2020/10/12

Monday, October 12, 2020 2:44 PM

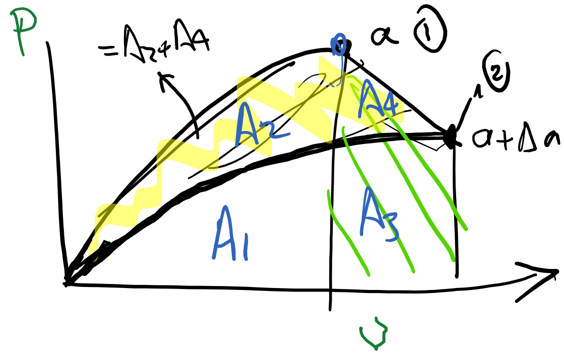
From last time ☆

Summary

$$G = \frac{1}{B}\left(-\frac{dU_e}{da} + P\frac{du}{da}\right)$$

$$= \frac{1}{B}\frac{dU_e^{*}}{da} \quad \text{fixed load}$$

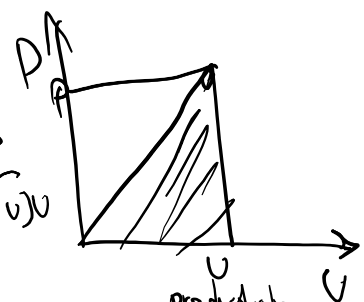
$$= -\frac{1}{B}\frac{dU_e}{da} \quad \text{" grip}$$



Does ☆ match the linear results if the material is linear.

Linear case:

$$U_e = U_e^{*} = \frac{1}{2}Pu = \frac{1}{2}(\underbrace{Ku}_{P})u$$



$$U_e = \frac{1}{2}Ku^2$$

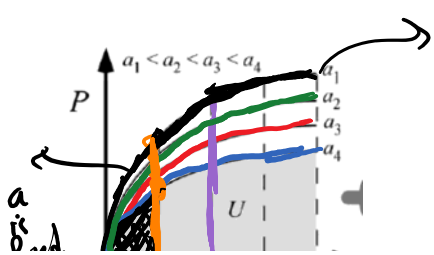
☆ $$G = \frac{1}{B}\left(-\frac{dU_e}{da} + P\frac{du}{da}\right) = \frac{1}{B}\left(-\frac{d\,\overbrace{\tfrac{1}{2}Ku^2}^{\text{product rule}}}{da} + P\frac{du}{da}\right)$$

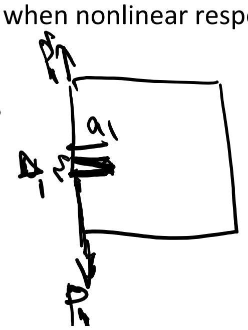
$$= \frac{1}{B}\left(-\frac{1}{2}\frac{dK}{da}\cdot u^2 - \cancel{\underbrace{Ku}_{P}\frac{du}{da}} + \cancel{P\frac{du}{da}}\right)$$

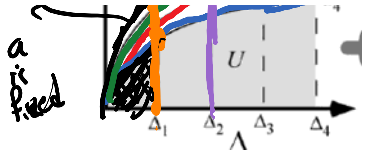
$$\boxed{G = -\frac{u^2}{2B}\frac{dK}{da}}$$ ✓

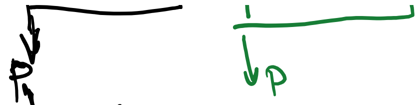
This matches our result from linear analysis

How to compute G experimentally when nonlinear response is present?

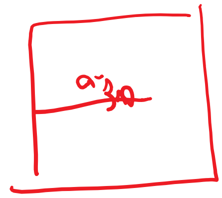






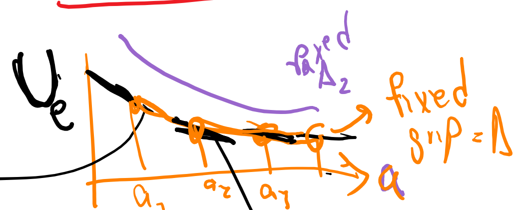


increase P or Δ while keeping a fixed



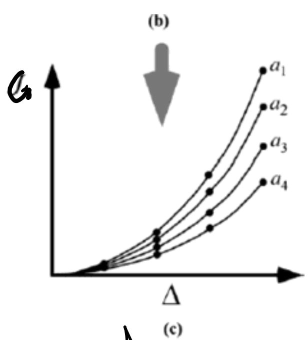
In none of these experiments a is increasing, that is none of the experiments involve crack propagation.

Still, we want to represent a fixed grip crack propagation set up.



For a fixed grip set up we have $\boxed{G = -\frac{1}{B}\frac{dU_e}{da}}$

$\frac{-\text{slope}}{B} = G$



The issue with this experiment is that we need SEVERAL P-U curves for fixed a's.

This is cumbersome

Improvement:

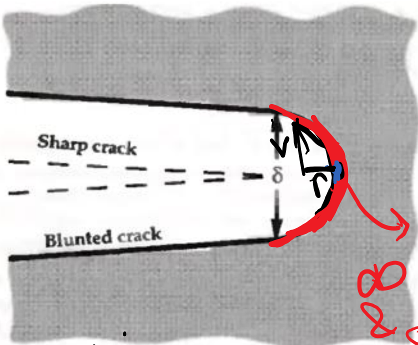
- Rice proposes a method to obtain J with <u>only one test</u> for certain geometries

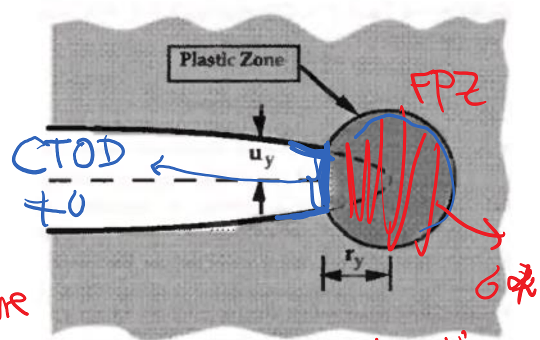
cf. Anderson 3.2.5 for details

Crack Tip Opening Displacement (CTOD) versus J-integral

5.4. Crack tip opening displacement (CTOD), relations with J and G

LEFM theory





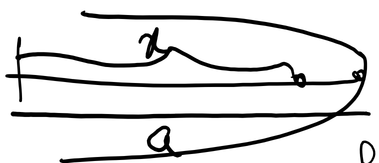
∞ slope & curvature that we have for LEFM $\to \delta \to \infty$ as $r \to 0$

In reality we have yielding & nonlinear response in FPZ

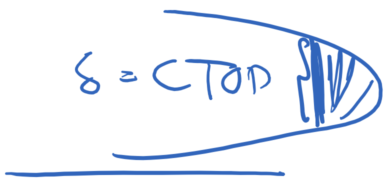
the asymptotic expansion for v: $\cancel{v \propto \sqrt{r}}$

$$v = \frac{\kappa + 1}{4\mu} \sigma \sqrt{a^2 - x^2}$$

exact displacement (NOT the asymptotic) for a mid-crack specimen



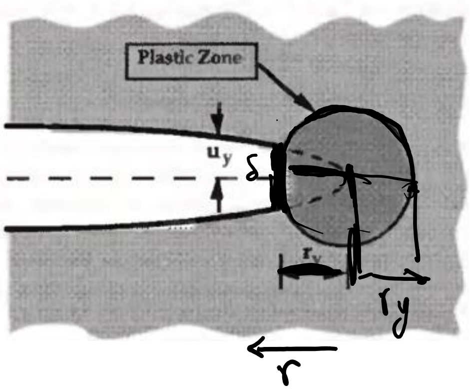
Parallel to Rice's work in the USA, Wells in the UK looked at CTOD as a measure of nonlinear material response in the FPZ



$\delta \nearrow \rightarrow$ larger nonlinear response

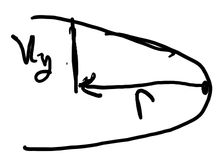
is δ related to $\mathcal{G} = G$ energy release it?

Estimates for CTOD:



without stress redistribution see if we can get a scale for δ

$$u_y = \frac{(\kappa+1) K_I}{2\mu} \sqrt{\frac{r}{2\pi}}$$



$$r \to r_y \to u_y = \frac{\delta}{2}$$

$$\frac{\delta}{2} = \frac{(\kappa+1)}{2\mu} \sqrt{\frac{r_y}{2\pi}} \longrightarrow$$

$2\mu \quad \frac{1}{2\pi}$

$$r_y = \left(\frac{K_I}{\sigma_y}\right)^2$$

$$\delta = \frac{\kappa+1}{\sqrt{2\pi}} \frac{K_I^2}{\mu\, \sigma_{ys}}$$

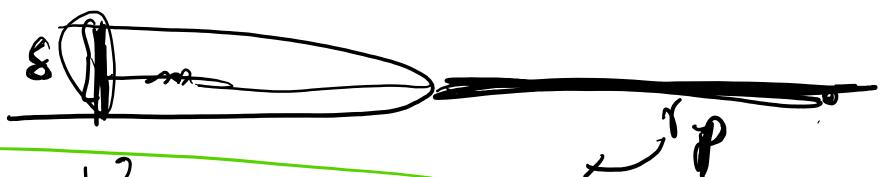
$\kappa = \frac{3-\nu}{1+\nu}$ p. stain

$$\mu = \frac{E}{2(1+\nu)}$$

$$\longrightarrow \boxed{\delta \approx \frac{4}{\pi} \frac{K_I^2}{\sigma_{ys} E}}$$

estimate for CTOD

$\bar{\delta}$



$\bar{\delta}$

$$\delta \propto \frac{K_I^2}{E\, \sigma_y}$$

$\wedge$

$$r_p = \frac{K_I^2}{\sigma_y^2} \longrightarrow \boxed{\frac{\delta}{r_p} = \frac{\sigma_y}{E}} \longrightarrow 0.01 \sim .001$$

$$\boxed{\delta \ll r_p}$$

for LEFM $\wedge$

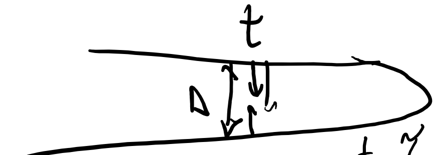
$$r_s = \frac{K_I^2}{\bar{\delta}^2}$$

for LEFM $\bar{\delta} \ll \sigma_y$

displacement scale of the fracture model $\ll$ FPZ size

In many instances, LEFM, PFM, Traction-separation laws, frictional laws, … FPZ size is much larger than the displacement scale of the model and typically the ratio is proportional to E/σ_y

Cohesive models



$$t = \tilde{\sigma} f\left(\frac{\Delta}{\tilde{\Delta}}\right)$$

displacement $\tilde{\Delta}$ scale

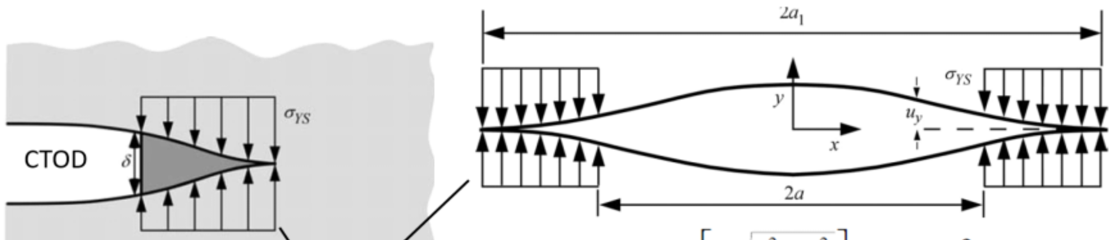
$$FPZ \propto \tilde{\Delta} \left(\frac{E}{\tilde{\sigma}}\right)$$

$$\text{FPZ} \propto \tilde{\Delta}\left(\frac{E}{\tilde{\sigma}}\right)$$

displacement scale

The above estimate for CTOD was very crude and did not consider the actual solution of the problem with nonlinear response.

Crack Tip Opening Displacement: Strip yield model



Stresses that yielded K = 0

K = 0 $\Rightarrow$ $Z = \dfrac{2\sigma_{YS}}{\pi}\left[\dfrac{k}{z}\sqrt{\dfrac{z^2-a_1^2}{1-k^2}}\right] \Rightarrow \tilde{Z} = \dfrac{2\sigma_{YS}}{\pi}[z\omega_1 - a\omega_2]$

$$\Rightarrow u_y = \frac{2}{E}\mathrm{Im}\tilde{Z} = \frac{4\sigma_{YS}}{\pi E}\left[a\coth^{-1}\left(\frac{1}{a_1}\sqrt{\frac{a_1^2-z^2}{1-k^2}}\right) - z\coth^{-1}\left(\frac{k}{z}\sqrt{\frac{a_1^2-z^2}{1-k^2}}\right)\right]$$

$$\Rightarrow u_y = \frac{2}{E}\mathrm{Im}\tilde{Z} = \frac{4\sigma_{YS}}{\pi E}\left[a\coth^{-1}\left(\frac{1}{a_1}\sqrt{\frac{a_1^2-z^2}{1-k^2}}\right) - z\coth^{-1}\left(\frac{k}{z}\sqrt{\frac{a_1^2-z^2}{1-k^2}}\right)\right]$$

$$z = a \Rightarrow \delta = 2u_y = \frac{8\sigma_{YS}a}{\pi E}\ln\left(\frac{1}{k}\right) = \frac{8\sigma_{YS}a}{\pi E}\left[\frac{1}{2}\left(\frac{\pi}{2}\frac{\sigma}{\sigma_{YS}}\right)^2 + \frac{1}{12}\left(\frac{\pi}{2}\frac{\sigma}{\sigma_{YS}}\right)^4 + \cdots\right] \Rightarrow$$

For $\sigma/\sigma_{YS} \to 0$ $\quad \boxed{\delta = \dfrac{K_I^2}{\sigma_{YS}E}}$ $\rightarrow$ this matches our crude earlier estimate

$$\delta = \frac{4}{\pi}\frac{K_I^2}{\sigma_{ys}E}$$

$\frac{4}{\pi} \approx 1$

Is there a relation between J and CTOD?
Both can measure the extent of nonlinear response, but are they related?

$$\left.\begin{aligned}\delta &= \frac{K_I^2}{E\sigma_{ys}} \\ J &= G = \frac{K_I^2}{E'} \\ E &\sim E'\end{aligned}\right\} \rightarrow \delta = \frac{J}{\sigma_y} \rightarrow$$

$J = G = \frac{K_I^2}{E'}$

$E \simeq E'$

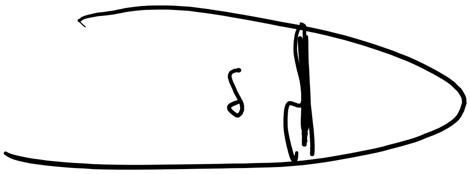
σ_y

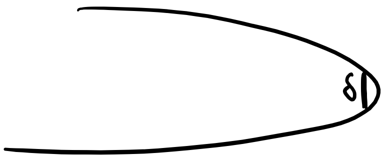
they are related

$$\boxed{\begin{gathered} J = \delta \sigma_y \\ \delta = \frac{J}{\sigma_y} \end{gathered}}$$

$J \nearrow$ (higher loads → higher energy release rate)

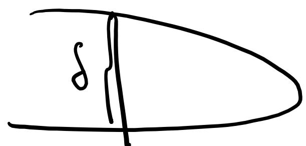
OR if $\underline{\sigma_y} \searrow$ (more ductility) → $\boxed{\delta \nearrow}$





LEFM good

σ_y high quasi-brittle material



ductile

σ_y low

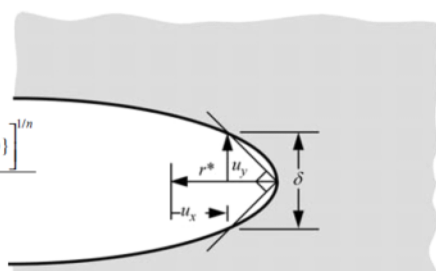
CTOD-J relation

- When SSY is satisfied G = J so we expect:

$$G = m\sigma_y\delta \quad \Rightarrow \quad \boxed{J = m\sigma_y\delta}$$

- In fact this equation is valid well beyond validity of LEFM and SSY
- E.g. for HRR solution Shih showed that:

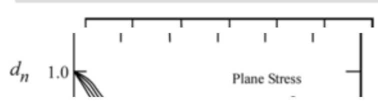
$$u_i = \frac{\alpha\sigma_o}{E}\left(\frac{EJ}{\alpha\sigma_o^2 I_n r}\right)^{\frac{n}{n+1}} r\tilde{u}_i(\theta, n) \qquad d_n = \frac{2\tilde{u}_y(\pi,n)\left[\frac{\alpha\sigma_o}{E}\{\tilde{u}_x(\pi,n)+\tilde{u}_y(\pi,n)\}\right]^{1/n}}{I_n}$$



- δ is obtained by 90 degree method:
  Deformed position corresponding to $r^* = r$ and $\varphi = -\pi$ forms 45 degree w.r.t crack tip)

$$\frac{\delta}{2} = u_y(r^*, \pi) = r^* - u_x(r^*, \pi)$$

σ_y CTOD ↑



φ = -π forms 45 degree w.r.t crack tip)

$$\frac{\delta}{2} = u_y(r^*, \pi) = r^* - u_x(r^*, \pi)$$

CTOD

$\approx \sigma_y$

$$\boxed{r^* = \left(\frac{\alpha\sigma_o}{E}\right)^{1/n} \{\tilde{u}_x(\pi, n) + \tilde{u}_y(\pi, n)\}^{\frac{n+1}{n}} \frac{J}{\sigma_o I_n}} \Rightarrow \boxed{J = m\sigma_0\delta}$$

for $\boxed{m = \dfrac{1}{d_n},\ d_n = \dfrac{2\tilde{u}_y(\pi,n)\left[\dfrac{\alpha\sigma_o}{E}\{\tilde{u}_x(\pi,n) + \tilde{u}_y(\pi,n)\}\right]^{1/n}}{I_n}}$

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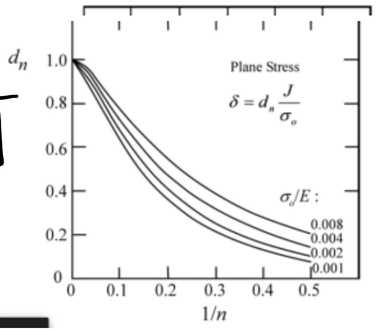
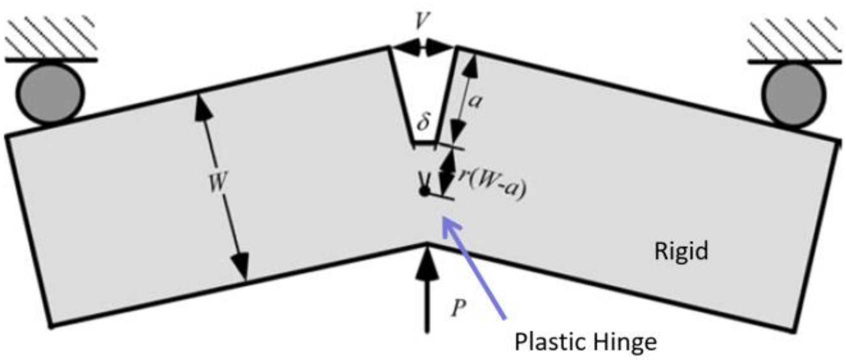


Experimentally, one can measure CTOD and from there from the relation above compute J (this is another way to experimentally measure J)
For the description of the experiment, please refer to Anderson:



$$\frac{\delta_t}{CMOD} = \frac{r(W-a)}{r(W-a)+a}$$ similarity of triangles

r : rotational factor [-], between 0 and 1

For high elastic deformation contribution, elastic corrections should be added

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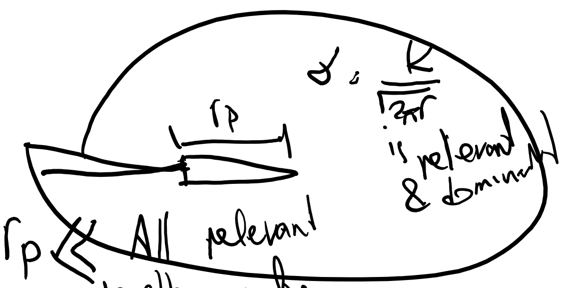
When can we use LFEM, PFM, ...?

SSY is satisfied → LEFM

necessary condition (but not sufficient)



including r_s

$$r_p \ll r_s$$

$$\left(\frac{K_I}{\sigma_y}\right)^2 \quad \left(\frac{K_I}{\sigma}\right)^2 \rightarrow \left(\frac{\sigma}{\sigma_y}\right)^2 \ll 1 \quad \text{necessary condition}$$

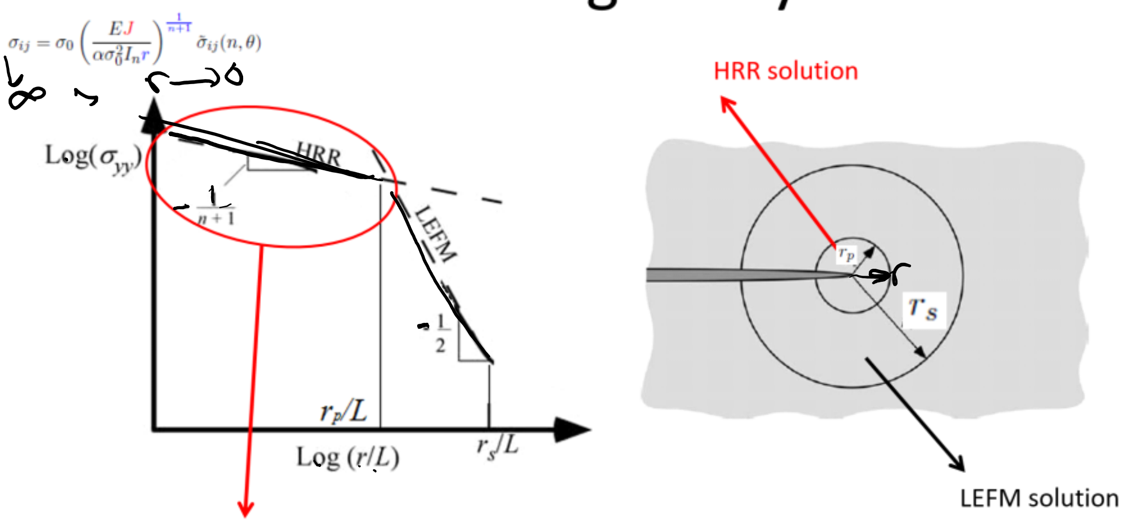
if SSY is violated we need to use PFM

$\begin{cases} G = J \text{ still valid (as long as there is not much plastic unloading)} \\ \sigma = J^{\alpha} r^{\beta} \text{ stress solution} \end{cases}$

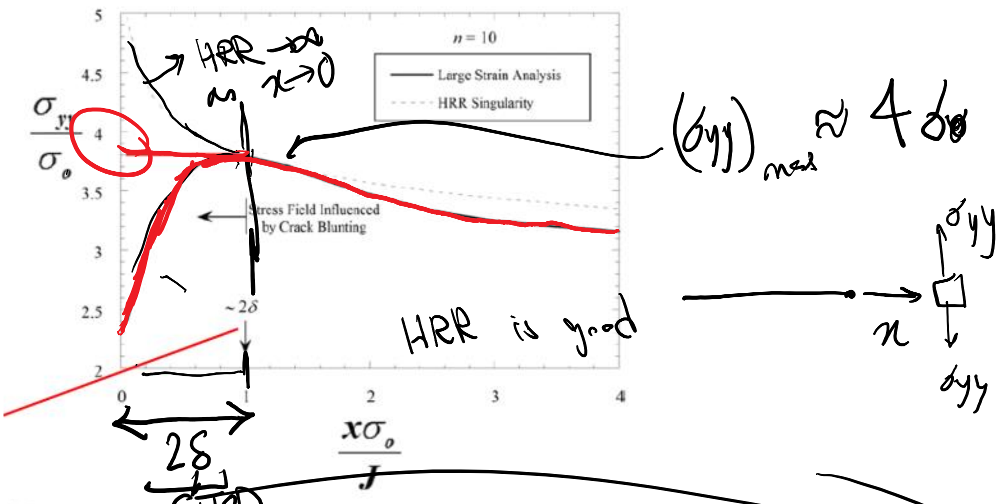
HRR solution

So, when PFM HRR solution is no longer relevant?

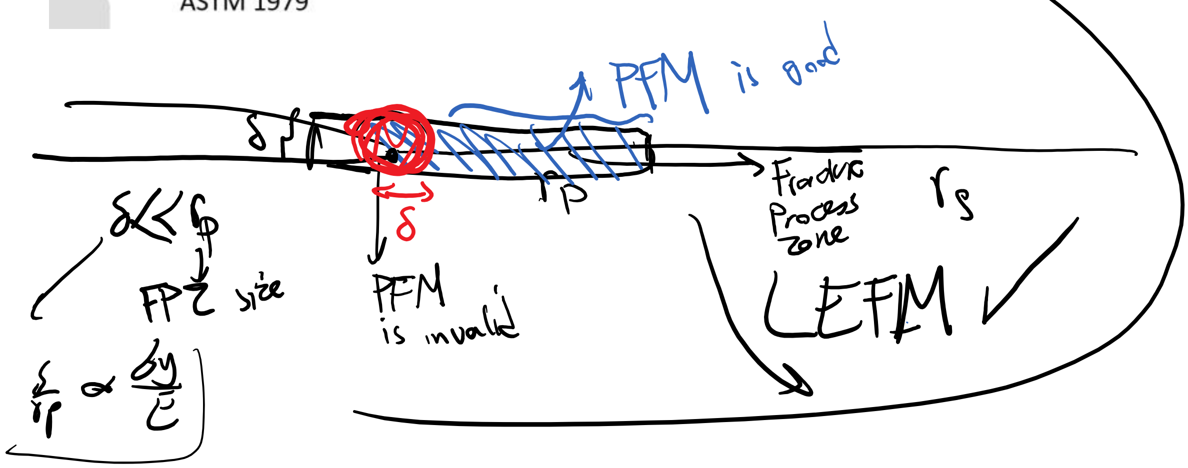
HRR solution: Stress singularity



Stress is still singular but with a weaker power of singularity!



McMeeking and Parks, ASTM STP 668, ASTM 1979



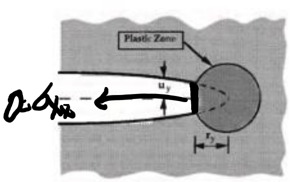
Limitations of HRR analysis

- Small strain: $\epsilon = \frac{1}{2}(\nabla \mathbf{u} + \nabla^T \mathbf{u})$ (accurate for $\epsilon \lessapprox 0.1$)
- Small deformation theory (*e.g.*, not using PK stresses, etc)
- Elastic HRR model instead of plastic model
- Crack tip blunting: $\Rightarrow \sigma_{xx} = 0$

otherwise we need to use large-strain definitions

$G = \frac{1}{2}(C^T C - I)$

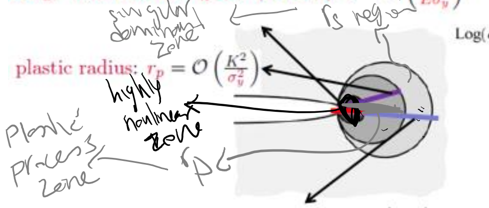
$C = \nabla u + I$



From SSY to LSY

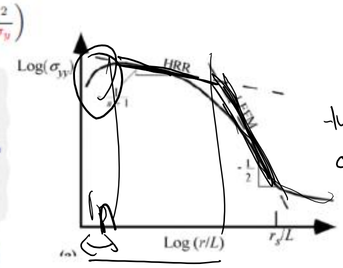
Large strain radius $r_n \propto \delta$ (CTOD): $\delta = \mathcal{O}\left(\frac{K^2}{E\sigma_y}\right)$

plastic radius: $r_p = \mathcal{O}\left(\frac{K^2}{\sigma_y^2}\right)$



K-dominant radius: $r_s = \mathcal{O}\left(\frac{K^2}{\bar{\sigma}^2}\right)$

$\bar{\sigma}$: applied stress

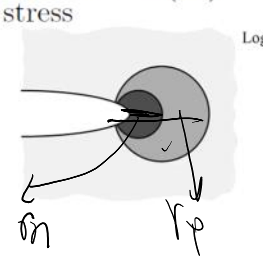


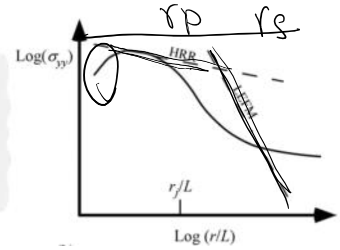
SSY (Small Scale Yielding)

$r_n \ll r_p \ll r_s \quad \Rightarrow$

typically $\propto \delta$

$$\boxed{\frac{r_p}{r_s} \propto \left(\frac{\bar{\sigma}}{\sigma_y}\right)^2 \ll 1}$$

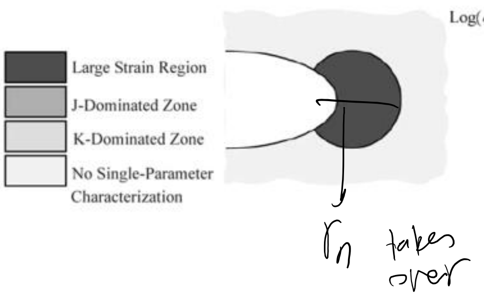


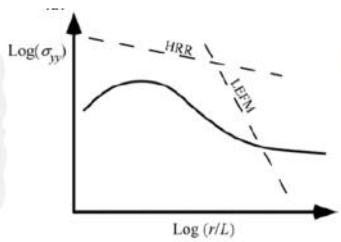


Elastic plastic condition

$r_n \ll r_p \approx r_s \quad \Rightarrow$

$$\boxed{\frac{r_n}{r_p} \ll 1, \quad \frac{r_p}{r_s} \propto \left(\frac{\bar{\sigma}}{\sigma_y}\right)^2 \approx 1}$$





LSY (Large Scale Yielding)

$r_n \approx r_p$

Note that $\frac{\delta}{r_p} \propto \left(\frac{\sigma_y}{E}\right)$

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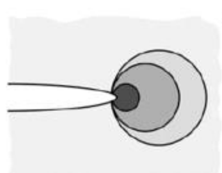
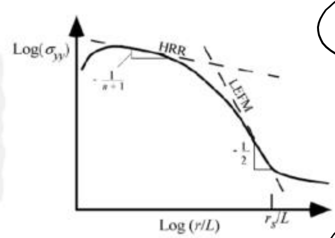
Fracture process zone



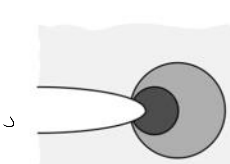


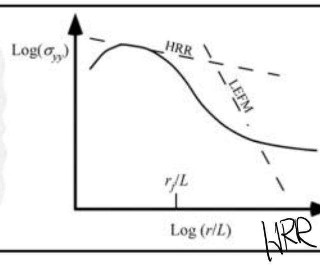
LEFM: SSY satisfied and gene ~~have~~ Generally have

$$\bar{\sigma} \ll \sigma_y$$

Relevant parameters:

G (energy) K (stress)



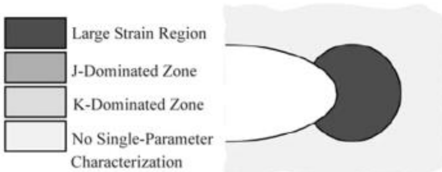


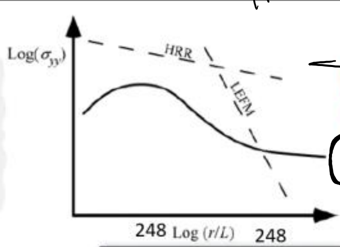
PFM (or NFM): SSY is gradually violated and

$$\bar{\sigma} \approx \sigma_y$$

Relevant parameters:

J (energy & used for stress)





LSY condition:

No single parameter can characterize fracutre!

J + other parameters (e.g. T stress, Q-J, etc)

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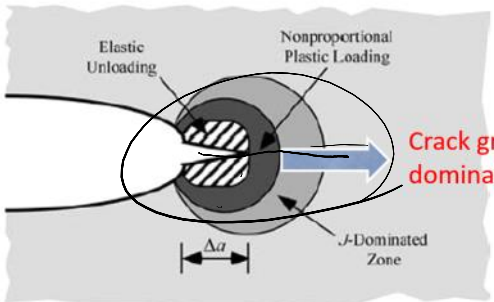
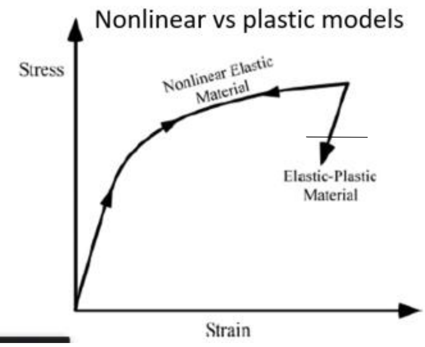
LSY: When a single parameter (G, K, J, CTOD) is not enough?

- Under considerable plastic deformation and crack propagation when unloading and non-proportional zones grow out of J dominant zone with crack propagation. Reasons are:
  - Unloading: In J integral analysis plastic model was replaced by a
  -





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