$$

Einstein notation $\quad \Sigma \sigma_{ij} n_j$



$$J_k = \int_\Gamma W(\varepsilon) n_k \, d\Gamma - \int_\Gamma \sigma_{ij} n_j \frac{\partial u_i}{\partial x_k} d\Gamma$$

divergence theorem

$$\boxed{\int_{\partial A = \Gamma} f \, n_j \, d\Gamma = \int_A f_{,j} \, dA}$$

$$J_k = \int_A W(\varepsilon)_{,k} - \left(\sigma_{ij} \frac{\partial u_i}{\partial x_k}\right)_{,j} dA$$

$$\frac{\partial W(\varepsilon)}{\partial x_k} = \frac{\partial W(\varepsilon)}{\partial \varepsilon_{mn}} \frac{\partial \varepsilon_{mn}}{\partial x_k} \qquad \varepsilon_{mn} = \frac{1}{2}(u_{m,n} + u_{n,m})$$

$$\varepsilon_{mn} = \tfrac{1}{2}(u_{m,n}+u_{n,m})$$

A

$$-\boxed{\frac{\partial W(\varepsilon)}{\partial x_k} = \frac{\partial W(\varepsilon)}{\partial \varepsilon_{mn}}\frac{\partial \varepsilon_{mn}}{\partial x_k}} = \sigma_{mn}\left(\frac{1}{2}\frac{\partial^2 u_m}{\partial x_k \partial x_n} + \frac{1}{2}\frac{\partial^2 u_n}{\partial x_k \partial x_m}\right)$$

$$= \sigma_{mn}\frac{\partial^2 u_m}{\partial x_k \partial x_n}$$

using $\sigma_{mn} = \sigma_{nm}$ (stress is sym)

$$-\left(\sigma_{ij}\frac{\partial u_i}{\partial x_k}\right)_{,j} = \cancel{\sigma_{ij,j}}\frac{\partial u_i}{\partial x_k} + \sigma_{ij}\frac{\partial^2 u_i}{\partial x_j \partial x_k} = \sigma_{ij}\frac{\partial^2 u_i}{\partial x_k \partial x_j}$$

$$(\nabla\cdot\sigma)_i = 0$$

$$\nabla\cdot\sigma + \cancel{\rho b} = \cancel{\rho\ddot{u}}$$

$$J_k = \int_A \left(\sigma_{mn}\frac{\partial^2 u_m}{\partial x_k \partial x_n} - \sigma_{ij}\frac{\partial^2 u_i}{\partial x_k \partial x_j}\right) dA = 0$$

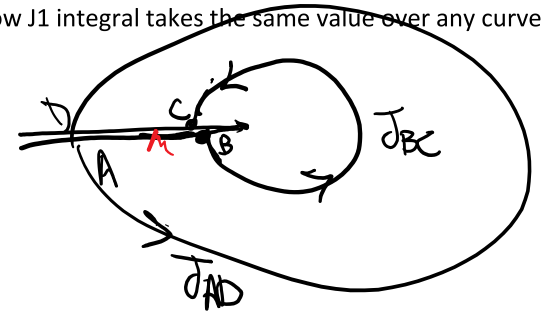
$m \to i$, $n \to j$

$$\sigma_{ij}\frac{\partial^2 u_i}{\partial x_k \partial x_j}$$

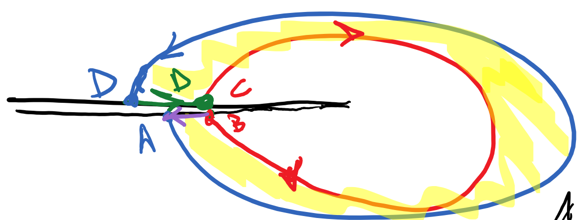
they cancel out

Eshelby and Cherepanov showed that this integral is zero over closed curves for C1 smooth solution.

Show J1 integral takes the same value over any curve around the crack tip:



$$0 \neq J_{AD} = J_{BC}\ !$$



$$J_{ADCBA} = 0 = J_{AD} + \cancel{J_{DC}} + J_{CB} + \cancel{J_{BA}}$$

are zero!

why

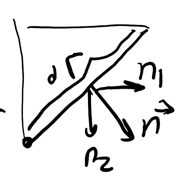
$$(J)_{DC} = (J_1)_{DC} = \int_D^C \left[W(\varepsilon)\,n_1 - \vec{t}\cdot\frac{\partial u}{\partial x_1}\right] d\Gamma$$

$= -n_2\, d\Gamma$

dx

$n_1\, d\Gamma$

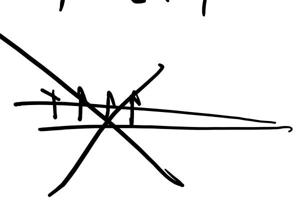
dy



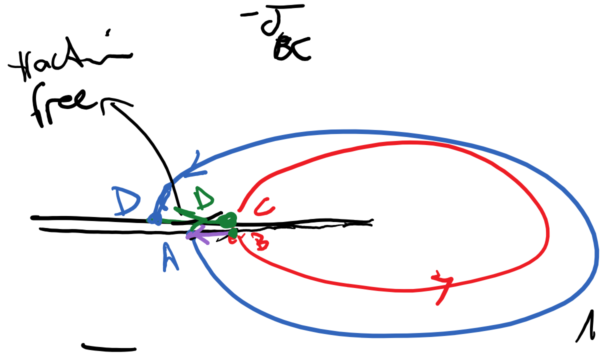
$$\int_D^C w(\varepsilon)\, \underbrace{n_1 d\Gamma}_{dy = 0} - \int \vec{t} \cdot \frac{\partial u}{\partial x}\, d\Gamma$$

traction-free crack surface

(Hydr. frac) X



$$J_{AD} + \underbrace{J_{CB}}_{= -J_{BC}} = 0 \rightarrow \boxed{J_{AD} = J_{BC}}$$



J. Rice $\quad J = G$

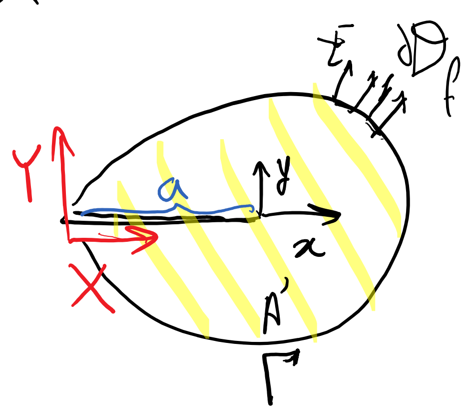
$$G = -\frac{\partial \Pi}{\partial A} = -\frac{\partial \Pi}{\partial Ba} \qquad B = 1$$

∂A: crack surface

$$\Pi = \underbrace{U_e}_{\text{internal energy}} - \underbrace{W}_{\text{external energy}}$$

int. ene. density

$$U_e = \int w(\varepsilon)\, dv$$



$$U_\varepsilon = \int_{A'} \mathcal{W}(\varepsilon)\, dv$$

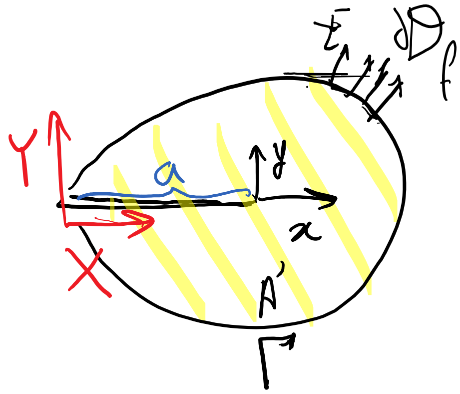
$$W = \int_{A} u\, \cancel{\rho b}\, dv + \int_{\partial D_f} u \bar{t}\, d\Gamma$$

$\rho b = 0$

$$G = -\frac{d}{da}\left\{ \int_{A'} \mathcal{W}(\varepsilon)\, dv - \int_{\partial D_f} u \bar{t}\, ds \right\}$$

$$= -\int_{A'} \frac{d\mathcal{W}(\varepsilon)}{da}\, dv + \int_{\partial D_f} \frac{du \bar{t}}{da}\, ds$$

skipping a few steps



$$G = -\int_{A'} \frac{d\mathcal{W}(\varepsilon)}{da}\, dv + \int_{\Gamma} \frac{du}{da} \vec{t}\, ds$$

$\left.\frac{\partial}{\partial a}\right|_{x \text{ is fixed}}$ vs $\left.\frac{d}{da}\right|_{X \text{ is fixed}}$ — fixed coordinate system

we move with the crack tip & take the derivative

$$\frac{df}{da} = \frac{\partial f}{\partial a} - \frac{\partial f}{\partial x}$$