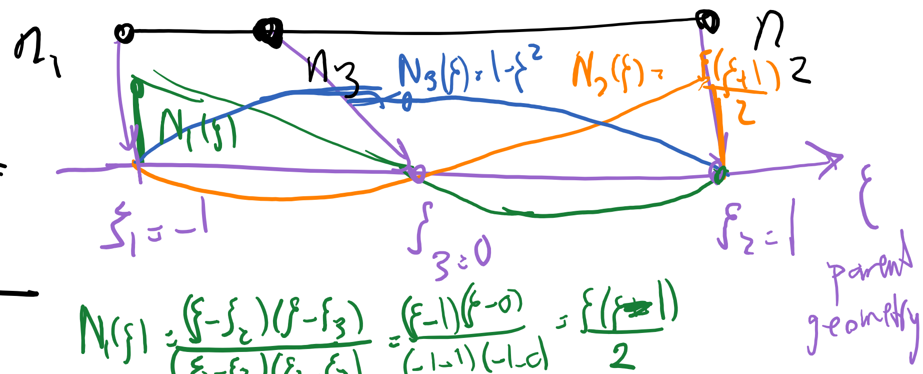
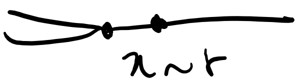
2020/10/1

Friday, October 16, 2020 2:51 PM

Real geometry

$\alpha = \frac{1}{4}$ $\quad \varepsilon \propto \frac{1}{\sqrt{x}}$





$$N_1(\xi) = \frac{(\xi-\xi_2)(\xi-\xi_3)}{(\xi_1-\xi_2)(\xi_1-\xi_3)} = \frac{(\xi-1)(\xi-0)}{(-1-1)(-1-0)} = \frac{\xi(\xi-1)}{2}$$

$$\boxed{U(\xi) = u_1 N_1(\xi) + u_2 N_2(\xi) + u_3 N_3(\xi)}$$

$$\varepsilon = \frac{du}{dx} = \frac{du(\xi)}{d\xi} \cdot \frac{d\xi}{dx}$$

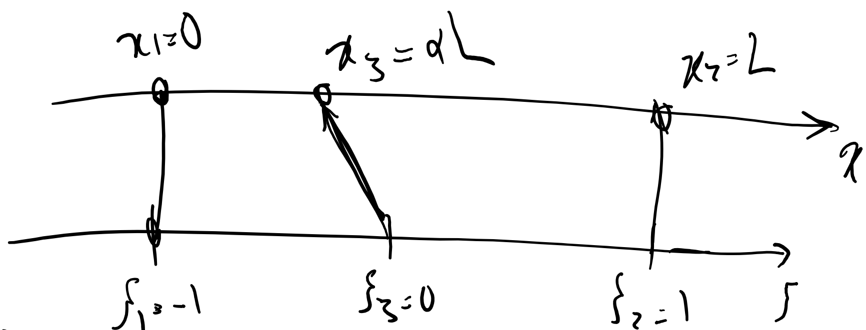
$\xi(x)$ does not exist, instead we'll have $x(\xi)$

$$x(\xi) = x_1 N_1(\xi) + x_2 N_1(\xi) + x_3 N_3(x)$$

$$x(\xi_1) = x_1 \underbrace{N_1(\xi_1)}_{1} + x_2 \underbrace{\cancel{N_2(\xi_1)}}_{0} + x_3 \underbrace{\cancel{N_3(\xi_1)}}_{0}$$

$$= x_1$$

Similarly $\quad x(\xi_2) = x_2 \quad , \quad x(\xi_3) = x_3$

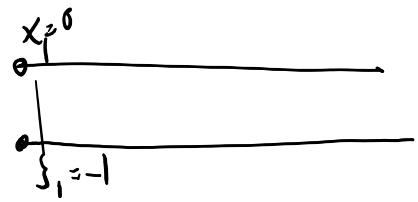


$$x = \underbrace{x_1}_{0} \overbrace{\frac{\xi(\xi-1)}{2}}^{N_1} + \underbrace{x_2}_{L} \left(\overbrace{\frac{\xi(\xi+1)}{2}}^{N_3} \right) + \underbrace{\alpha L}_{x_3} \left(1-\xi^2 \right)$$

$$\boxed{x = L\left(\alpha + \frac{\xi}{2} + \xi^2\left(\frac{1}{2} - \alpha\right)\right)} \quad (\star)$$

$$u = u_1 \frac{\xi(\xi-1)}{2} + u_2 \frac{\xi(\xi+1)}{2} + u_3(1-\xi^2)$$

$$\varepsilon(\xi) = \frac{du/d\xi}{dx/d\xi} = \_\_\_\_\_\_$$



at x_1 $(\xi = -1)$ we want

$\varepsilon \to \infty$

$$\frac{dx}{d\xi} = L\left(\frac{1}{2} + 2\xi\left(\frac{1}{2} - \alpha\right)\right)$$

$$\frac{dx}{d\xi}(\xi=-1) = L\left(\frac{1}{2} - 1(1-2\alpha)\right) = 0 \Rightarrow \boxed{\alpha = \frac{1}{4}}$$

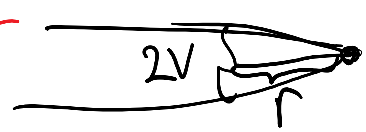
for $\alpha = \frac{1}{4}$ in $(\star)$ we get

$$\boxed{x = \frac{L}{4}(\xi+1)^2 \rightarrow \xi = 2\sqrt{x/L} - 1}$$

$$u = u_1\left(\frac{\xi(\xi-1)}{2} + u_2\, \xi\frac{(\xi+1)}{2} + u_3(1-\xi^2)\right)$$

$$\rightarrow \quad u = u_1 + \sqrt{\frac{x}{L}}\left(-3u_1 - u_2 + 4u_3\right) + 2\frac{x}{L}\left(u_1 + u_2 - 2u_3\right)$$

$$2v = \frac{8K_I}{\sqrt{2\pi}E'}\sqrt{r} \quad (\mathrm{I})$$



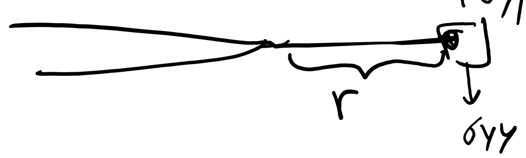
$\varepsilon = \frac{du}{dx}$ in (I)

$$\varepsilon = \frac{1}{\quad}\left(-3u_1 - u_2 + 2(\ldots)\right) + \frac{1}{\quad}\left(u_1 + u_2 - 2u_3\right)$$

$$\varepsilon = \frac{1}{\sqrt{\alpha L}}\left(\frac{-3}{2}u_1 - \frac{u_2}{2} + 2u_3\right) + \frac{1}{L}\left(u_1 + u_2 - 2u_3\right)$$

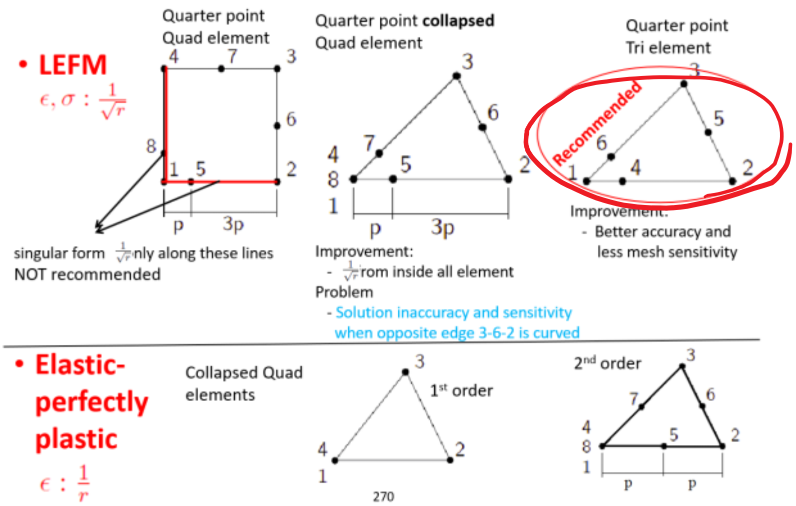
$$\sigma = E\varepsilon = \frac{1}{\sqrt{xL}}$$

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}$$

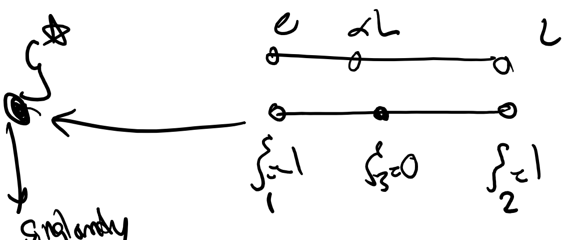


This is why 1/4 position 2nd elements, can produce LEFM singularity

Isoparametric singular elements



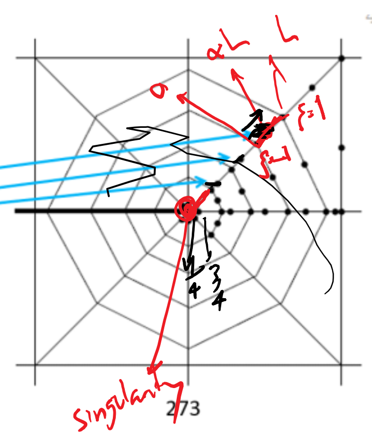
How should the mesh look like around the crack tip?



singularity starts @ $\xi = -1$ and moves to $\xi = -\infty$

- Transition elements:
  According to this analysis
  mid nodes of next layers move to ½ point from ¼ point

Lynn and Ingraffea 1977)



$$\varepsilon = \frac{\frac{du}{d\xi}}{\frac{dx}{d\xi}} \qquad \frac{dx}{d\xi} = 0 \text{ @ } \xi^* \qquad \frac{dx}{d\xi} = L\left(\frac{1}{2} + \xi(1-2\alpha)\right)$$

$$\rightarrow \alpha = \frac{\pm 1}{4\xi^*} + \frac{1}{2}$$

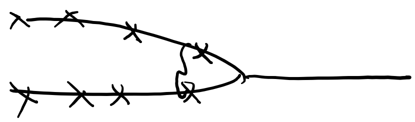
Recall $\rho^* = -1 \to \alpha = \frac{1}{4}$

$\rho^* = -\infty \to \boxed{\alpha = \frac{1}{2}}$

As we will see, even with inferior meshes we can get decent solutions for K, J, G if the right method is used.

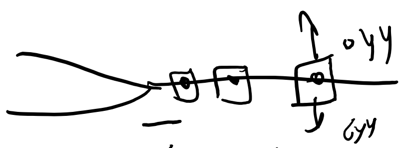
6.1.3. Extraction of K (SIF), G

— u_y



$2u_y \propto \sqrt{r}\, K \to K = ?$

— σ_y

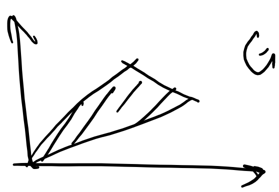


$\sigma_{yy} = \frac{K}{\sqrt{2\pi r}}$

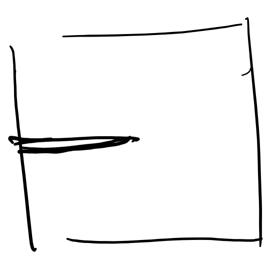
Direct extraction of K

— K calculated from G or (J)

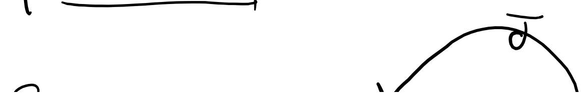
models G $= \frac{K_I^2}{E'} \to K_I = \sqrt{GE'}$

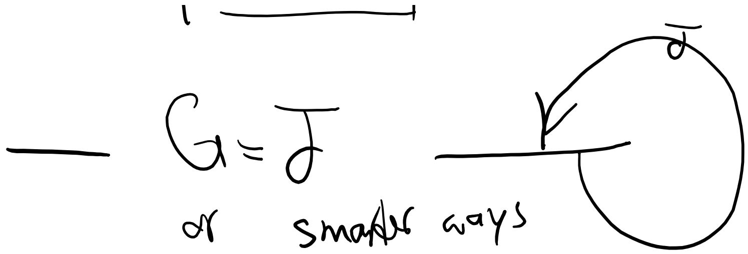


$G = \frac{\text{Shaded area}}{B\Delta a} = \frac{P^2}{2B}\frac{dC}{da}$



$G = -\frac{d\Pi}{B\,da}$ (potential energy)



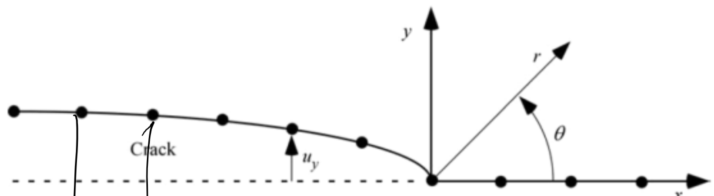


$G = J$

or smarter ways to calculate J

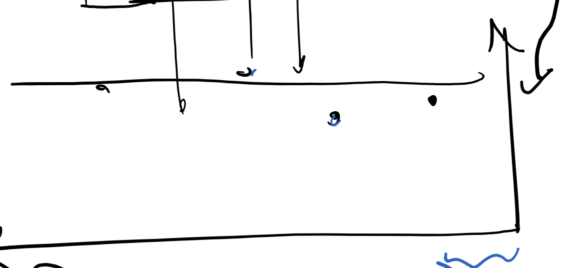
1. K from local fields

1. Displacement



$$u_y(r, \theta = \pi) = \frac{4K_I\sqrt{r}}{\sqrt{2\pi}E'} \rightarrow \boxed{K_I = \frac{u_y(r)\sqrt{2\pi}E'}{4\sqrt{r}}}$$

$r \rightarrow 0$



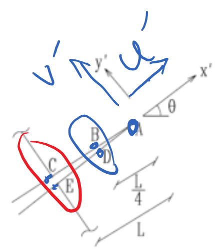
too far

higher order asymptotic terms pollute the solution.

to close to CT is bad

large numerical errors "discretization"

or alternatively from the first quarter point element:



$$\left.\begin{aligned} v &= K_I \frac{\kappa+1}{2G}\sqrt{\frac{r}{2\pi}} \\ u' &= \bar{u}'_A + \left(-3\bar{u}'_A + 4\bar{u}'_B - \bar{u}'_C\right)\sqrt{\tfrac{r}{L}} + \left(2\bar{u}'_A + 2\bar{u}'_C - 4\bar{u}'_B\right)\tfrac{r}{L} \\ v' &= \bar{v}'_A + \left(-3\bar{v}'_A + 4\bar{v}'_B - \bar{v}'_C\right)\sqrt{\tfrac{r}{L}} + \left(2\bar{v}'_A + 2\bar{v}'_C - 4\bar{v}'_B\right)\tfrac{r}{L} \end{aligned}\right\}$$

$$K_I = \frac{2G}{\kappa+1}\sqrt{\frac{2\pi}{L}}\left(-3\bar{v}'_A + 4\bar{v}'_B - \bar{v}'_C\right)$$

Recall for 1D

$$u = u_1 + \frac{\sqrt{x}}{\sqrt{L}}\left(-3u_1 - u_2 + 4u_3\right) + \frac{2x}{L}\left(u_1 + u_2 - 2u_3\right)$$

U_{CE}

$= \Delta U_{BD}$

$$\begin{Bmatrix} K_I \\ K_{II} \end{Bmatrix} = \frac{1}{2}\frac{2G}{\kappa+1}\sqrt{\frac{2\pi}{L}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -3\bar{u}'_A + 4\left(\bar{u}'_B - \bar{u}'_D\right) - \left(\bar{u}'_C - \bar{u}'_E\right) \\ -3\bar{v}'_A + 4\left(\bar{v}'_B - \bar{v}'_D\right) - \left(\bar{v}'_C - \bar{v}'_E\right) \end{bmatrix}$$

$$u = u_1 + \frac{\sqrt{x}}{\sqrt{L}}(-3u_1 - u_2 + 4u_3) + \frac{x}{L}(u_1 + u_2 - 2u_3)$$

$$\left\{\begin{array}{c} K_I \\ K_{II} \end{array}\right\} = \frac{1}{2}\frac{2G}{\kappa+1}\sqrt{\frac{2\pi}{L}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -3\overline{u}'_A + 4\left(\overline{u}'_B - \overline{u}'_D\right) - \left(\overline{u}'_C - \overline{u}'_E\right) \\ -3\overline{v}'_A + 4\left(\overline{v}'_B - \overline{v}'_D\right) - \left(\overline{v}'_C - \overline{v}'_E\right) \end{bmatrix}$$

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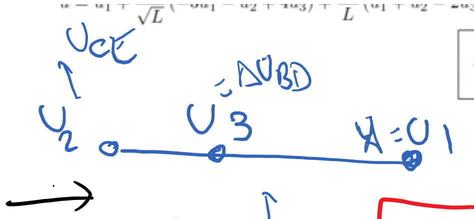
Mixed mode generalization:



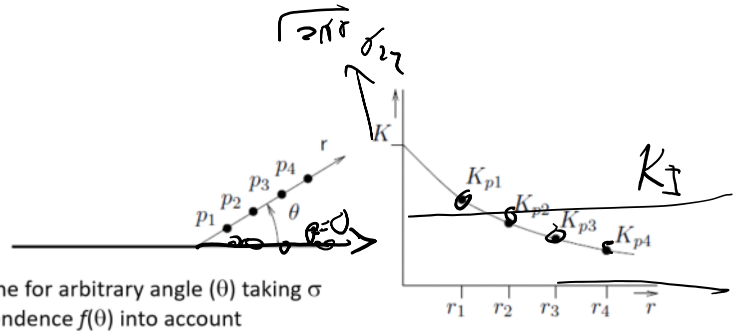
$$U = U_1 + \sqrt{\frac{x}{L}}\left(-3U_1 - U_3 + 4U_2\right) + 2\frac{x}{L}\left(U_1 + U_2 - 2U_3\right)$$

1. K from local fields

2. Stress

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \times \sqrt{2\pi r}$$

$$K_I = \lim_{r\to 0}\left(\sqrt{2\pi r}\;\sigma_{22}|_{\theta=0}\right) \quad ; \quad K_{II} = \lim_{r\to 0}\left(\sqrt{2\pi r}\;\sigma_{12}|_{\theta=0}\right)$$



or can be done for arbitrary angle (θ) taking σ angular dependence $f(\theta)$ into account

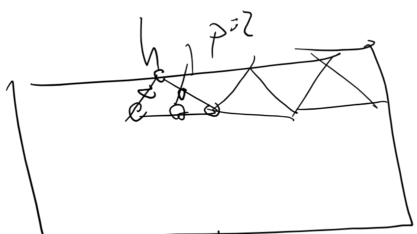
Reasons why σ approach is not preferred, compared to u

1 — $\sigma \propto \frac{1}{\sqrt{r}} \to \infty \qquad u \propto \sqrt{r} \to 0$

Even with quarter point elements & spider web mesh $r \to 0$

It's much more difficult to capture σ field because $\sigma \to \infty$

2 — FEM



discret error $= C\,h^{p+1}$ for displacement u

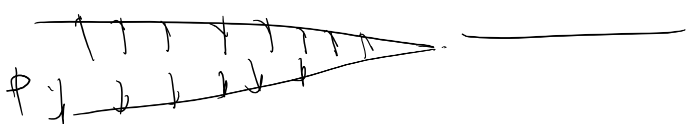
$= Ch$ displacement

$\varepsilon, (\sigma = C\varepsilon)$ $= Ch^p$ stress solution has a lower convergence rate

$= \nabla u + \nabla u^T$

stress solution in general is less accurate.

3. This method is sensitive to crack surface tractions



2. K from energy approaches

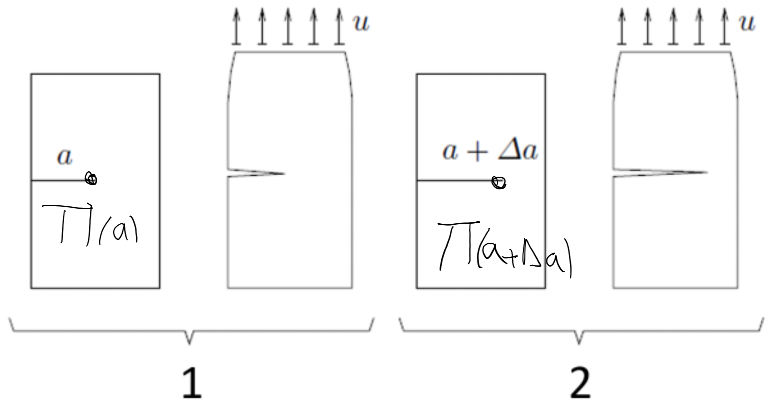
1. Elementary crack advance (two FEM solutions for a and $a + \Delta a$)
2. Virtual Crack Extension: Stiffness derivative approach
3. J-integral based approaches (next section)

After obtaining G (or $J=G$ for LEFM) K can be obtained from

$$K_I^2 = E'G \qquad E' = \begin{cases} E & \text{plane stress} \\ \frac{E}{1-\nu^2} & \text{plane strain} \end{cases}$$

2.1 Elementary crack advance

For fixed grip boundary condition perform two simulations (1, a) and (2, $a+\Delta a$):
All FEM packages can compute strain (internal) energy U_i



$$G = -\frac{d\Pi}{B\,da} \rightarrow \text{use Finite Difference}$$

$$G \simeq -\frac{\Pi(a+\Delta a) - \Pi(a)}{B\,\Delta a}$$

Cons: – 2 solutions needed

– Like any FD calculation we can have large errors in $\frac{d\Pi}{da}$ if Δa is large and large numerical cancelation if Δa is small

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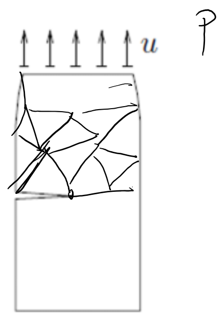
2.2 Virtual crack extension

n dofs

$$\Pi = U_e - W$$

U_e: internal energy, W: external work

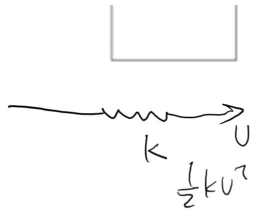


$$\Pi = \frac{1}{2} U^t K U - P u$$

[illegible]

$\Pi = \frac{1}{2} U^t K U - P U$

vector of unknowns — unknown vector — stiffness matrix



B=1 (assume)

$$G = -\frac{d\Pi}{da} = -\left(\frac{d\,\frac{1}{2} U^t K U}{da} - \frac{dPu}{da}\right)$$

$$G = \left(-\frac{1}{2}\left(\frac{du}{da}\right)^t K u - \frac{1}{2} U^t \frac{dK}{da} U - \frac{1}{2} U^t K \frac{du}{da}\right) + \frac{dP}{da} U + P \frac{du}{da}$$

$$\left(\frac{du}{da}\right)^t K U = U K \frac{du}{da} \qquad \leftarrow K = K^t$$

$$G = \left(-\frac{du}{da} \underbrace{K u}_{P} + P \frac{du}{da}\right) - \frac{1}{2} U^t \frac{dK}{da} U + \frac{dP}{da} U$$

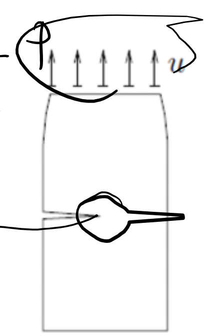
FEM solution satisfies $Ku = P$

$$\boxed{G(a) = -\frac{1}{2} U^t \frac{dK}{da} U + \frac{dP}{da} U}$$

typically zero

— Solve 1 problem

$Ku = P \to U$



typically don't depend on a

Elements changed only here

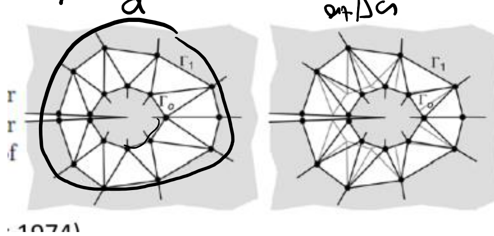
— Calculate $\frac{dK}{da}$

$$\frac{dK}{da} = \frac{K(a+\Delta a) - K(a)}{\Delta a}$$

very few

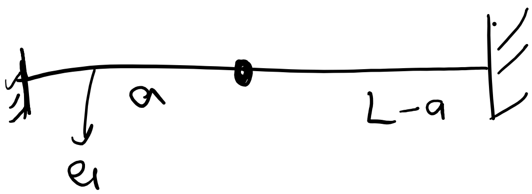
Mesh for a

Mesh a+Δa



very few entries are changed

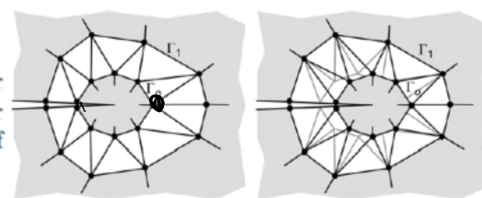
We can even do better than this, by analytically computing $\frac{dK}{da}$



$$\overset{e_1}{K}(a) = \frac{AE}{a}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \qquad \frac{d\overset{e_1}{k}(a)}{da} = -\frac{AE}{a^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

If we compute dK/da analytically, the virtual crack extension becomes equivalent to "Equivalent Domain Integration (EDI)" which is a very robust method for calculating J

- Only the few elements that are distrorted contribute to $\frac{\partial K}{\partial a}$
- We may not even need to form elements and assemble K for a and $a+\Delta a$ to obtain $\frac{\partial K}{\partial a}$. We can explicitly obtain $\frac{\partial k^e}{\partial a}$ for elements affected by crack growth by computing derivatives of actual geometry of the element to parent geometry.



- This method is equivalent to J integral method (Park 1974)

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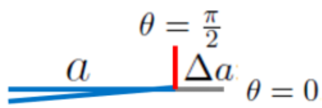
2.2 Virtual crack extension: Mixed mode

- For LEFM energy release rates G_1 and G_2 are given by

$$
\begin{aligned}
J_1 &= G_1 = \frac{K_I^2 + K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu} \\
J_2 &= G_2 = \frac{-2K_I K_{II}}{E'}
\end{aligned}
$$

- Using Virtual crack extension (or elementary crack advance) compute G_1 and G_2 for crack lengths a, $a + \Delta a$

$$
\begin{aligned}
J_1 &= G_1 = \frac{K_I^2 + K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu} \\
J_2 &= G_2 = \frac{-2K_I K_{II}}{E'}
\end{aligned}
$$



- Obtain K_I and K_{II} from:

$$
\boxed{
\begin{aligned}
K_I &= \frac{s \pm \sqrt{s^2 + \frac{8G_2}{\alpha}}}{4} \\
K_{II} &= \frac{s \mp \sqrt{s^2 + \frac{8G_2}{\alpha}}}{4}
\end{aligned}
}
$$

Note that there are two sets of solutions!

$s = 2\sqrt{\frac{G_1 - G_2}{\alpha}}$ and $\alpha = \frac{(1+\nu)(1+\kappa)}{E}$

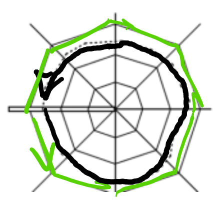
6.1.4. J integral

Methods to evaluate J integral:

1. Contour integral:

$$
J_1 = \int_\Gamma \left(w\,dy - \mathbf{t}\frac{\partial \mathbf{u}}{\partial x} d\Gamma \right)
$$

$$
J_2 = \int_\Gamma \left(-w\,dx - \mathbf{t}\frac{\partial \mathbf{u}}{\partial y} d\Gamma \right)
$$

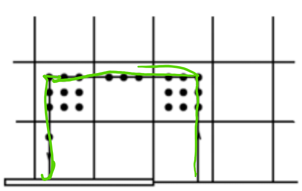


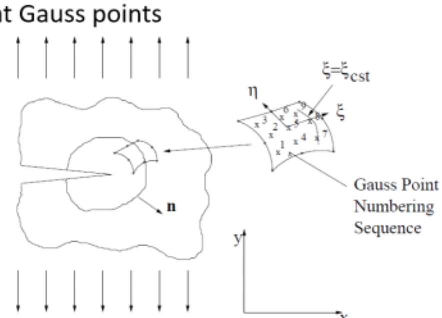
G = J even for nonlinear elasticity,

Contour integrals are difficult to implement in FEMs:

J integral: 1.Contour integral

- Stresses are available and also more accurate at Gauss points
- Integral path goes through Gauss points





$$J = \int_\Gamma w dy - \mathbf{t}\cdot\frac{\partial \mathbf{d}}{\partial x} ds \Rightarrow J^e = \int_{-1}^{1}\left\{\underbrace{\frac{1}{2}\left[\sigma_x\frac{\partial u}{\partial x} + \tau_{xy}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) + \sigma_y\frac{\partial v}{\partial y}\right]}_{w}\underbrace{\frac{\partial y}{\partial \eta}}_{dy} - \underbrace{\left[(\sigma_x n_1 + \tau_{xy} n_2)\frac{\partial u}{\partial x} + (\tau_{xy} n_1 + \sigma_y n_2)\frac{\partial v}{\partial x}\right]}_{\mathbf{t}\cdot\frac{\partial \mathbf{d}}{\partial x}}\underbrace{\sqrt{\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2}}_{ds}\right\} d\eta = \int_{-1}^{1} I d\eta$$

Cumbersome to formulate the integrand, evaluate normal vector, and integrate over lines (2D) and surfaces (3D)

Not commonly used

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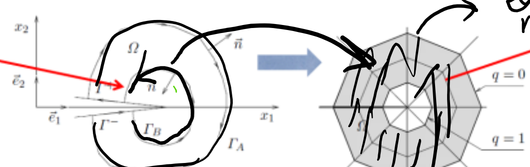
2. Equlvalent (Energy) **domain** integral (**EDI**):
   - Gauss theorem: line/surface (2D/3D) integral → surface/volume integral
   - Much simpler to evaluate computationally → easier to implement, more accurate & general
   - Easy to incorporate plasticity, crack surface tractions, thermal strains, *etc.*
   - Prevalent method for computing J-integral

Line integral



Volume integral

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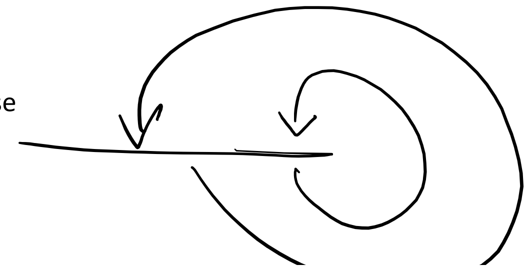
Assumptions we have for J = G, and that we could use any contour plots

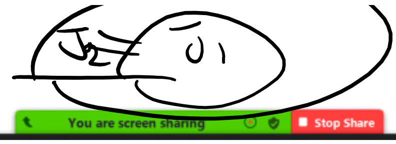


Energy release rate of J integral: Assumptions

1. Homogeneous body
2. Linear or non-linear elastic solid (No plastic unloading)
3. No inertia, or body forces; no initial stresses $\nabla\cdot\sigma + \rho b = \cancel{\rho a}$ static
4. No thermal loading & no plastic unloading $\sigma = \frac{\partial W(\varepsilon)}{\partial \varepsilon}$
5. 2-D stress and deformation field
6. Plane stress or plane strain
7. Mode I loading $J_2 = J_1$
8. Stress free crack

7. Mode I loading

8. Stress free crack



Generalization of J integral

- Dynamic loading
- Surface tractions on crack surfaces
- Body force
- Initial strains (e.g. thermal loading)
- Initial stress from pore pressures

cf. Saouma 13.11 & 13.12 for details

General form of J integral

$$w = \int_0^{\varepsilon_{kl}^m} \sigma_{ij} \, d\varepsilon_{ij}^m$$

$$J = \lim_{\Gamma_o \to 0} \int_{\Gamma_o} \left[(w + T)\delta_{1i} - \sigma_{ij} \frac{\partial u_j}{\partial x_1} \right] n_i \, d\Gamma$$

Inelastic stress

Can include (visco-) plasticity, and thermal stresses

Kinetic energy density

$$T = \frac{1}{2}\rho \frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t}$$

$$\varepsilon_{ij}^{total} = \varepsilon_{ij}^{e} + \varepsilon_{ij}^{p} + \alpha\Theta\delta_{ij} = \varepsilon_{ij}^{m} + \varepsilon_{kk}^{t}$$

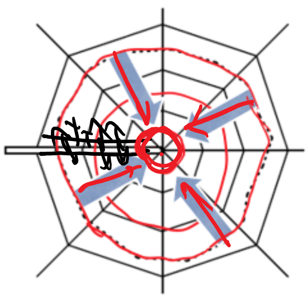
nonlinear

Elastic

we had before

Plastic

Thermal (Θ temperature)



$\Gamma_o \to 0$: J contour approaches Crack tip (CT)

Accuracy of the solution deteriorates at CT

Inaccurate/Impractical evaluation of J using contour integral

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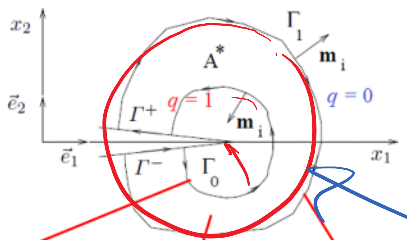
This formula is great: Crack surface traction, plasticity, dynamics, can even have body force

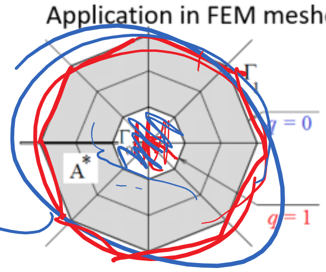
The catch is that the contour integral should be at the limit of a point around the crack tip ->

Computationally, this is where we have the worst numerical solution (very close to the crack tip)

Divergence theorem: Line/Surface (2D/3D) integral **~~Surface~~/Volume Integral**



Application in FEM meshes



Original J integral contour

$\Gamma_o \to 0$ 2D mesh covers crack tip

Surface integral after using divergence theorem

- Contour integral added to create closed surface
- By using $q = 0$ this integral in effect is zero

$$J = \int_{\Gamma_o} \left[(w+T)\delta_{1i} - \sigma_{ij} \frac{\partial u_j}{\partial x_1} \right] n_i d\Gamma = \int_{\Gamma^*} \left[\sigma_{ij} \frac{\partial u_j}{\partial x_1} - w\delta_{1i} \right] q m_i d\Gamma - \int_{\Gamma_+ + \Gamma_-} \sigma_{2j} \frac{\partial u_j}{\partial x_1} q d\Gamma$$

Zero integral on Γ_1 ($q = 0$)

Divergence theorem

$$\boxed{J = \int_{A^*} \frac{\partial}{\partial x_i} \left\{ \left[\sigma_{ij} \frac{\partial u_j}{\partial x_1} - w\delta_{1i} \right] q \right\} dA - \int_{\Gamma_+ + \Gamma_-} \sigma_{2j} \frac{\partial u_j}{\partial x_1} q d\Gamma}$$

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