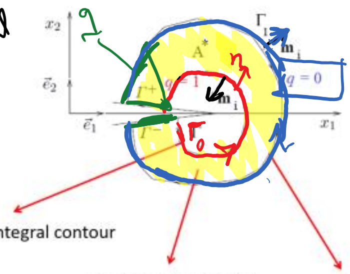
J integral: 2. EDI: Derivation

Divergence theorem: Line/Surface (2D/3D) integral ~~Surface~~/Volume Integral

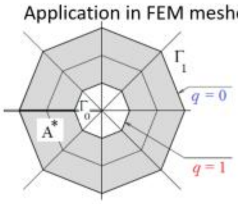
m_i normal to ∂A^*



Original J integral contour

Surface integral after using divergence theorem

Application in FEM meshes



$\Gamma_o \rightarrow 0$ 2D mesh covers crack tip

- Contour integral added to create closed surface
- By using $q=0$ this integral in effect is zero

$$J = \int_{\Gamma_0}\left[(w+T)\delta_{i1}-\sigma_{ij}\frac{\partial u_j}{\partial x_1}\right]\underbrace{n_i}_{=-m_i}d\Gamma$$

on Γ_0 $q(\vec{x})=1$

$q=0$ on Γ_1 will be 0

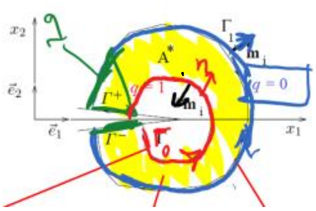
$$-J = \int_{\Gamma_0}\underbrace{\left[(w+T)\delta_{i1}-\sigma_{ij}\frac{\partial u_j}{\partial x_1}\right]}_{I}m_i q\,d\Gamma + \int_{\Gamma_1} I m_i q\,d\Gamma$$

$$+\int_{\Gamma^-\cup\Gamma^+} I m_i q\,d\Gamma$$

$$+\int_{\Gamma^-\cup\Gamma^+} I m_i q\,d\Gamma = \int_{\partial A^*}\left[(w+T)\delta_{i1}-\sigma_{ij}\frac{\partial u_j}{\partial x_1}\right]q\,m_i\,d\Gamma$$

Use Gauss theorem

$$= -J + \int_{\Gamma^-\cup\Gamma^+}\left[(w+T)\delta_{i1}-\sigma_{ij}\frac{\partial u_j}{\partial x_1}\right]q\,m_i\,d\Gamma$$



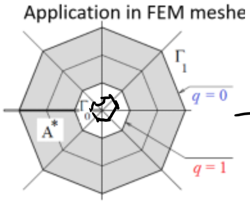
$$\boxed{J = \int_{A^*}\frac{\partial}{\partial x_i}\left\{\left[\sigma_{ij}\frac{\partial u_j}{\partial x_1}-(w+T)\delta_{i1}\right]q\right\}dV + \int_{\Gamma^-\cup\Gamma^+}\left[\cancel{(w+T)\delta_{i1}}-\sigma_{ij}\frac{\partial u_j}{\partial x_1}\right]q\,m\,d\Gamma}$$

0

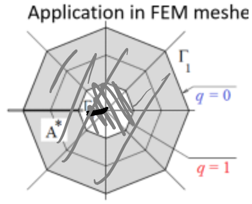
$\equiv$ div $= 0$

Let $\Gamma_0 \rightarrow 0$

Application in FEM meshe



Application in FEM meshe



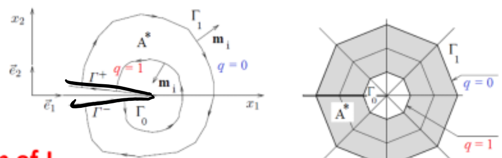
crack surface contribution

$$J=\int_{A^*}\frac{\partial}{\partial x_i}\left\{\left[\sigma_{ij}\frac{\partial u_j}{\partial x_1}-w\delta_{1i}\right]q\right\}dA-\int_{\Gamma_++\Gamma_-}\sigma_{2j}\frac{\partial u_j}{\partial x_1}q\,d\Gamma$$

$$J=\int_{A^*}\frac{\partial}{\partial x_i}\left\{\left[\sigma_{ij}\frac{\partial u_j}{\partial x_1}-w\delta_{1i}\right]q\right\}dA-\int_{\Gamma_++\Gamma_-}\sigma_{2j}\frac{\partial u_j}{\partial x_1}q\,d\Gamma$$

$$\underbrace{\frac{\partial}{\partial x_i}\left(\sigma_{ij}\frac{\partial u_j}{\partial x_1}-w\delta_{1i}\right)}_{\text{Often is zero}}q+\left(\sigma_{ij}\frac{\partial u_j}{\partial x_1}-w\delta_{1i}\right)\frac{\partial q}{\partial x_j}$$

$(q) \qquad (\qquad) \left(\frac{\partial q}{\partial x_i}\right)$



General form of J

$$J=\int_{A^*}\left\{\left[\sigma_{ij}\frac{\partial u_j}{\partial x_1}-w\delta_{1i}\right]\frac{\partial q}{\partial x_i}+\left[\sigma_{ij}\frac{\partial \varepsilon_{ij}^p}{\partial x_1}-\frac{\partial w^p}{\partial x_1}+\alpha\sigma_{ii}\frac{\partial\Theta}{\partial x_1}-F_i\frac{\partial u_i}{\partial x_1}\right]q\right\}dA-\int_{\Gamma_++\Gamma_-}\sigma_{2j}\frac{\partial u_j}{\partial x_1}q\,d\Gamma$$

Plasticity effects

Thermal effects

Body force

Nonzero crack surface traction

often these terms are zero

many problem these go away

thermal effects

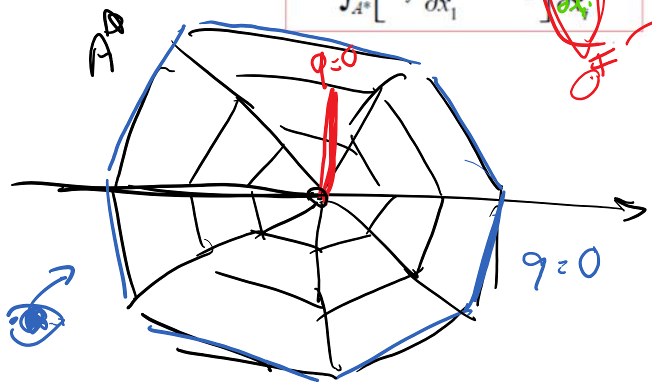
surface traction

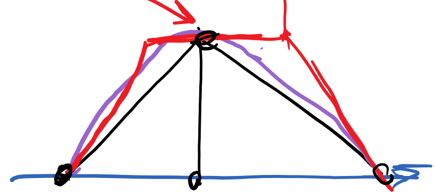
Simplified Case:

(Nonlinear) elastic, no thermal strain, no body force, traction free crack surfaces

$$J=\int_{A^*}\left[\sigma_{ij}\frac{\partial u_j}{\partial x_1}-w\delta_{1i}\right]\frac{\partial q}{\partial x_i}dA$$

$\neq 0$

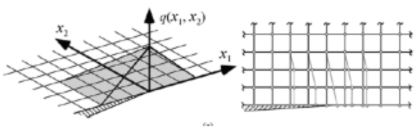




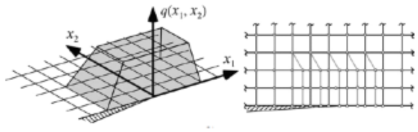
for simplified case

$$J=\int_{A^*}\left[\sigma_{ij}\frac{\partial u_j}{\partial x_1}-w\delta_{1i}\right]\frac{\partial q}{\partial x_i}dA$$

- Shape of decreasing function *q*:

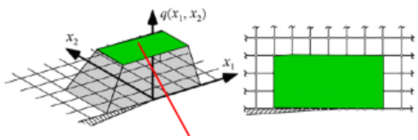


Pyramid *q* function

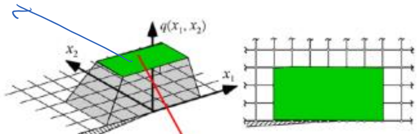


Plateau *q* function

- Plateau q function useful when inner elements are not very accurate: e.g. when singular/quarter point elements are not used



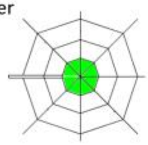
bad solution does not hurt us



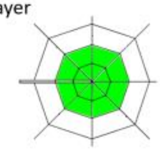
$$J=\int_{A^*}\left[\sigma_{ij}\frac{\partial u_j}{\partial x_1}-w\delta_{1i}\right]\frac{\partial q}{\partial x_i}dA \qquad \frac{\partial q}{\partial x_i}=0$$

These elements do not contribute to J

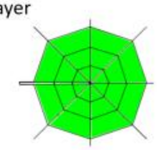
- Since $J_0 \to 0$ the inner J_0 collapses to the crack tip (CT)
- J_1 will be formed by element edges
- By using spider web (rozet) meshes any reasonable number of layers can be used to compute J:



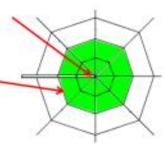
1 layer



2 layer



3 layer



- Spider web (rozet) mesh:
  - One layer of triangular elements (preferably singular, quadrature point elements)
  - Surrounded by quad elements