CurvilinearCoordinates

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Motivation on coordinate transformation:
PML: Perfectly matched layer:

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A Perfectly Matched Layer for the Absorption of Electromagnetic Waves

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infinite domain:

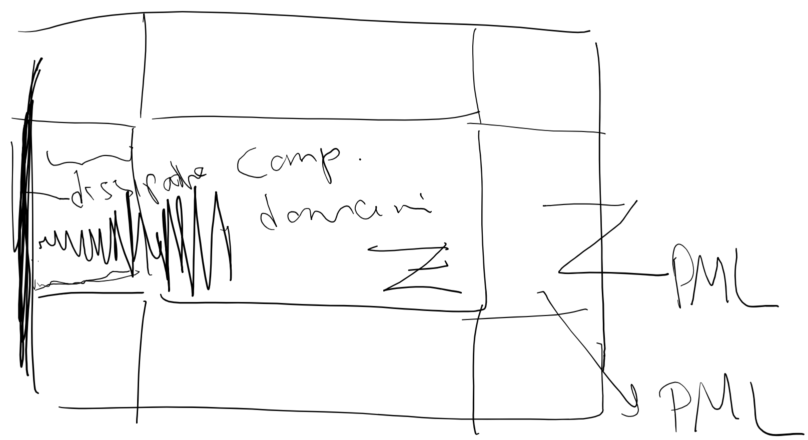
impedance match

$Z^- \,|\, Z^+$

$\rightsquigarrow \longrightarrow Z^- = Z^+$

no reflection

$Z = Z_{PML}$



Relation to coordinate transformation

A 3D PERFECTLY MATCHED MEDIUM FROM MODIFIED MAXWELL'S EQUATIONS WITH STRETCHED COORDINATES

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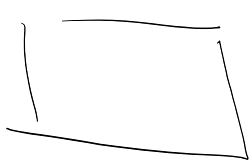
stretch the material
in complex plane:



$1 + \frac{i\sigma}{\omega}$

frequency

① impedance doesn't change

② corresponds to loss

frequency (2) Corresponds to loss

(2) matches have energy loss

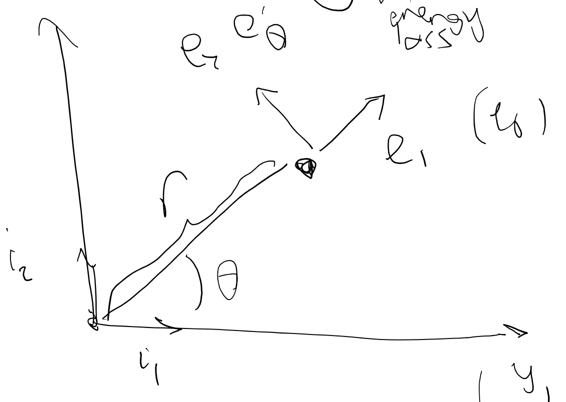
$$M\ddot{U} + C\dot{U} + KU = F$$

$$-M\omega^2\hat{U} + i\omega C\hat{U} + K\hat{U} = \hat{F}$$

Curvilinear coordinate systems

$y_1 = y_1(x_1, x_2)$

$y_2 = y_2(x_1, x_2)$



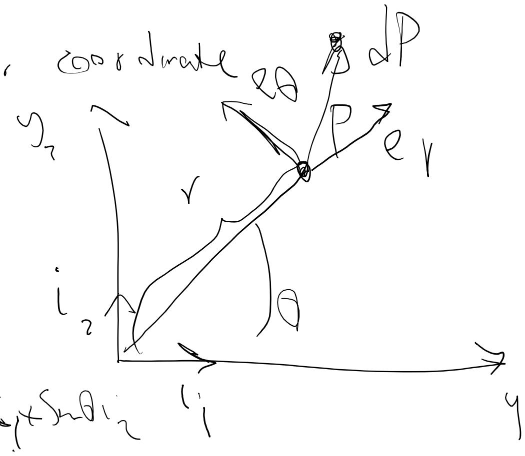
Example $\quad y_1 = r\cos\theta = x_1 \cos x_2$

Polar coordinate system $\quad y_2 = r\sin\theta = x_1 \sin x_2$

x_1, x_2 are curvilinear coordinate system

How to define e_r, e_θ

Brute-force calculation of grad in polar coordinate



$$P = r e_r$$

$$d\vec{P} = d(re_r) = (dr)e_r + r(de_r)$$

$$e_r = \cos\theta\, i_1 + \sin\theta\, i_2$$

$$de_r = \underbrace{(-\sin\theta\, i_1 + \cos\theta\, i_2)}_{e_\theta}\, d\theta$$

$$\boxed{dP = (dr)e_r + (r\,d\theta)e_\theta}$$

?

Also $\quad \boxed{\begin{aligned} de_r &= e_\theta\, d\theta \\ de_\theta &= -e_r\, d\theta \end{aligned}}$

$$d\phi(r,\theta) = \frac{\partial\phi}{\partial r}dr + \frac{\partial\phi}{\partial\theta}d\theta$$

$$\boxed{d\phi(r,\theta) = \frac{\partial\phi}{\partial r}\underbrace{dr} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\underbrace{(r\,d\theta)}}$$

$$d\phi(r,\theta) = \frac{\partial \phi}{\partial r}\,\underbrace{dr} + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\,\underbrace{\{r\,d\theta\}}$$

$$\boxed{dP = \underbrace{dP_r}_{dr}\, e_r + \underbrace{dP_\theta}_{r\,d\theta}\, e_\theta}$$

$$d\phi = \begin{bmatrix} (\nabla\phi)_r & (\nabla\phi)_\theta \end{bmatrix} \begin{bmatrix} (dP)_r \\ dP_\theta \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\partial \phi}{\partial r} & \dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta} \end{bmatrix} \begin{bmatrix} dr \\ r\,d\theta \end{bmatrix}$$

$$\Rightarrow \boxed{\nabla\phi = \begin{bmatrix} \dfrac{\partial \phi}{\partial r} & \underbrace{\dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta}} \end{bmatrix}}$$

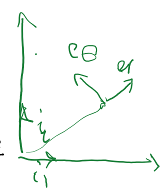
Gradient of a vector

$$dP = d(r e_r) = (dr)\, e_r + (r\,d\theta)\, e_\theta$$
$$= (dP)_r\, e_r + (dP_\theta)\, e_\theta$$

$$V = V_r e_r + V_\theta e_\theta$$

$$dV = (V_r e_r + V_\theta e_\theta)_{,r}\, \underbrace{dr}_{dP_r} + (V_r e_r + V_\theta e_\theta)_{,\theta}\, \underbrace{d\theta}_{\frac{dP_\theta}{r}}$$

$$= \{V_{r,r} e_r + V_{\theta,r} e_\theta + V_r e_{r,r} + V_\theta e_{\theta,r}\}\, dP_r + \{V_{r,\theta} e_r + V_{\theta,\theta} e_\theta + V_r \overbrace{e_{r,\theta}}^{e_\theta} + V_\theta \overbrace{e_{\theta,\theta}}^{-e_r}\}\, \frac{dP_\theta}{r}$$



note $e_r = \cos\theta\, i_1 + \sin\theta\, i_2$
$e_\theta = -\sin\theta\, i_1 + \cos\theta\, i_2$ $\Big\} \Rightarrow$
$e_{r,r} = 0 \quad e_{\theta,r} = 0$
$e_{r,\theta} = -\sin\theta\, i_1 + \cos\theta\, i_2 = e_\theta$
$e_{\theta,\theta} = -\cos\theta\, i_1 - \sin\theta\, i_2 = -e_r$

$$= \{V_{r,r} e_r + V_{\theta,r} e_\theta\}\, dP_r + \left\{\left(\frac{V_{r,\theta} - V_\theta}{r}\right) e_r + \left(\frac{V_{\theta,\theta} + V_r}{r}\right) e_\theta\right\} dP_\theta$$

$$\Rightarrow dV = \begin{bmatrix} e_r & e_\theta \end{bmatrix} \underbrace{\begin{bmatrix} V_{r,r} & \dfrac{V_{r,\theta} - V_\theta}{r} \\ V_{\theta,r} & \dfrac{V_{\theta,\theta} + V_r}{r} \end{bmatrix}}_{\nabla V} \begin{bmatrix} dP_r \\ dP_\theta \end{bmatrix}$$

$$\Rightarrow \quad \boxed{V = V_r e_r + V_\theta e_\theta \;\Rightarrow\; \nabla V = \begin{bmatrix} V_{r,r} & \dfrac{V_{r,\theta} - V_\theta}{r} \\ V_{\theta,r} & \dfrac{V_{\theta,\theta} + V_r}{r} \end{bmatrix}}$$

General orthonormal curvilinear coordinate systems:

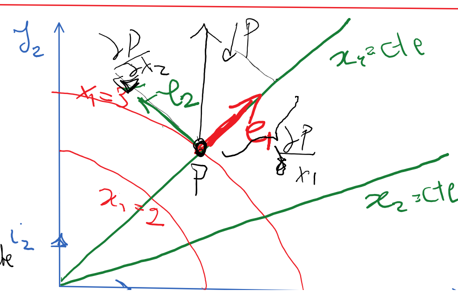
In general we have statements in the form

$$\boxed{\begin{aligned} y_1 &= y_1(x_1, x_2) \\ y_2 &= y_2(x_1, x_2) \end{aligned}}$$

eg $x_1 = r \quad y_1 = r\cos\theta$
$x_2 = \theta \quad y_2 = r\sin\theta \rightarrow$

$$\boxed{\begin{aligned} x_1 &= r \\ x_2 &= \theta \\ y_1 &= x_1 \cos x_2 \\ y_2 &= x_1 \sin x_2 \end{aligned}}$$

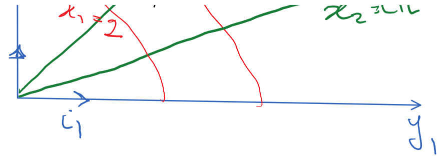
Polar coordinate example



$$y_2 = y_2(x_1, x_2)$$

$y_2 = x_1 \sin x_2$

polar coordinate example



$$P = y_1 i_1 + y_2 i_2$$

$$\frac{dP}{dx_1} = \frac{\partial y_1}{\partial x_1} i_1 + \frac{\partial y_2}{\partial x_1} i_2$$

$$\frac{\partial P}{\partial x_2} = \frac{\partial y_2}{\partial x_1} i_1 + \frac{\partial y_2}{\partial x_2} i_2$$

$$e_1 = \frac{\frac{\partial P}{\partial x_1}}{\left|\frac{\partial P}{\partial x_1}\right|} = \frac{\frac{\partial P}{\partial x_1}}{h_1} \qquad h_1 = \left|\frac{\partial P}{\partial x_1}\right|$$

$$e_2 = \frac{\frac{\partial P}{\partial x_2}}{\left|\frac{\partial P}{\partial x_2}\right|} = \frac{\frac{\partial P}{\partial x_2}}{h_2} \qquad h_2 = \left|\frac{\partial P}{\partial x_2}\right|$$

Example

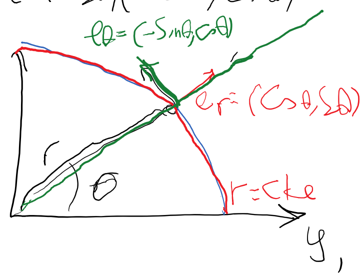
$$\frac{\partial P}{\partial x_1} = \cos x_2 i_1 + \sin x_2 i_2 = (\cos\theta, \sin\theta)$$

$$\frac{\partial P}{\partial x_2} = -x_1 \sin x_2 i_1 + x_1 \cos x_2 i_2 = x_1(-\sin\theta, \cos\theta)$$

Example

$h_1 = 1 \qquad \underset{\sim}{e_1} = (\cos\theta, \sin\theta)$ (e_r)

$h_2 = x_1 = r \qquad \underset{\sim}{e_2} = (-\sin\theta, \cos\theta)$ (e_θ)



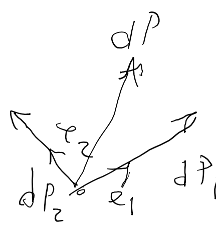
Now
$$dP = \frac{\partial P}{\partial x_1} dx_1 + \frac{\partial P}{\partial x_2} dx_2$$

$$= \underbrace{\left(\frac{\frac{\partial P}{\partial x_1}}{\left|\frac{\partial P}{\partial x_1}\right|}\right)}_{e_1} \underbrace{\left|\frac{\partial P}{\partial x_1}\right|}_{h_1} dx_1 + \underbrace{\left(\frac{\frac{\partial P}{\partial x_2}}{\left|\frac{\partial P}{\partial x_2}\right|}\right)}_{e_2} \underbrace{\left|\frac{\partial P}{\partial x_2}\right|}_{h_2} dx_2$$

$$\left|\frac{\partial P}{\partial x_i}\right| = h_i$$

$$= (h_1 dx_1) e_1 + (h_2 dx_2) e_2$$

$$= dP_1 \, e_1 + dP_2 \, e_2$$



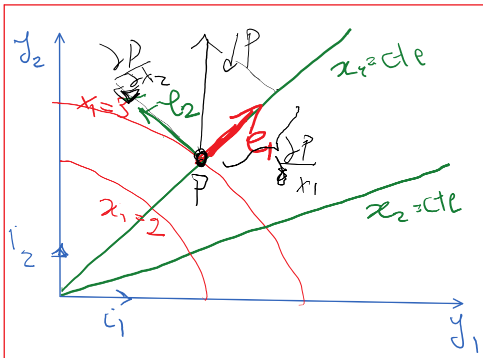
$\longrightarrow$

$$dP_1 = h_1 dx_1 \qquad dx_1 = \frac{dP_1}{h_1}$$
$$dP_2 = h_2 dx_2 \qquad dx_2 = \frac{dP_2}{h_2}$$

Polar coordinate

$$dP_r = dr \longleftrightarrow dr = dP_r$$
$$dP_\theta = r\,d\theta \longleftrightarrow d\theta = \frac{dP_\theta}{r}$$

Summary:



$y_1 = y_1(x_1, x_2)$

$y_2 = y_2(x_1, x_2)$

$P = y_1 i_1 + y_2 i_2$

$h_i = \left|\frac{\partial P}{\partial y_i}\right|$ scale modulus

$e_i = \frac{\frac{\partial P}{\partial y_i}}{\left|\frac{\partial P}{\partial y_i}\right|} = \frac{\frac{\partial P}{\partial y_i}}{h_i}$ no summation on i

$dP_i = h_i \, dy_i$ //

$\Longleftrightarrow$

$dy_i = \frac{dP_i}{h_i}$ //

$$\nabla = \sum_{i=1}^{d} \frac{e_i}{h_i} \frac{\partial}{\partial x_i}$$

Example of grad for a scalar

$\phi(x_1, x_2)$

$h_i \, dx_i = dP_i$ no summation on i

$$d\phi = \frac{\partial \phi}{\partial x_1} dx_1 + \frac{\partial \phi}{\partial x_2} dx_2$$

$$= \frac{\partial \phi}{\partial x_1} \frac{dP_1}{h_1} + \frac{\partial \phi}{\partial x_2} \frac{dP_2}{h_2}$$

$$= \underbrace{\left[\frac{1}{h_1}\frac{\partial \phi}{\partial x_1}, \frac{1}{h_2}\frac{\partial \phi}{\partial x_2}\right]}_{\nabla \phi} \begin{pmatrix} dP_1 \\ dP_2 \end{pmatrix}$$

Example $x_1 = r$, $x_2 = \theta$ $\rightarrow$ $h_1 = 1$, $h_2 = r$

$$\nabla \phi = \left[\frac{\partial \phi}{\partial r}, \frac{1}{r}\frac{\partial \phi}{\partial \theta}\right]$$ ✓

$$\boxed{\nabla = \sum_{i=1}^{d} \frac{1}{h_i} e_i \frac{\partial}{\partial y_i}}$$

Gradient of vectors and the concept of "Balanced derivatives"

a. unit vectors of orthonormal curvilinear coordinate system

Gradient of vectors and the concept of "Balanced derivatives"

e_i unit vectors of orthonormal curvilinear coordinate system

$$\boxed{\begin{aligned} V &= V_i e_i \\ d\vec{V} &= \frac{\partial \vec{V}}{\partial x_1} dx_1 + \frac{\partial \vec{V}}{\partial x_2} dx_2 \\ &= \frac{1}{h_1}\frac{\partial \vec{V}}{\partial x_1} dP_1 + \frac{1}{h_2}\frac{\partial \vec{V}}{\partial x_2} dP_2 \end{aligned}}$$

$h_i dx_i = dP_i$

no summation on i

Now we need to compute

$$\frac{\partial \vec{V}}{\partial x_\ell} = \frac{\partial (V_K e_K)}{\partial x_\ell} = V_{K,\ell}\, e_K + V_K \frac{\partial e_K}{\partial x_\ell}$$

vector

$$\frac{\partial e_K}{\partial x_\ell} = \left(\frac{\partial e_K}{\partial x_\ell} \cdot e_m\right) e_m$$

$$= \underbrace{\Gamma^m_{K\ell}}_{\text{Christoffel symbol}}\, e_m$$

$$\Gamma^m_{K\ell} = \frac{\partial e_K}{\partial x_\ell} \cdot e_m$$

$$\sum_K V_{K,\ell}\, e_K + \sum_K V_K \sum_m \Gamma^m_{K\ell}\, e_m$$

$K \to m$

$m \to K$

$$\frac{\partial \vec{V}}{\partial x_\ell} = \sum_K V_{K,\ell}\, e_K + \sum_m V_m \sum_K \Gamma^K_{m\ell}\, e_K$$

$$\frac{\partial V}{\partial x_\ell} = \sum_K \left(V_{K,\ell} + \sum_m V_m \Gamma^K_{m\ell}\right) e_K$$

$$= \sum_K V_{K;\ell}\, e_\ell$$

$$\underbrace{V_{K;\ell}}_{\text{balanced derivative}} = V_{K,\ell} + \sum_m V_m \Gamma^K_{m\ell}$$

$$\Gamma^K_{m\ell} = \frac{\partial e_m}{\partial x_\ell} \cdot e_K$$

because unit vectors are not constant

$$\Longrightarrow \quad \nabla V = \begin{bmatrix} \frac{1}{h_1} V_{1;1} & \frac{1}{h_2} V_{1;2} \\ \frac{1}{h_1} V_{\cdot} & \frac{1}{h_2} V \end{bmatrix}$$

Since $dV = \sum_{i=1}^{d} \frac{1}{h_i} \frac{\partial \vec{V}}{\partial x_i} dP_i$

$$\boxed{\Longrightarrow \quad \nabla v = \begin{bmatrix} \frac{1}{h_1} v_{1;1} & \frac{1}{h_2} v_{1;2} \\ \frac{1}{h_1} v_{2;1} & \frac{1}{h_2} v_{2;2} \end{bmatrix}}$$

$$dv = \sum_{i=1}^{d} \frac{1}{h_i} \frac{\partial \vec{v}}{\partial x_i} d P_i$$

Example:
Polar coordinate:

$x_1 = r \qquad x_2 = \theta$

$e_1 = \cos x_2 \, i_1 + \sin x_2 \, i_2 \qquad \frac{\partial e_1}{\partial x_1} = 0 \qquad \frac{\partial e_1}{\partial x_2} = e_2$

$e_2 = -\sin x_2 \, i_1 + \cos x_2 \, i_2 \qquad \frac{\partial e_2}{\partial x_1} = 0 \qquad \frac{\partial e_2}{\partial x_2} = -e_1$

$\Gamma_{11}^{1} = \frac{\partial e_1}{\partial x_1} \cdot e_1 = 0 \qquad \Gamma_{11}^{2} = \frac{\partial e_1}{\partial x_1} \cdot e_2 = 0$

$\Gamma_{12}^{1} = \frac{\partial e_1}{\partial x_2} \cdot e_1 = e_2 \cdot e_1 = 0 \qquad \Gamma_{12}^{2} = \frac{\partial e_1}{\partial x_2} \cdot e_2 = e_2 \cdot e_2 = 1$

$\Gamma_{21}^{1} = \frac{\partial e_2}{\partial x_1} e_1 = 0 \qquad \Gamma_{21}^{2} = \frac{\partial e_2}{\partial x_1} \cdot e_2 = 0$

$\Gamma_{22}^{1} = \frac{\partial e_2}{\partial x_2} \cdot e_1 = (-e_1) \cdot e_1 = -1 \qquad \Gamma_{22}^{2} = \frac{\partial e_2}{\partial x_2} \cdot e_2 = (-e_1) \cdot e_2 = 0$

$$\boxed{\Gamma_{12}^{2} = 1 \qquad \Gamma_{22}^{1} = -1}$$

$$v_{1;1} = v_{1,1} + v_1 \Gamma_{11}^{1} + v_2 \Gamma_{21}^{1} = v_{1,1} + 0 + 0 = v_{1,1}$$

$$v_{1;2} = v_{1,2} + v_1 \Gamma_{12}^{1} + v_2 \underbrace{\Gamma_{22}^{1}}_{-1} = v_{1,2} + 0 - v_2$$

$$v_{2;1} = v_{2,1} + v_1 \Gamma_{11}^{2} + v_2 \Gamma_{21}^{2} = v_{2,1} + 0 + 0 = v_{2,1}$$

$$v_{2;2} = v_{2,2} + v_1 \underbrace{\Gamma_{12}^{2}}_{1} + v_2 \Gamma_{22}^{2} = v_{2,2} + 1 \cdot v_1 + 0$$

$$\nabla v = \begin{bmatrix} \frac{1}{h_1} v_{1;1} & \frac{1}{h_2} v_{1;2} \\ \frac{1}{h_1} v_{2;1} & \frac{1}{h_2} v_{2;2} \end{bmatrix}$$

$1 \to r$
$2 \to \theta$
$h_1 = 1$
$h_2 = r = x_1$

$$\Rightarrow \boxed{\nabla v = \begin{bmatrix} v_{r,r} & \frac{v_{r,\theta} - v_\theta}{r} \\ v_{\theta,r} & \frac{v_{\theta,\theta} + v_r}{r} \end{bmatrix}}$$

Properties of Christoffel symbol:

$$\Gamma^m_{kl} = \binom{m}{k\,l}$$

$$\boxed{\binom{m}{k\,l} = \frac{1}{h_k}\frac{\partial h_l}{\partial x_k}\delta_{lm} - \frac{1}{h_m}\frac{\partial h_k}{\partial x_m}\delta_{kl}.} \qquad \text{(C.22)}$$

If k, l and m are all different, $k \neq l \neq m$, then

$$\binom{m}{k\,l} = \binom{k}{k\,k} = \binom{k}{k\,l} = 0. \qquad \text{(C.23)}$$

The last equality is the consequence of the fact that the vector $\partial\vec{e}_k/\partial x_l$ is orthogonal to the x_k-coordinate line and, thus, has no component in the direction of $\vec{e}_k$ (but may have components in both directions orthogonal to $\vec{e}_l$). Because of (C.23), at most 12 of the 27 Christoffel symbols are non-zero:

$$\boxed{\binom{l}{k\,l} = \frac{1}{h_k}\frac{\partial h_l}{\partial x_k}, \qquad \binom{l}{k\,k} = -\frac{1}{h_l}\frac{\partial h_k}{\partial x_l} \qquad \text{if} \quad k \neq l.} \qquad \text{(C.24)}$$

Of these, only six can be independent since it holds:

$$\binom{l}{k\,l} = -\binom{k}{l\,l}. \qquad \text{(C.25)}$$

$$\boxed{\operatorname{grad}\phi = \sum_k \frac{1}{h_k}\frac{\partial\phi}{\partial x_k}\vec{e}_k.} \qquad \text{(C.36)}$$

$$\boxed{\operatorname{div}\vec{v} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3v_1) + \frac{\partial}{\partial x_2}(h_3h_1v_2) + \frac{\partial}{\partial x_3}(h_1h_2v_3)\right].} \qquad \text{(C.40)}$$

$$\boxed{\operatorname{rot}\vec{v} = \sum_m \left[\sum_{kl} \frac{\varepsilon_{klm}}{h_kh_l}\frac{\partial(h_lv_l)}{\partial x_k}\right]\vec{e}_m.} \qquad \text{(C.44)}$$

$$\operatorname{rot}\vec{v} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\vec{e}_1 & h_2\vec{e}_2 & h_3\vec{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1v_1 & h_2v_2 & h_3v_3 \end{vmatrix}. \qquad \text{(C.45)}$$

$$
\boxed{(\mathrm{grad}\,\vec{v})_{kl} = \begin{cases} \dfrac{1}{h_k}\left(\dfrac{\partial v_k}{\partial x_k} + \displaystyle\sum_{\substack{m \\ m \neq k}} \dfrac{1}{h_m}\dfrac{\partial h_k}{\partial x_m} v_m\right) & \text{if} \quad l = k, \\ \dfrac{1}{h_k}\left(\dfrac{\partial v_l}{\partial x_k} - \dfrac{1}{h_l}\dfrac{\partial h_k}{\partial x_l} v_k\right) & \text{if} \quad l \neq k. \end{cases}} \tag{C.50}
$$

$$
\boxed{(\mathrm{div}\,\boldsymbol{T})_l = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 h_3 T_{1l}) + \frac{\partial}{\partial x_2}(h_3 h_1 T_{2l}) + \frac{\partial}{\partial x_3}(h_1 h_2 T_{3l})\right] + \sum_k \frac{1}{h_k h_l}\left(\frac{\partial h_l}{\partial x_k} T_{lk} - \frac{\partial h_k}{\partial x_l} T_{kk}\right).} \tag{C.53}
$$