2020/11/02

Monday, November 2, 2020 2:50 PM

4.3 Mixed mode fracture

4.3.1 Crack propagation criteria

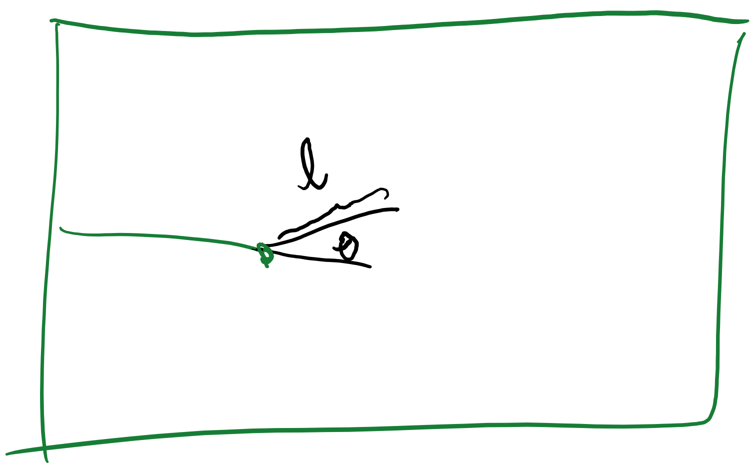
a) Maximum Circumferential Tensile Stress

b) Maximum Energy Release Rate

c) Minimum Strain Energy Density

4.3.2 Crack Nucleation criteria

Propagation:

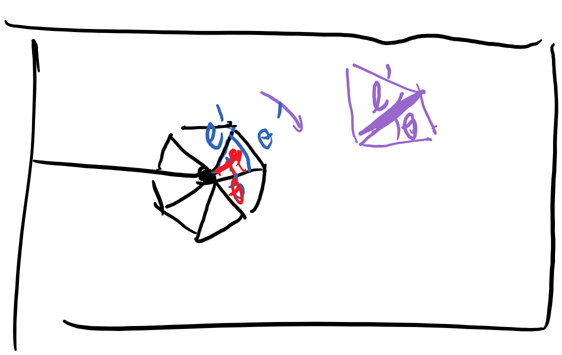


i) Does the crack grow?

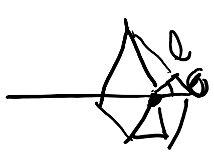
ii) What direction it grows (θ)?

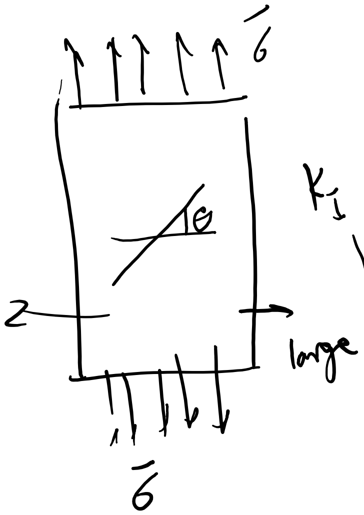
iii) How far it propagates (l) ?

Assuming that we know the answer, achieving this computationally is very difficult



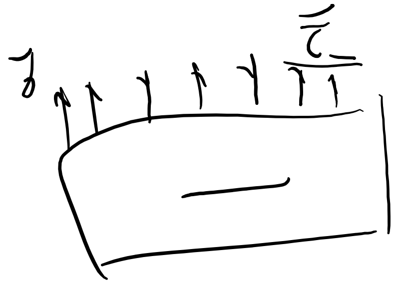
Mesh adaptive schemes can capture one or both correctly





$K_I = \sigma\sqrt{\pi a}\cos^2\theta$

$K_{II} = \sigma\sqrt{\pi a}\cos\theta\sin\theta$



$K_I = \sigma\sqrt{\pi a}$

$K_{II} = \tau\sqrt{\pi a}$

pure mode I, crack grows when $K_I = K_{Ic} \rightarrow \frac{K_I}{K_{Ic}} = 1$

" " II " " " $K_{II} = K_{IIc} \rightarrow \frac{K_{II}}{K_{IIc}} = 1$

$K_I \neq 0,\ K_{II} \neq 0$

- Pure Mode I fracture:

$$K_{\mathrm{I}} \geq K_{\mathrm{Ic}}$$

- Mixed mode fracture (in-plane)

$$F\left(K_{\mathrm{I}}, K_{\mathrm{II}}, K_{\mathrm{Ic}}\right) = 0$$

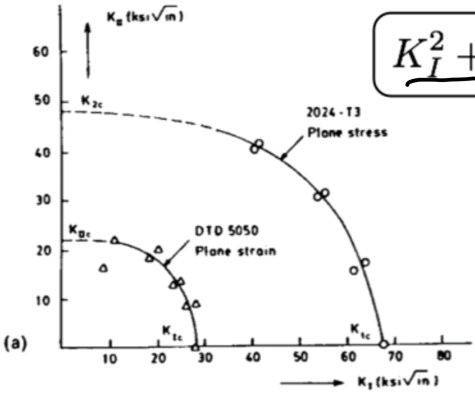
- Note the similarity with yield surface plasticity model:

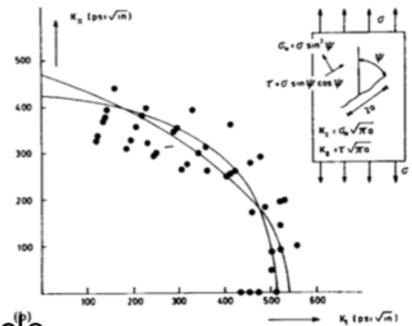
$$F_{yld}(\sigma_1, \sigma_2, \sigma_y) = 0 \qquad \sigma_v = \sigma_y \quad \text{for} \quad \sigma_v = \sqrt{\frac{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2}{2}}$$

Example: von Mises yield criterion

$$\boxed{K_I^2 + K_{II}^2 = K_{Ic}^2}$$

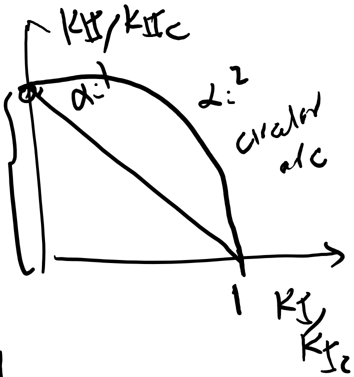
a circle in KI, KII plane





Data points do not fall exactly on the circle.

$$\left(\frac{\langle K_I \rangle_+}{K_{Ic}}\right)^{\alpha} + \left|\frac{K_{II}}{K_{IIc}}\right|^{\alpha} = 1$$



phenomenological model.

we'll discuss a few models that determine when the crack grows

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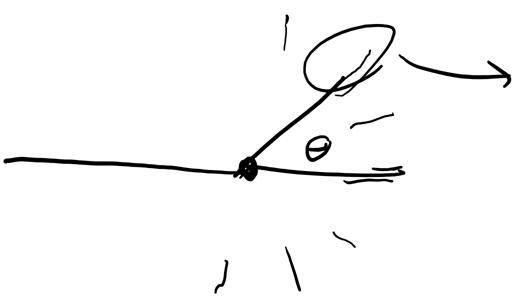
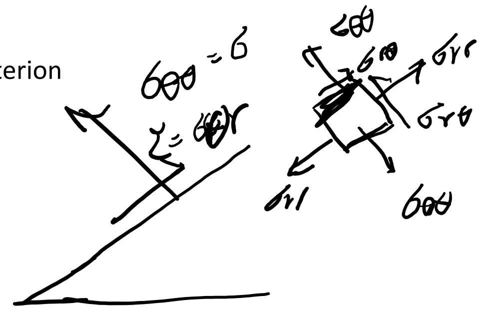
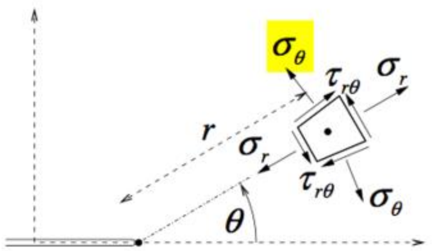
1. Maximum circumferential (hoop) stress criterion





crack propagates in a direction θ_c for which $\sigma_{\theta\theta}(\theta)$ is maximum (among all $\sigma_{\theta\theta}(\theta)$ $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$)

Erdogan and Sih



maximum circumferential stress criterion
(maximum hoop stress criterion):
crack propagates in the direction perpendicular to the maximum circumferential stress
(evaluated on a circle of a small diameter centered at the tip)

the direction of propagation is given by the angle θ_c for which

$$\sigma_\theta(r,\theta_c) = \max_{-\pi<\theta<\pi} \sigma_\theta(r,\theta)$$

(from M. Jirasek)

principal stress $\longrightarrow \tau, \boxed{\tau_{r\theta} = 0}$

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$$\sigma_r = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{5}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right) \quad (7.35a)$$

$$\sigma_\theta = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{3}{4}\sin\frac{\theta}{2} - \frac{3}{4}\sin\frac{3\theta}{2}\right) \quad (7.35b)$$

4 x

$$\tau_{r\theta} = \frac{K_I}{\cancel{\sqrt{2\pi r}}}\left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right) + \frac{K_{II}}{\cancel{\sqrt{2\pi r}}}\left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right) = 0 \quad (7.35c)$$

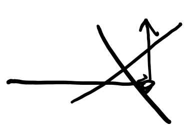
$$\tau_{r\theta} = 0 \longrightarrow \quad K_I\left(\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right) + K_{II}\left(\cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2}\right) = 0$$

$$\longrightarrow \theta_c = 2\arctan\frac{1}{4}\left(K_I/K_{II} \pm \sqrt{(K_I/K_II)^2 + 8}\right)$$

crack propagati direchi

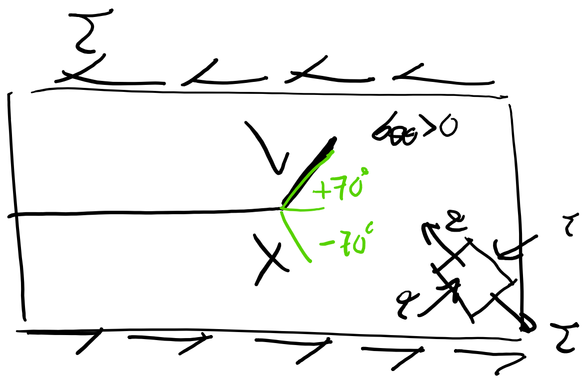
$K_I \neq 0 \quad K_{II} = 0$
pure mode I $\longrightarrow$ $K_I \, \Sigma(\frac{\theta}{2}) + \Sigma \frac{3\theta}{2} = 0$
$\theta = 0$ ✓

pure mode II
$K_I = 0 \quad K_{II} \neq 0$

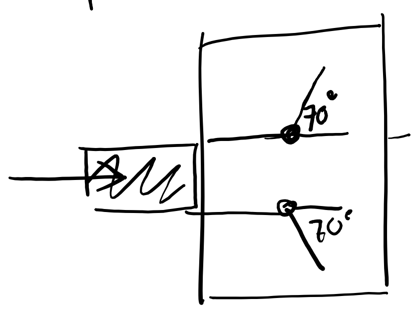


$K_I = 0$

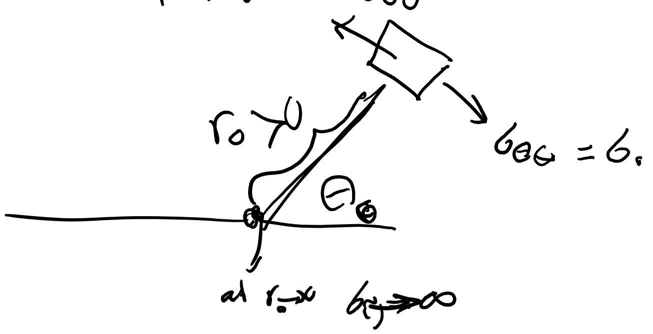
$\theta_c = 2\, tg^{-1} \frac{1}{4}(\pm\sqrt{8}) = \pm 70^\circ$



Kalthoff's example



i) Does it propagate



$$\underbrace{\sigma_{\theta\theta\,max}}_{\sigma_0} = \frac{1}{\sqrt{2\pi r_0}} \cos\frac{\theta_0}{2}\left[\cos^2\frac{\theta_0}{2} K_I - \frac{3}{2}\sin\theta_0 \, K_{II}\right]$$

$$\rightarrow \underbrace{\sigma_0\sqrt{2\pi r_0}}_{K_{Ic}} = \cos\frac{\theta_0}{2}\left[\cos^2\frac{\theta_0}{2} K_I - \frac{3}{2}\sin\theta_0 K_{II}\right] \quad (\triangle)$$

how about when $K_{II} = 0$? $\longrightarrow \theta_0 = 0$

$$(\triangle) \rightarrow \sigma_0\sqrt{2\pi r_0} = K_I = K_{Ic}$$

from LEFM crack prog. perspective (pure mode I)

Maximum allowable traction $\sigma_{\theta max}$ is reached at angle $\theta = \theta max$ and distance from crack tip r_0:

$$\sigma_{\theta max}\sqrt{2\pi r_0} = K_{\mathrm{Ic}} = \underbrace{\cos\frac{\theta_0}{2}\left[K_{\mathrm{I}}\cos^2\frac{\theta_0}{2} - \frac{3}{2}K_{\mathrm{II}}\sin\theta_0\right]}$$

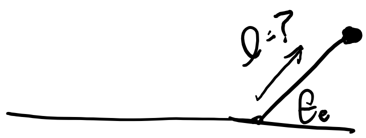
$$K_{Ic} = K_{eq}(K_I, K_{II})$$

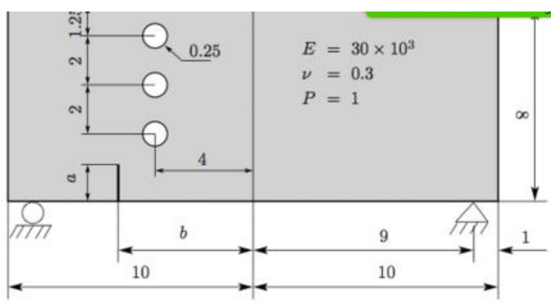
a) Find the angle

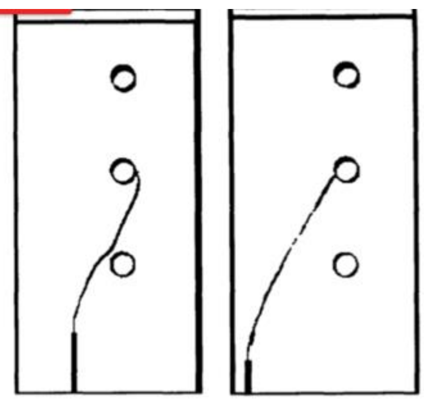
$$\boxed{\tan\frac{\theta_0}{2} = \frac{1}{4}\frac{K_{\mathrm{I}}}{K_{\mathrm{II}}} \pm \frac{1}{4}\sqrt{\left(\frac{K_{\mathrm{I}}}{K_{\mathrm{II}}}\right)^2 + 8}}$$

b) See if K_eq satisfy crack propagation condition:

$$K_{\mathrm{Ic}} = \cos\frac{\theta_0}{2}\left[K_{\mathrm{I}}\cos^2\frac{\theta_0}{2} - \frac{3}{2}K_{\mathrm{II}}\sin\theta_0\right]$$





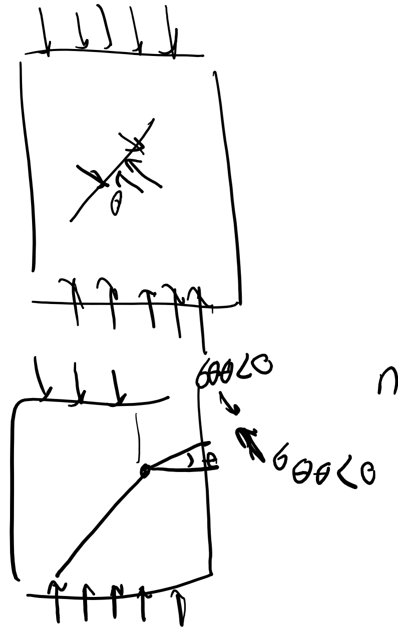


XFEM

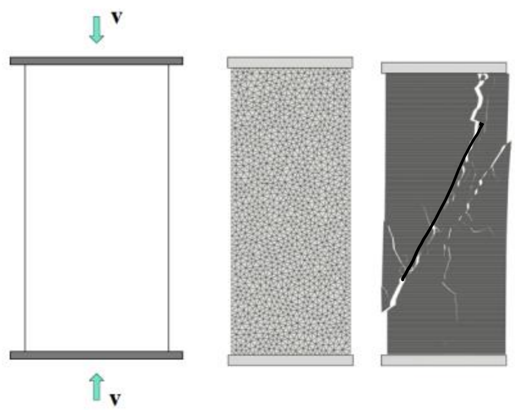
$$\theta_c = 2\arctan\frac{1}{4}\left(K_I/K_{II} \pm \sqrt{(K_I/K_I I)^2 + 8}\right)$$

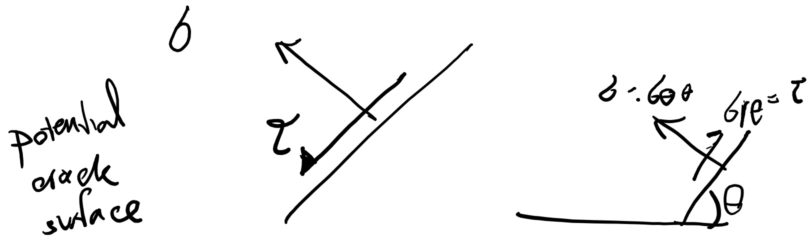
MCS criterion can also be used outside LEFM, e.g. cohesive models, etc.

Modifications to maximum circumferential stress criterion



no tensile fracture





tensile fracture

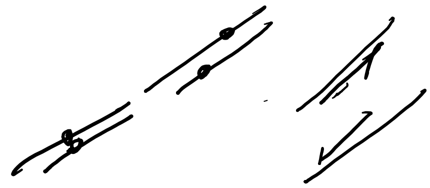
$\sigma_{eff} = \sigma$ original Max Cir. Str. ✓

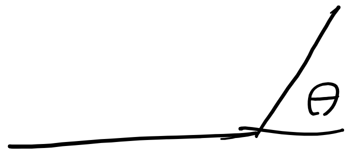
$$\sigma_{eff} = \sqrt{\underbrace{\langle \sigma \rangle_+^2} + (\beta\tau)^2}$$

$\sigma<0$

$\sigma_{eff} = \beta\tau$

slips eventually





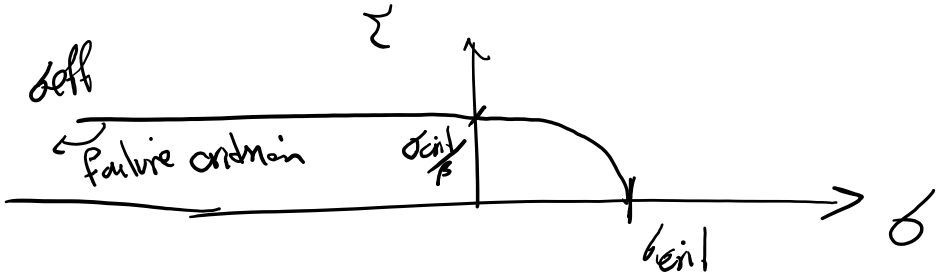
find θ_c for which

$$\sigma_{eff}(\theta) = \sqrt{\sigma(\theta)_+^2 + \beta^2 \tau(\theta)^2}$$

is maximum for θ_c

then the crack propagates if

$$\boxed{\sigma_{eff}(\theta_c) = \sigma_{crit}}$$



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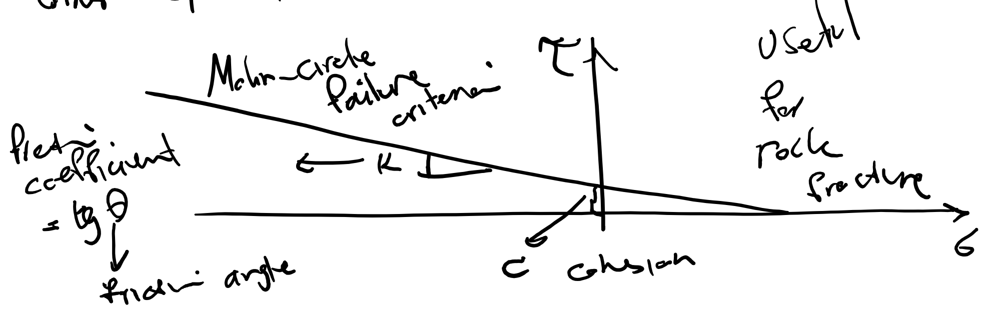
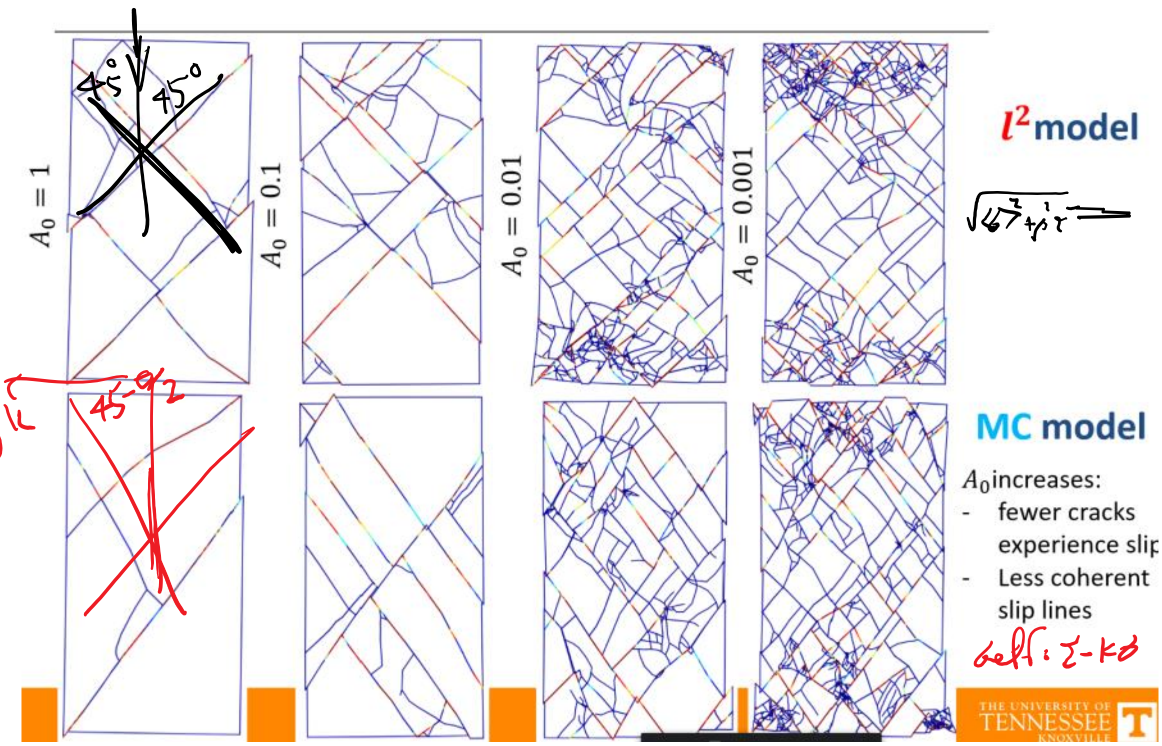
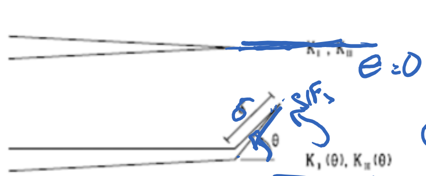
other equivalent stress formulas



$$\sigma_{eff} = \tau + k\sigma$$



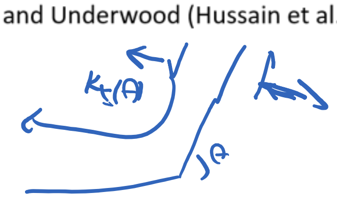
Maximum Energy Release Rate



$$G(\theta = 0) = \frac{K_I^2 + K_{II}^2}{E'}$$

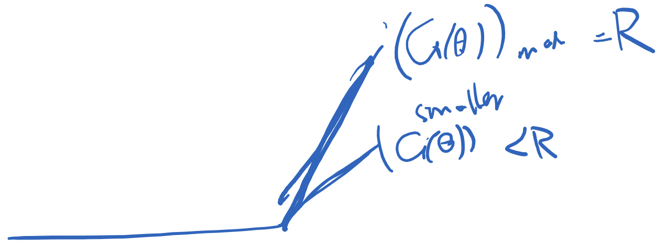
$$G(\theta) = \frac{K_I^2(\theta) + K_{II}^2(\theta)}{E'}$$

Stress intensity factors for **kinked crack extension**:
Hussain, Pu and Underwood (Hussain et al. 1974)



$K_{II}(\theta)$ causes shearing in this direction

$\underline{G(\theta)}$



find θ_c for which $G(\theta)$ is max over θ

see if $G(\theta_c) = R$

then the crack propagates in direction θ_c

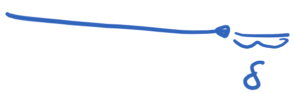
Hussain, Pu and Underwood (Hussain et al. 1974)

$$\left\{ \begin{array}{c} K_I(\theta) \\ K_{II}(\theta) \end{array} \right\} = \left(\frac{4}{3+\cos^2\theta}\right)\left(\frac{1-\frac{\theta}{\pi}}{1+\frac{\theta}{\pi}}\right)^{\frac{\theta}{2\pi}} \left\{ \begin{array}{c} K_I\cos\theta + \frac{3}{2}K_{II}\sin\theta \\ K_{II}\cos\theta - \frac{1}{2}K_I\sin\theta \end{array} \right\}$$

example $\theta = 0$

$$\left\{ \begin{array}{c} K_I(\theta) \\ K_{II}(\theta) \end{array} \right\} = \left(\frac{4}{3+1}\right)\left(\frac{1-0}{1+0}\right)^{0} \left\{ \begin{array}{c} K_I \times 1 + K_{II} \times 0 \\ K_{II} \times 1 - K_I \times 0 \end{array} \right\}$$

$$= \left[\begin{array}{c} K_I \\ K_{II} \end{array} \right] \quad \checkmark$$



$$\left(\begin{array}{c} K_I \\ K_{II} \end{array} \right)(\theta = 0) = \left[\begin{array}{c} K_I \\ K_{II} \end{array} \right]$$

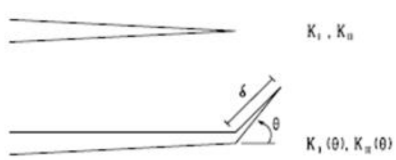
$$G(\theta) = \frac{1}{E'}\left(K_I^2(\theta) + K_{II}^2(\theta)\right)$$

$$G(\theta) = \frac{4}{E'}\left(\frac{1}{3+\cos^2\theta}\right)^2\left(\frac{1-\frac{\theta}{\pi}}{1+\frac{\theta}{\pi}}\right)^{\frac{\theta}{\pi}} \left[(1+3\cos^2\theta)K_I^2 + 8\sin\theta\cos\theta K_I K_{II} + (9-5\cos^2\theta)K_{II}^2\right]$$

Maximization condition

$$\frac{\partial G(\theta)}{\partial \theta} = 0$$

$$\frac{\partial^2 G(\theta)}{\partial \theta^2} < 0$$



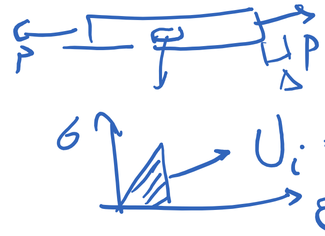
$$G(\theta) = \frac{4}{E'}\left(\frac{1}{3+\cos^2\theta}\right)^2\left(\frac{1-\frac{\theta}{\pi}}{1+\frac{\theta}{\pi}}\right)^{\frac{\theta}{\pi}} \left[(1+3\cos^2\theta)K_I^2 + 8\sin\theta\cos\theta K_I K_{II} + (9-5\cos^2\theta)K_{II}^2\right]$$

$$\boxed{4\left(\frac{1}{3+\cos^2\theta_0}\right)^2\left(\frac{1-\frac{\theta_0}{\pi}}{1+\frac{\theta_0}{\pi}}\right)^{\frac{\theta_0}{\pi}} \left[\left(1+3\cos^2\theta_0\right)\left(\frac{K_{\mathrm{I}}}{K_{\mathrm{Ic}}}\right)^2 + 8\sin\theta_0\cos\theta_0\left(\frac{K_{\mathrm{I}}K_{\mathrm{II}}}{K_{\mathrm{Ic}}^2}\right) + \left(9-5\cos^2\theta_0\right)\left(\frac{K_{\mathrm{II}}}{K_{\mathrm{Ic}}}\right)^2\right] = 1}$$

Strain Energy Density (SED) criterion

Sih 1973

strain energy density



1D

$U_i = \frac{1}{2}\sigma\cdot\varepsilon = \frac{1}{2}\sigma\cdot\frac{\sigma}{E} =$

1D $\quad \frac{1}{2}\frac{\sigma^2}{E}$

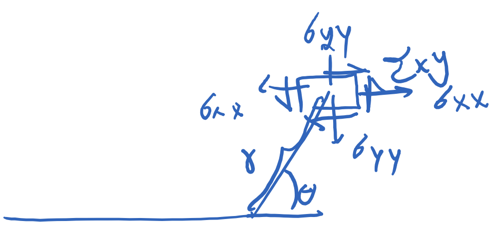
$$U_i = \int_0^{\epsilon_{ij}} \sigma_{ij} d\epsilon_{ij} \qquad U_i = \frac{1}{4\mu}\left[\frac{\kappa+1}{4}(\sigma_x^2+\sigma_y^2) - 2(\sigma_x\sigma_y - \tau_{xy}^2)\right]$$

strain energy density

$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) - \frac{K_{II}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\left(2+\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)$$

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} \quad (7.13)$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2} + \frac{K_{II}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right).$$



$$U_i = \frac{S(\theta)}{r}$$

$$U_i(\theta) = \frac{S(\theta)}{r}$$

$S(\theta)$

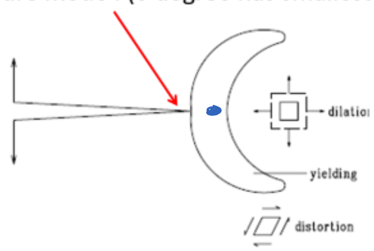
the crack propagates along the direction for which $S(\theta)$ is minimum.

- Crack direction θ_0 which **minimizes** the strain energy density S
- Crack Extends when S reaches a critical value at a distance r_0

Minimization condition

$$\frac{\partial S}{\partial \theta} = 0$$
$$\frac{\partial^2 S}{\partial \theta^2} > 0$$

Pure mode I (0 degree has smallest S)



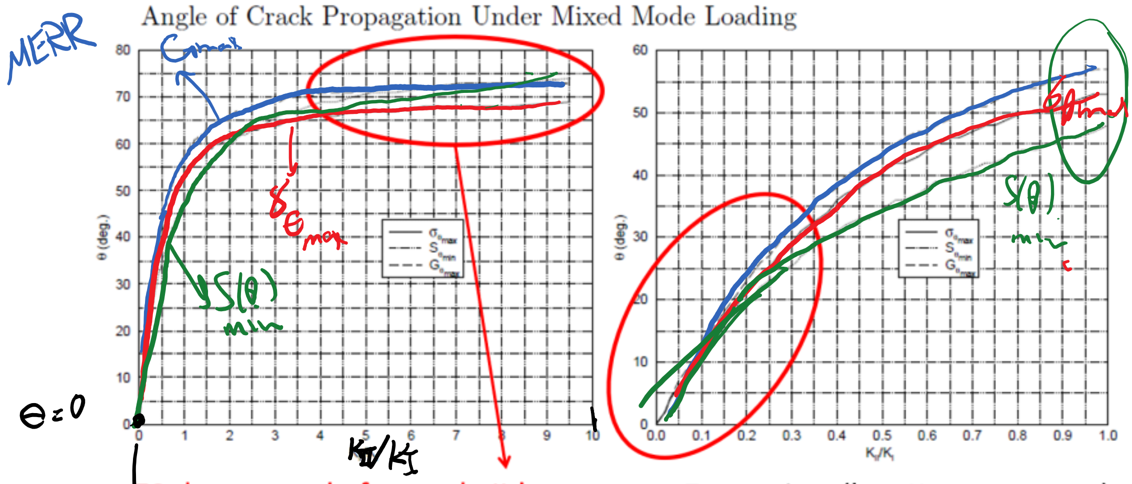
$$\boxed{\frac{8\mu}{(\kappa-1)}\left[a_{11}\left(\frac{K_{\mathrm{I}}}{K_{\mathrm{Ic}}}\right)^2 + 2a_{12}\left(\frac{K_{\mathrm{I}}K_{\mathrm{II}}}{K_{\mathrm{Ic}}^2}\right) + a_{22}\left(\frac{K_{\mathrm{II}}}{K_{\mathrm{Ic}}}\right)^2\right] = 1}$$

$$\begin{aligned}
a_{11} &= \frac{1}{16\mu}\left[(1+\cos\theta)(\kappa-\cos\theta)\right] \\
a_{12} &= \frac{\sin\theta}{16\mu}\left[2\cos\theta-(\kappa-1)\right] \\
a_{22} &= \frac{1}{16\mu}\left[(\kappa+1)(1-\cos\theta)+(1+\cos\theta)(3\cos\theta-1)\right]
\end{aligned}$$

$$\kappa = \frac{3-\nu}{1+\nu} \quad \text{(plane stress)}$$
$$\kappa = 3-4\nu \quad \text{(plane strain)}$$

θ is the θ that minimizes $S(\theta)$

a) Crack Extension angle

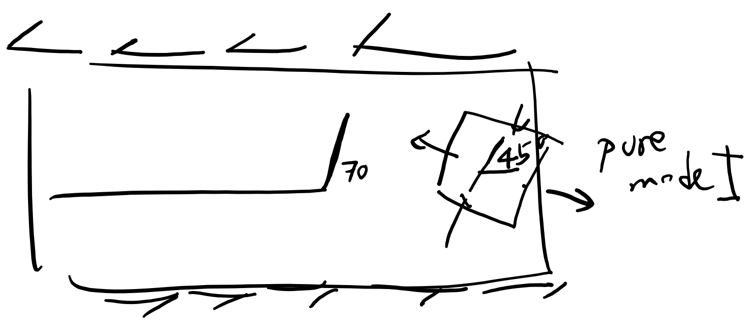


~ 70 degree angle for mode II !

Zoom view (low K_{II} component)

$K_{II}=0$ pure mode I

as $K_{II}/K_I \rightarrow \infty \rightarrow \theta \rightarrow 70^\circ$



1. First crack extension θ_0 is obtained followed by on whether crack extends in θ_0 direction or not.
2. Strain Energy Density (SED) and Maximum Circumferential Tensile Stress require an r_0 but the final crack propagation locus is independent of r_0.
3. SED theory depends on Poisson ratio ν.
4. All three theories give identical results for **small ratios of K_{II}/K_I** and diverge slightly as this ratio increases
5. Crack will always extend in the direction which attempts to minimize K_{II}/K_I.
6. For practical purposes during crack propagation all three theories yield very similar paths as from 4 and 5 cracks extend mostly in mode I where the there is a better agreement between different criteria

b) Locus of crack propagation

MERR
Gθmax

Least conservative

Most conservative
Sθmin