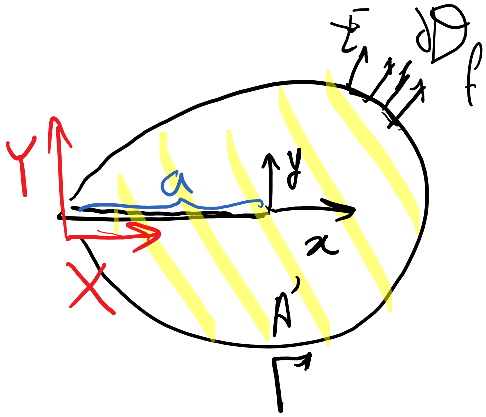
2020/10/07

Wednesday, October 7, 2020 2:37 PM

$$G = -\frac{\partial \Pi}{\partial A} = -\frac{\partial \Pi}{\partial B a} \qquad B=1$$

∂A → crack surface



$$\Pi = \underbrace{U_e}_{\text{internal energy}} - \underbrace{W}_{\text{external energy}}$$

int. ene. density

$$U_e = \int_{A'} \mathcal{W}(\varepsilon)\, dV$$

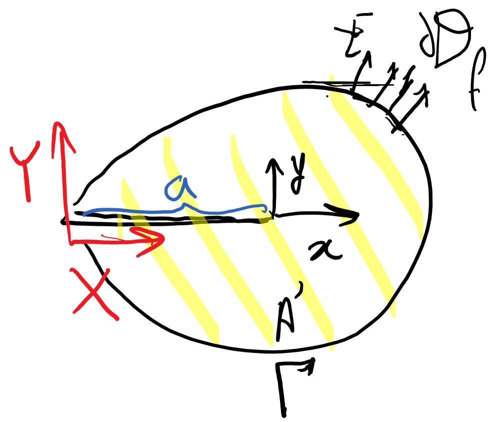
$$W = \cancel{\int_{A} u \rho b\, dV} + \int_{\partial D_f} u \bar{t}\, d\Gamma$$

$\rho b = 0$

$$G = -\frac{d}{da}\left\{ \int_{A'} \mathcal{W}(\varepsilon)\, dV - \int_{\partial D_f} u \bar{t}\, ds \right\}$$

$$= -\int_{A'} \frac{d\mathcal{W}(\varepsilon)}{da}\, dV + \int_{\partial D_f} \frac{d u \bar{t}}{da}\, ds$$

skipping a few steps



$$G = -\int_{A'} \frac{d\mathcal{W}(\varepsilon)}{da}\, dV + \int_{\Gamma} \frac{du}{da} \vec{t}\, ds$$

$$\left.\frac{\partial}{\partial a}\right|_{x\text{-is fixed}} \quad \text{vs} \quad \underbrace{\left.\frac{d}{da}\right|_{X \text{ is fixed}}}_{\text{fixed}}$$

fixed coordinate system

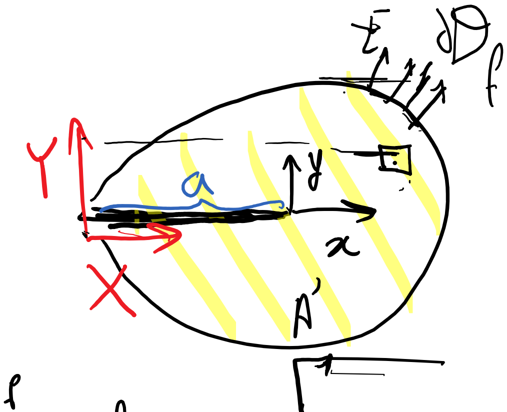
we move with the crack tip & take the derivative

$$\frac{df}{da} = \frac{\partial f}{\partial a} - \frac{\partial f}{\partial x}$$

Since the J-integral is computed in the coordinate attached to the crack tip, we need to change the coordinate system used for expressing terms above and compute the derivatives in the local coordinate system:

$$X = a + x \quad \cancel{+ cte}$$

$$x = X - a$$



$$\left.\frac{df}{da}\right|_{X \text{ is fixed}} = \left.\frac{d f(x,a)}{da}\right|_{X \text{ fixed}}$$

$$= \frac{\partial f}{\partial x}\underbrace{\frac{\partial x}{\partial a}}_{-1} + \frac{\partial f}{\partial a}\underbrace{\frac{\partial a}{\partial a}}_{1} = \underbrace{\frac{\partial f}{\partial a} - \frac{\partial f}{\partial x}}_{\text{when quantity } f \text{ is represented in local coordinate system}}$$

$$G = \int_A -\underbrace{\frac{dW}{da}}_{\frac{\partial W}{\partial a} - \frac{\partial W}{\partial x}} dA + \int_\Gamma \underbrace{\frac{du}{da}}_{\frac{\partial u}{\partial a} - \frac{\partial u}{\partial x}} \vec{t}\, ds$$

$$\rightarrow G = \int_{A'} -\frac{\partial W}{\partial x} dA + \int_\Gamma t \frac{\partial u}{\partial x} ds$$

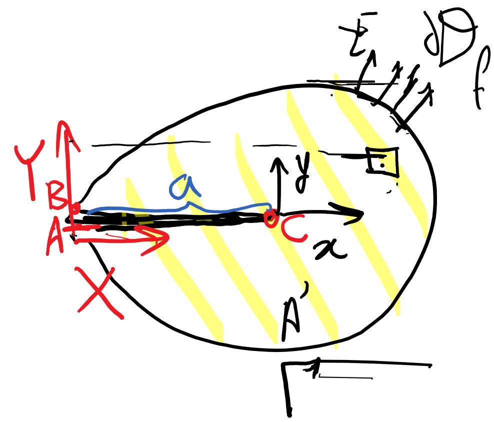
$$+ \boxed{\int_{A'} \frac{\partial W}{\partial a} dA - \int_\Gamma t \frac{\partial u}{\partial a} ds} = 0$$

similar to the proof we had for $J_{\Gamma} = 0$

http://rezaabedi.com/wp-content/uploads/Courses/FractureMechanics/J=G.pdf

$$G = \int -\frac{\partial W}{\partial x} dA + \int t \frac{\partial u}{\partial x} ds$$

$$G = \underbrace{\int_{A'} -\frac{\partial W}{\partial x}\, dA}_{\text{divergence theorem}} + \int_{\Gamma} t \frac{\partial u}{\partial x}\, ds$$



$$= \int_{\Gamma} -W n_1\, ds + \int_{\Gamma} t \frac{\partial u}{\partial x}\, ds =$$

$$= \int_{A\to B} \left(-W n_1 + t \frac{\partial u}{\partial x}\right) ds$$

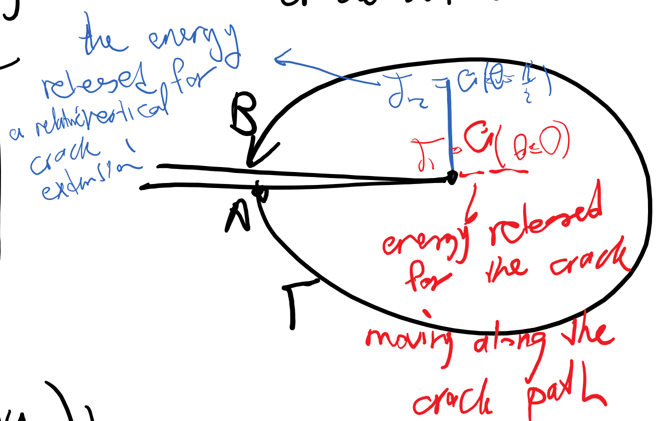
$$\left.\begin{aligned} &+ \int_{B\to C} - \quad \ldots \quad ds \\ &+ \int_{C\to A} - \quad - \quad ds \end{aligned}\right\} = 0$$

$n_1 = 0$

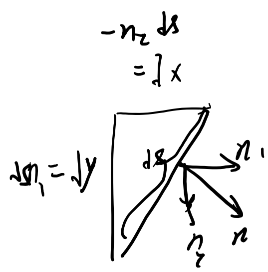
$t = 0$ (traction-free crack surface)

$$\boxed{J = J_1 = \int_{A\to B} \left(-W n_1 + \vec{t} \frac{\partial u}{\partial x}\right) ds}$$

$= G$



$$J_K = \int_{\Gamma} \left(-W n_K + t \frac{\partial u}{\partial x_K}\right) ds$$



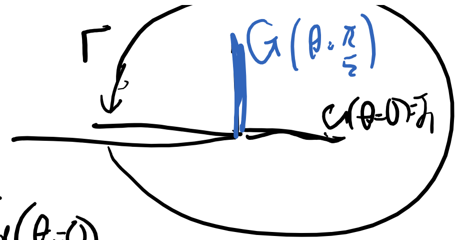
$$\rightarrow J = J_1 = \int_{\Gamma} \left(-W n_1 + t \frac{\partial u}{\partial x}\right) ds =$$

$$J_1 = \int_{\Gamma} -W\, dy + t \frac{\partial u}{\partial x}\, ds$$

$$J_2 = \int -W n_2\, ds + t \frac{\partial u}{\partial y}\, ds = \int_{\Gamma} \left(W\, dx + t \frac{\partial u}{\partial y}\right) ds$$

Summary

Summary



$$J_1 = \int_\Gamma \left(-W\,dy + t\frac{\partial u}{\partial x}\,ds\right) = G(\theta=0)$$

$$J_2 = \int_\Gamma \left(W\,dx + t\frac{\partial u}{\partial y}\right) = G\left(\theta=\frac{\pi}{2}\right)$$

Relation between Ks and J:

$$J = J_1 = G = \frac{K_I^2 + K_{II}^2}{E'} \quad (i)$$

mode I $\rightarrow$ $K_I = \sqrt{JE'}$ $\qquad J_1 \longleftrightarrow K_I$

What about mixed mode $K_I \neq 0$, $K_{II} = 0$

we need another equation

$$J_2 = \frac{-K_I K_{II}}{E'} \quad (ii)$$

(ii) $\rightarrow$ (i) $\qquad K_I^2 + \left(\frac{J_2 E'}{K_I}\right)^2 = J_1$

$$\rightarrow (K_I^2)^2 - J_1(K_I^2) + (J_2 E') = 0$$

$$\overset{Z}{Z^2} - J_1 Z + J_2 E' = 0 \rightarrow \text{solve for } Z = K_I^2$$

$K_I = a \qquad K_{II} = b$

all the following are valid solutions

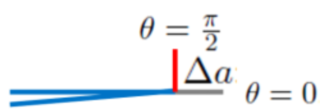
$K_I = \mp a \qquad K_{II} = \mp b \qquad$ 2 s's

$K_I = \mp b \qquad K_{II} = \mp a \qquad$ 2 "

In fact both J1 (J) and J2 are related to SIFs:

$$J_1 = \int_\Gamma \left(w dy - \mathbf{t}\frac{\partial \mathbf{u}}{\partial x} d\Gamma \right)$$
$$J_2 = \int_\Gamma \left(-w dx - \mathbf{t}\frac{\partial \mathbf{u}}{\partial y} d\Gamma \right)$$

J1 & J2: crack advance for (θ = 0, 90) degrees



$$\begin{aligned} J &= J_1 - iJ_2 \\ &= \frac{(1+\nu)(1+\kappa)}{4E}(K_I^2 + K_{II}^2 + 2iK_I K_{II}) \end{aligned}$$

Hellen and Blackburn (1975)

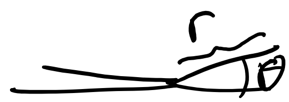
$$\boxed{J_1 = \frac{K_I^2 + K_{II}^2}{E'} \qquad J_2 = \frac{-2K_I K_{II}}{E'}}$$

$$E' = \begin{cases} E & \text{plane strain} \\ \frac{E}{1-\nu^2} & \text{plane stress} \end{cases}$$

Note that if $K_I = a, K_{II} = b$ is a solution the general solution is:
$K_I = \pm a, K_{II} = \pm b$ and $K_I = \pm b, K_{II} = \pm a$

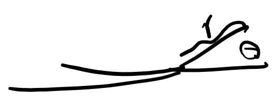
— local importance of J integral

PFM



$\sigma_{ij} = J^{?} f_{ij}(\theta, \ldots)$

Generalization of



$\sigma_{ij}^{I} = \frac{K_I}{\sqrt{2\pi r}} f_I^{ij}(\theta)$

LEFM

5.3. 5. Plastic crack tip fields; Hutchinson, Rice and Rosengren (HRR) solution

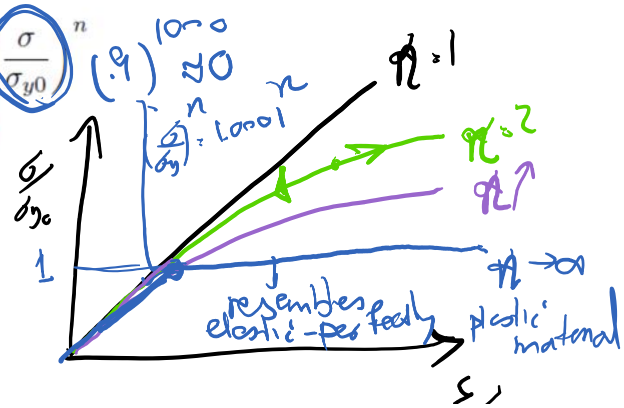
Nonlinear-Elasticity models

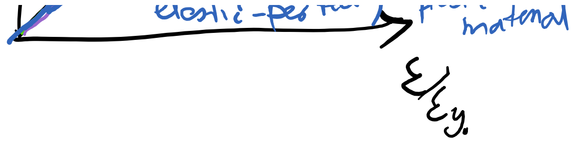
Ramberg–Osgood model

$$\frac{\varepsilon}{\varepsilon_{y0}} = \frac{\sigma}{\sigma_{y0}} + \alpha\left(\frac{\sigma}{\sigma_{y0}}\right)^n$$

Compare with $\epsilon = \epsilon^e + \epsilon^p$

$(.9)^{1000}$ → 0





$$\left(\frac{\varepsilon}{\varepsilon_{y_0}}\right) = \frac{\sigma}{\sigma_{y_0}} + \alpha\left(\frac{\sigma}{\sigma_{y_0}}\right)^n$$

very large around the crack tip

$$\frac{\varepsilon}{\varepsilon_{y_0}} \approx \alpha\left(\frac{\sigma}{\sigma_{y_0}}\right)^n$$

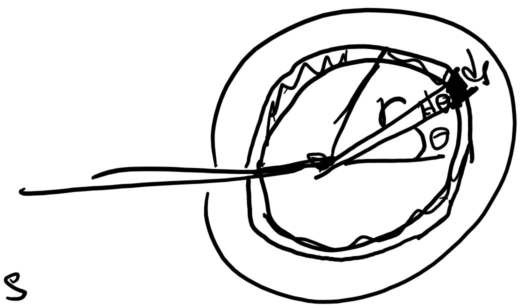
$$\longrightarrow \quad \boxed{\varepsilon \propto \sigma^n}$$

around the crack tip

looking for powers of singularity for the first strain & stress expansion terms in HRR solution

$$\sigma = \frac{c_1}{r^x} \longrightarrow \text{singularity power for stress}$$

$$\varepsilon = \frac{c_2}{r^y}$$



$$\mathcal{E} = \int \underbrace{W(\varepsilon)}_{\propto\, \sigma\varepsilon} \underbrace{ds}_{r\,dr\,d\theta}$$

$$\int \underbrace{\frac{1}{r^x} \cdot \frac{1}{r^y}} \cdot r \, dr\, d\theta$$

$$\int r^{\underbrace{1-x-y}} dr\, d\theta$$

$$1 - x - y \geq 0 \longrightarrow x + y \leq 1$$

In fact $\boxed{x + y = 1}$

$$\varepsilon \sim \sigma^n$$

In fact $\boxed{x+y=1}$

(*) $\varepsilon \propto \sigma^n$

$\varepsilon = \dfrac{c_3}{r^x}$, $\quad \sigma^n = \left(\dfrac{c_1}{r^y}\right)^n \longrightarrow \boxed{x = ny}$

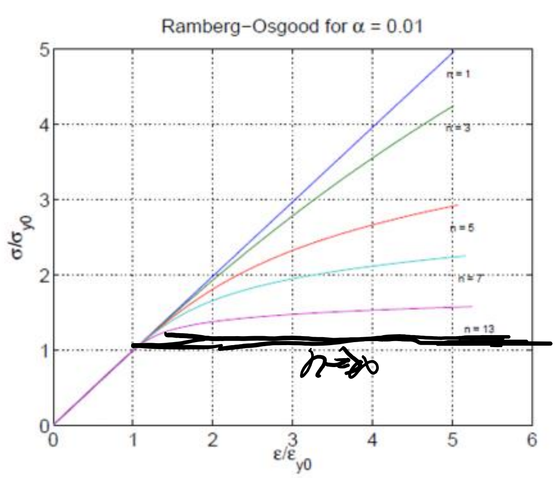
$(n+1)\,y = 1 \longrightarrow \boxed{x = \dfrac{n}{n+1}, \quad y = \dfrac{1}{n+1}}$

$$\boxed{\sigma \propto \frac{1}{r^{\frac{1}{n+1}}} \qquad \varepsilon \propto \frac{1}{r^{\frac{n}{n+1}}}}$$

How about $n = 1$?
linear Elastic material

$$\varepsilon \propto \frac{1}{\sqrt{r}} \qquad \sigma \propto \frac{1}{\sqrt{r}}$$

Recall LEFM $\sigma_{ij}^{I} = \dfrac{K_I}{\sqrt{2\pi r}} f_{ij}(\theta)$ ✓

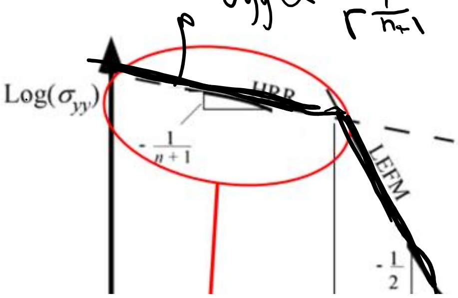


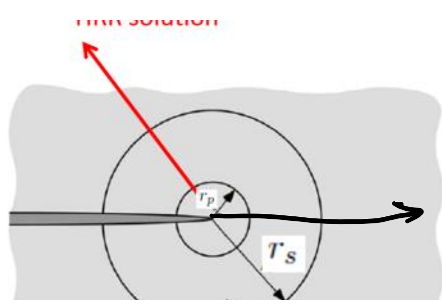
$n \to \infty$

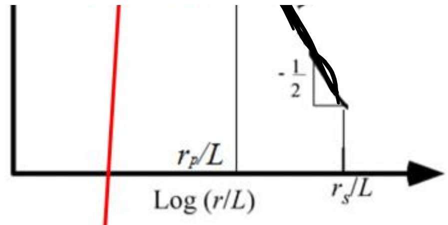
$\underbrace{\sigma \to \frac{1}{r^0} \to \text{const}}_{\text{stress becomes bounded by } \sigma_{y0}}$ $\quad \underbrace{\varepsilon \to \frac{1}{r}}_{\text{all singularity goes to strain}}$

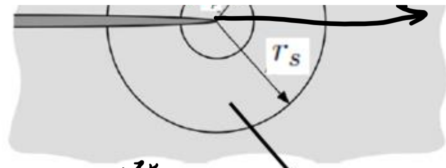
Not singular anymore

$\sigma_{yy} \propto \dfrac{1}{r^{\frac{1}{n+1}}}$









$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \rightarrow$ LEFM

$\log(\sigma_{yy}) = \log \frac{K_I}{\sqrt{2\pi}} - \frac{1}{2}\log r$

LEFM solution

Stress is still singular but with a weaker power of singularity!

This model predicts a weaker singularity power than LEFM for stress (but a higher one for strain)

- Final form of HRR solution:

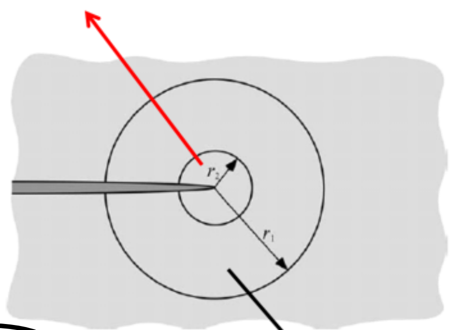
dimensionless

$$\sigma_{ij} = \sigma_0 \left(\boxed{\frac{EJ}{\alpha\sigma_0^2 I_n r}}\right)^{\frac{1}{n+1}} \tilde{\sigma}_{ij}(n, \theta)$$

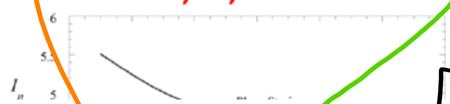
$$\epsilon_{ij} = \frac{\alpha\sigma_0}{E}\left(\frac{EJ}{\alpha\sigma_0^2 I_n r}\right)^{\frac{n}{n+1}} \tilde{\epsilon}_{ij}(n, \theta)$$

J plays the role of K for local σ, ε, u fields

HRR solution

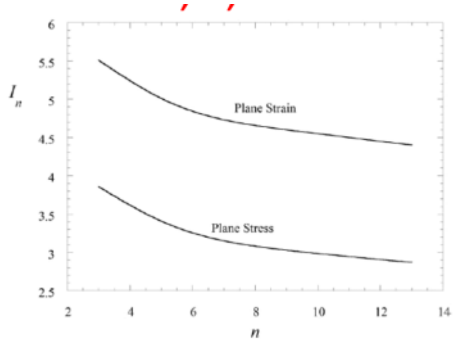


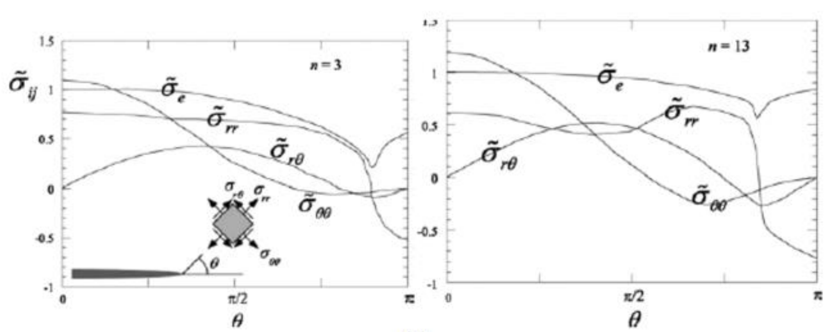
LEFM solution



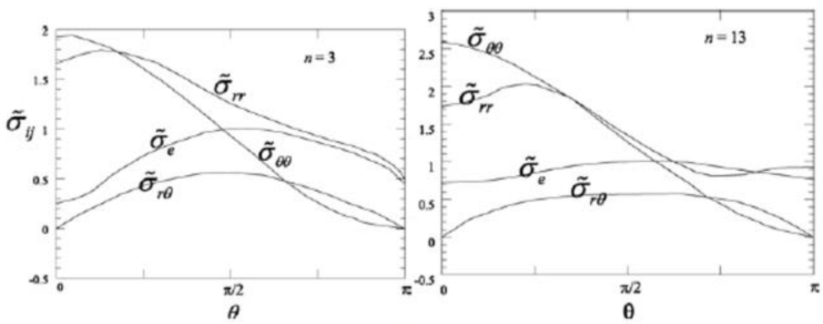
$$\sigma_{ij} = \frac{K_I}{\sqrt{2\pi r}} f_{ij}^{I}(\theta)$$

$$\left[\frac{E\,J}{\underset{\sigma_{ys}\text{ of RO model}}{\alpha\,\sigma_0^2}\, I_n(n)\, r}\right] = \frac{[\sigma]\overbrace{[\sigma][L]}^{J}}{\underbrace{[\sigma]^2}_{\sigma_0^2}[L]} = 1$$





(a)

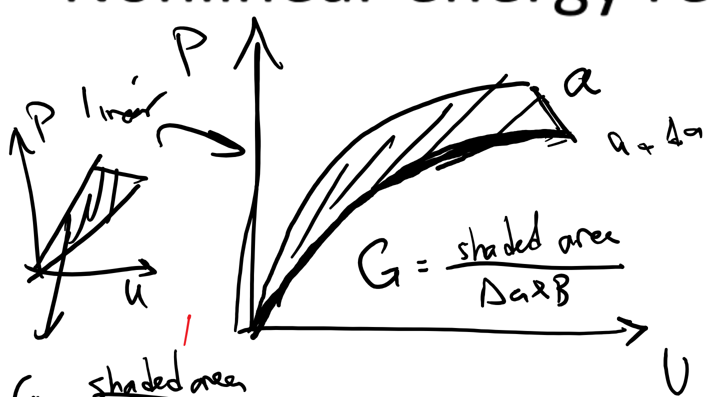


(b)

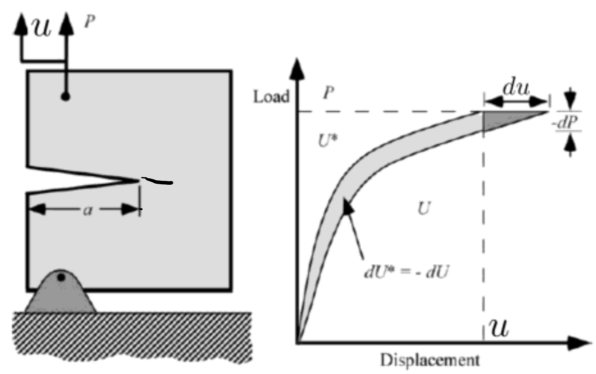
Extending the analysis of P-u systems to nonlinear response:

5.3. 4. Energy Release Rate, crack growth and R curves

Nonlinear energy release rate



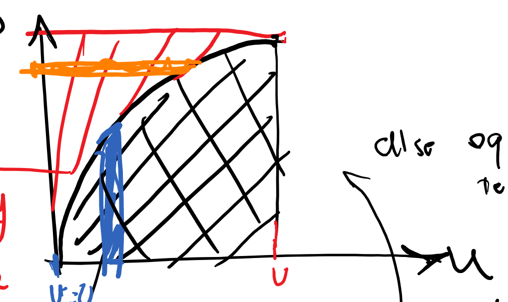
$G = \dfrac{\text{shaded area}}{\Delta a \times B}$



$G = \dfrac{\text{shaded area}}{B \, \Delta a}$

Definitions

Complementary internal energy $= U_e^*$



also equal to area under P-u

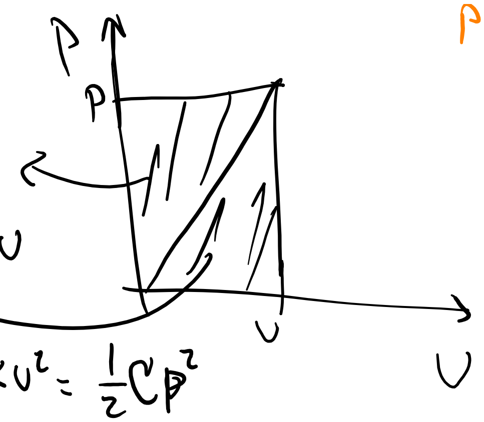
$\mathcal{W}(\varepsilon)$ Internal energy $U_e = \int_V \mathcal{W}(\varepsilon)\, dV$

$U_e = \int_0^u P(u)\, du$ P(u) is given

$U_e^* = P.u - U_e = \int_0^P u(P)\, dP$

easy to compute when U is expressed in terms of P

For linear elasticity

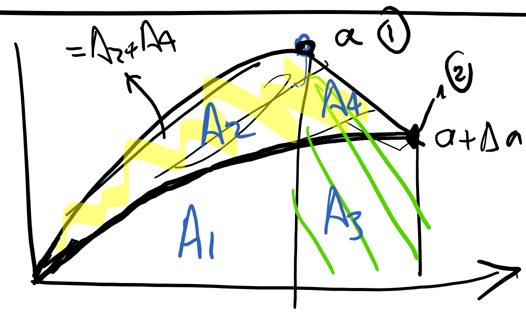


$U_e^* = U_e = \frac{1}{2}Pu$

$= \frac{1}{2}ku^2 = \frac{1}{2}CP^2$

Why?

$$G = \lim_{\Delta a \to 0} \frac{\text{shaded area}}{\Delta a \cdot B}$$



$$G = \frac{-d\Pi_e}{dA} = -\frac{d\Pi}{B\,da}$$

area

$$= \lim_{\Delta a \to 0} -\frac{(\Delta U_e - W_{12})}{B\,\Delta a}$$

want to show this the shaded area

$\Delta U_e = U_e(a + \Delta a)$

$- U_e(a)$

$= (A_1 + A_3) - (A_1 + A_2)$

$= A_3 - A_2$

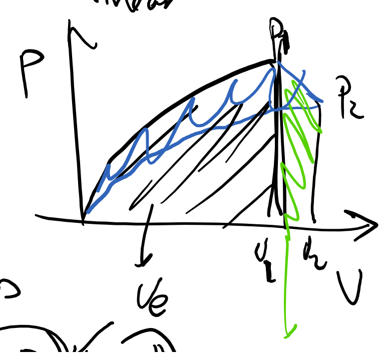
$$W_{12} = \int_1^2 P\,du = A_3 + A_4$$

$$G = \lim_{\Delta a \to 0} \frac{1}{B}\left(-(A_3 - A_2) + A_3 + A_4\right) = \lim_{\Delta a \to 0} \frac{A_2 + A_4}{B\,\Delta a} = \lim_{\Delta a \to 0} \frac{\text{shaded area}}{B\,\Delta a}$$

Do we have eqns similar to $\underbrace{G = \frac{P^2}{2B}\frac{dC}{da} = \frac{-u^2}{2B}\frac{dk}{da}}_{\text{linear}}$

Recall

$$G = \lim_{\Delta a \to 0} \frac{-\{(U_{e_2} - U_{e_1}) - W_{12}\}}{B\,\Delta a}$$



$$= \frac{1}{B}\Big\{ -\underbrace{\lim_{\Delta a \to 0} \frac{U_e(a+\Delta a) - U(a)}{\Delta a}}_{\frac{dU_e}{da}} + \lim_{\Delta a \to 0} \frac{\left(\frac{P_1+P_2}{2}\right)(U_2-U_1)}{\Delta a}$$

P $\quad U_e$

$$W_{12} \approx \left(\frac{P_1+P_2}{2}\right)(U_2-U_1)$$

$$\frac{dU}{da} = U(a+\Delta a) - U(a)$$

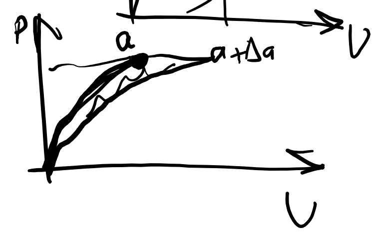
$$= \frac{1}{B}\left(-\frac{dU_e}{da} + P\frac{du}{da}\right)$$

General
$$\boxed{G = \frac{1}{B}\left(-\frac{dU_e}{da} + P\frac{du}{da}\right)} \quad G$$

special cases: 1 dead load, P is constant

$$G = \frac{1}{B}\left(-\frac{dU_e}{da} + \frac{d\overset{\nearrow \text{is const}}{P}U}{da}\right) = \frac{1}{B}\frac{d(PU - U_e)}{da} \qquad U_e^*$$

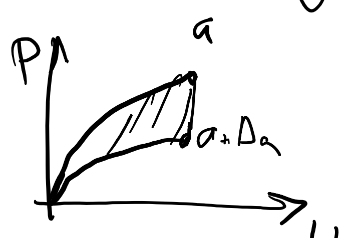
$$G = \frac{1}{B}\frac{dU_e^*}{da} \quad \text{dead load}$$



case 2 constant U (fixed grip)

$$G = \frac{1}{B}\left(-\frac{dU_e}{da} + P\cancel{\frac{dU}{da}}_{\to 0}\right)$$

$$\boxed{G = \frac{-1}{B}\frac{dU_e}{da}}$$



Summary $$G = \frac{1}{B}\left(-\frac{dU_e}{da} + P\frac{du}{da}\right)$$

$$= \frac{1}{B}\frac{dU_e}{da} \quad \text{fixed load}$$

$$= -\frac{1}{B}\frac{dU_e}{da} \quad \text{" grip}$$