Hyperelastic

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Hyperelastic material:

It's an elastic material that whose internal energy density depends on F

objectivity

General elastic $T = G(F) \longrightarrow T = F\bar{G}(C)F^t$

hyperelastic Material $\underbrace{e}_{\text{internal energy density}} = \breve{e}(F) \longrightarrow e = \bar{e}(C)$

$$2\rho F\frac{\partial\bar{e}}{\partial C}F^t : D = T : D \qquad T$$

to prove this

Start with Balance of linear momentum

$$\boxed{\frac{D}{Dt}P = F_{surface} + F_{body}}$$

$$\frac{D}{Dt}\int_{B_t}\underbrace{\rho v\, dV_y}_{P} = \int_{\partial B_t}\underbrace{\underbrace{t_y}_{T\cdot n_y}\, dA_y}_{dF_{surface}} + \int_{B_t}\underbrace{\rho b\, dV_y}_{dF_{body}}$$

$$\frac{D}{Dt}\int_{B_t}\vec{v}\underbrace{(\rho dV_y)}_{dm} = \int\frac{Dv}{Dt}(\rho dV_y)$$

reduced transport

$$\int_{B_t}\underbrace{\frac{Dv}{Dt}}_{\vec{a}}\rho\, dV_y = \underbrace{\int_{\partial B_t}T\cdot n_y\, dA_y}_{\downarrow} + \int_{B_t}\rho b\, dV_y$$

$$\int_{B_t}\rho\frac{Dv}{Dt}\, dV_y = \int_{B_t}\text{div}\, T\, dV_y + \int_{B_t}\rho b\, dV_y$$

$$\int_{B_t}\left(\rho\frac{Dv}{Dt} - \text{div}\, T - \rho b\right) dV_y = 0 \qquad B_t \text{ arbitrary, use localization}$$

$$\boxed{\rho\frac{Dv}{Dt} - \text{div}\, T - \rho b = 0} \quad (1a)$$

Balance of lin momentum x v ->
Turns it to sth similar to bal. of energy

Form the inner product of 1a and v

$$\boxed{\rho\,\vec{v}\,\frac{D\vec{v}}{Dt} - \vec{v}\cdot\text{div}\, T - \vec{v}\cdot\rho b = 0} \quad (1b)$$

Now, we want to use the strong form of the balance of energy to get an equation similar to 1b (thermal & electromagnetic effects ignored), equation is first written with thermal terms kept:

$$\frac{D}{Dt}\underbrace{\mathcal{E}}_{\text{energy}} = \int_{\partial B_t} \vec{v}\cdot d\vec{F}_s + \int_{B_t} \vec{v}\cdot \underbrace{\rho \vec{b}}_{d\vec{F}_v} \underbrace{- \int_{\partial B(t)} q\, dA_y + \int_{B(t)} Q\, dV_y}_{\text{thermal effects}}$$

$$\frac{D}{Dt}\int_{B_t}\Big(\underbrace{\frac{\rho|v|^2}{2}}_{\text{kinetic energy density}} + \rho \underbrace{e}_{\text{internal energy density}}\Big)\, dV_y = \int_{\partial B_t}\vec{v}\cdot Tn\, dA_y + \int_{B_t}\vec{v}\cdot\rho b\, dV_y$$

$$-\int_{B(t)} \nabla_y\cdot q\, dV_y + \int_{B(t)} Q\, dV_y$$

$$\frac{D}{Dt}\int_{B_t}\frac{\vec{v}\cdot\vec{v}}{2}\underbrace{(\rho\, dV_y)}_{dm} + \frac{D}{Dt}\int_{B_t}\bar{e}\underbrace{(\rho\, dV_y)}_{dm} = \int_{\partial B_t}(vT)n\, dA_y + \int_{B_t}\vec{v}\cdot\rho\vec{b}\, dV_y + \int_{B(t)}(Q-\nabla_y\cdot q)\, dV_y$$

reduced transport

$$\frac{D}{Dt}\int_{B_t} f\rho\, dV_y = \int_{B_t}\frac{Df}{Dt}\rho\, dV_y$$

$$\int_{B_t}\left(\frac{1}{2}\frac{D\,v\cdot v}{Dt} + \frac{D\bar{e}}{Dt}\right)\rho\, dV_y = \int_{B_t}\mathrm{div}(vT)\, dV_y + \int_{B_t} v\,\rho b\, dV_y + \int_{B(t)}(Q-\nabla_y\cdot q)\, dV_y$$

Note $\frac{1}{2}\frac{D\,\vec{v}\cdot\vec{v}}{Dt} = \frac{1}{2}\frac{D\vec{v}}{Dt}\cdot v + \frac{1}{2}\vec{v}\cdot\frac{Dv}{Dt} = v\cdot\underbrace{\frac{Dv}{Dt}}_{a}$

$$\int_{B_t}\left[\rho v\cdot\frac{Dv}{Dt} + \rho\frac{D\bar{e}}{Dt} - \mathrm{div}(vT) - \vec{v}\cdot\rho\vec{b} - Q + \nabla_y q\right] dV_y = 0$$

B_t arbitrary

localization

$\rightarrow$

$$\boxed{\rho\, v\cdot\frac{Dv}{Dt} + \rho\frac{D\bar{e}}{Dt} - \mathrm{div}(vT) - v\cdot\rho b - Q + \nabla_y\cdot q = 0} \quad (2)$$

(a) (b) (c) (d)

$$\boxed{\rho\vec{v}\frac{D\vec{v}}{Dt} - \vec{v}\cdot\mathrm{div}\,T - \vec{v}\cdot\rho b = 0} \quad (7b)$$

(a) (c') (d)

(2) - (7b)

$$\boxed{\rho\frac{D\bar{e}}{Dt} - \mathrm{div}(vT) + v\,\mathrm{div}\,T - Q + \nabla_y q = 0} \quad (3)$$

$$\mathrm{div}(vT) = (v_i T_{ij})_{,j} = \left(\frac{\partial v_i}{\partial y_j}\right)T_{ij} + v_i\left(\frac{\partial T_{ij}}{\partial y_j}\right)$$

der w.r.t y_j

$f_{i,j} = \frac{\partial f_i}{\partial y_j}$

spatial gradient of $v = L$

$(\nabla\cdot T)_i$

$$\mathrm{div}(vT) = L_{ij}T_{ij} + v_i\nabla T_i = L : T + v\cdot\mathrm{div}\,T$$

div T

y_j ... ij ... spatial gradient of $V = L$

$$\mathrm{div}\, vT = L_{ij} T_{ij} + v_i \nabla T_i = L : T + v \cdot \mathrm{div}\, T \qquad (\nabla \cdot T)_i \quad \mathrm{div}\, T$$

plug this in eqn 3

$$\rho \frac{D e}{D t} - (L : T + v \cancel{\mathrm{div}\, T}) + v \cancel{\mathrm{div}\, T} = 0$$

Finally $T : L = \overset{\text{sym sym}}{T : D} + \overset{\text{sym}}{T} : \overset{\text{skew}}{\cancel{W}} = T : D$

$$\boxed{\begin{aligned} &\rho \frac{D\bar{e}}{Dt} = T : D + Q - \nabla_y q && \text{with thermal effects} \\ &\rho \frac{D\bar{e}}{Dt} = T : D && \text{w/o thermal effects} \\ &\text{Note } T : D = T : L \end{aligned}} \quad (4)$$

For hyperelastic materials e can be expressed as a function of F

$e(F)$

But again consider two related observers: $\quad y^*(x,t) = c(t) + Q(t)\, y(x,t)$

$\Rightarrow \quad F^* = \nabla_x y^* = Q \nabla_x y = QF$

In this case, e is a scalar value and from y and y* the same e is observed:

$$\boxed{e(F^*) = e(QF) = e(F)}$$

Choose $Q = R^t$, for $F = RU \Rightarrow$

$e(R^t R U) = e(U) \Rightarrow \qquad e(F) = e(U)$

But $U = \sqrt{C} \Rightarrow$

$$e(F) = e(U) = e(\sqrt{C}) = \bar{e}(C)$$

(5)

Summary

hyperelastic

$e(F) \longrightarrow \bar{e}(C)$

from objectivity

$\bar{e}(C)$

$$\frac{D\bar{e}(C)}{Dt} = \frac{\partial \bar{e}(C)}{\partial C_{ij}} \left. \frac{D C_{ij}}{Dt} \right|_{x\text{-fixed}}$$

material time derivative

will be computed next

$\left(\frac{\partial \bar{e}(C)}{\partial C}\right)_{ij}$ will comment on this below

$$\frac{D\bar{e}(C)}{Dt} = \frac{\partial \bar{e}}{\partial C} : \frac{DC}{Dt} \qquad (6)$$

$\frac{DC}{Dt}$ term $\qquad \left(\frac{DC}{Dt}\right)_{ij} = \frac{D C_{ij}}{Dt} = ? \qquad , \; C = F^t F$

$$(7) \quad \boxed{\frac{DC}{Dt} = \frac{DF^t}{Dt} F + F^t \frac{DF}{Dt} = \left(\frac{DF}{Dt}\right)^t F + F^t \left(\frac{DF}{Dt}\right)}$$

$$\left(\frac{DF}{Dt}\right)_{ij} = \frac{D\left(\overbrace{\frac{\partial y_i}{\partial x_j}}^{F_{ij}}\right)}{Dt} = \left.\frac{\partial^2 y_i}{\partial t\, \partial x_j}\right|_{x\text{-fixed}} = \left.\frac{\partial^2 y_i}{\partial x_j\, \partial t}\right|_{x\text{-fixed}}$$

$$\left(\frac{DF}{Dt}\right)_{ij} = \frac{D\left(\frac{\partial y_i}{\partial x_j}\right)}{Dt} = \left.\frac{\partial^2 y_i}{\partial t\, \partial x_j}\right|_{x\text{-fixed}} = \left.\frac{\partial^2 y_i}{\partial x_j\, \partial t}\right|_{x\text{-fixed}}$$

material time rate

$$= \frac{\partial}{\partial x_j}\Big(\underbrace{\left.\frac{\partial y_i}{\partial t}\right|_{x\text{-fixed}}}_{v_i}\Big) = \frac{\partial v_i}{\partial x_j}$$

we want to express this with $\partial/\partial y$ terms

$$= \underbrace{\left(\frac{\partial v_i}{\partial y_k}\right)}_{L_{ik}}\underbrace{\left(\frac{\partial y_k}{\partial x_j}\right)}_{F_{kj}} \rightarrow \left(\frac{DF}{Dt}\right)_{ij} = L_{ik} F_{kj} \rightarrow \boxed{\frac{DF}{Dt} = LF}$$

$$\frac{DC}{Dt} = (LF)^t F + F^t(LF) = F^tL^tF + F^tLF = 2F^t\left(\frac{L+L^t}{2}\right)F$$

$D = \frac{L+L^t}{2}$ (sym L) stretching tensor

$W = \frac{L-L^t}{2}$ (skew L) spin tensor

(don't confuse D with stretch tensor U: F=RU)

$$\boxed{\frac{DC}{Dt} = 2F^tDF} \quad (8)$$

$$\frac{D\bar{e}(C)}{Dt} = \frac{\partial \bar{e}}{\partial C} : \frac{DC}{Dt} \quad \text{eqn } (6)$$

$$\frac{D\bar{E}(C)}{Dt} = \frac{\partial \bar{e}}{\partial C} : 2F^tDF$$

$$\rho\frac{D\bar{e}}{Dt} = L : T \quad \text{eqn } (4)$$

$$\rightarrow \boxed{2\rho\frac{\partial \bar{e}}{\partial C} : F^tDF = T:L} \quad (9)$$

$$\left(\qquad\right):D \qquad T:D$$

modifying eqn 9

$$T:L = T:\underbrace{\text{sym}L}_{D} + T:\underbrace{\text{skew}L}_{W} = T:D + \underbrace{T:W}$$

$$\underbrace{T_{ij}}_{\text{sym}}\underbrace{W_{ij}}_{\text{skew}} = 0 \quad (\text{HW1})$$

$T_{ij} = T_{ji}$ from balance of angular momentum

$$\boxed{2\rho\frac{\partial \bar{e}}{\partial C} : F^tDF = 2\rho\left(F\frac{\partial \bar{e}}{\partial C}F^t\right):D}$$

Why?

$$\frac{\partial \bar{e}}{\partial C} : F^tDF = \left(\frac{\partial \bar{e}}{\partial C}\right)_{ij}(F^tDF)_{ij} = \left(\frac{\partial \bar{e}}{\partial C}\right)_{ij} F^t_{im} D_{mn} F_{nj}$$

$$= \left(F\frac{\partial \bar{e}}{\partial C}F^t\right)_{mn} D_{mn} = \left(F\frac{\partial \bar{e}}{\partial C}F^t\right)_{mn} D_{mn} = \left(F\frac{\partial \bar{e}}{\partial C}F^t\right):D$$

$$\frac{\partial \bar{e}}{\partial C} : F^t D F = \left(\frac{\partial \bar{e}}{\partial C}\right)_{ij} (F^t D F)_{ij} = \left(\frac{\partial \bar{e}}{\partial C}\right)_{ij} F^t_{im} D_{mn} F_{nj}$$

$$= \left(F_{mi} \frac{\partial \bar{e}}{\partial C_{ij}} F^t_{jn}\right) D_{mn} = \left(F \frac{\partial \bar{e}}{\partial C} F^t\right)_{mn} D_{mn} = \left(F \frac{\partial \bar{e}}{\partial C} F^t\right) : D$$

plug in eqn 9

$$\boxed{\underbrace{\left(2\rho F \frac{\partial \bar{e}}{\partial C} F^t\right)}_{\text{sym (?)}} : \underset{\text{sym}}{D} = \underset{\text{sym}}{T} : D} \qquad 10$$

$$\frac{\bar{e}}{\partial C}$$

$$\left(2\rho F \frac{\partial \bar{e}}{\partial C} F^t\right)^t = 2\rho (F^t)^t \left(\frac{\partial \bar{e}}{\partial C}\right)^t F^t = 2F \underbrace{\left(\frac{\partial e}{\partial C}\right)^t}_{} F^t$$

will discuss how $\frac{\partial \bar{e}}{\partial C}$ is calculated and why it's sym.

$$\overbrace{\left(2\rho \underbrace{F \frac{\partial \bar{e}}{\partial C} F^t}_{\text{sym}} - T\right)}^{=0} : \underset{\text{sym}}{D} = 0$$

$A : B = 0$ B sym   & A is sym then $A = 0$

if

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} : \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{12} & B_{22} & B_{23} \\ B_{12} & B_{23} & B_{33} \end{bmatrix} = A_{11}B_{11} + A_{22}B_{22} + A_{33}B_{33} + 2A_{12}B_{12} + 2A_{13}B_{13} + 2A_{23}B_{23} = 0 \qquad *$$

choose $B = 0$ except $B_{11} = 1$ $\longrightarrow$ $A_{11} = 0$

similarly $A_{22} = 0$, $A_{33} = 0$

choose $B_{21} = B_{12} = 1$ $\longrightarrow$ $A_{21} = A_{12} = 0$; $A_{13} = 0$, $A_{23} = 0$

$\rightarrow$ $A = 0$ ∎

$$T = 2\rho F \frac{\partial \bar{e}(C)}{\partial C} F^t$$

$$= F \left(2\rho \frac{\partial \bar{e}(C)}{\partial C} \right) F^t$$

$$= F \left(\frac{2\rho_0}{\sqrt{\det C}} \frac{\partial \bar{e}(C)}{\partial C} \right) F^t = F \bar{G}(C) F^t$$

Hyperelastic material

$$\bar{G}(C) = \frac{2\rho_0}{\sqrt{\det C}} \frac{\partial \bar{e}(C)}{\partial C}$$

$$\mathcal{C} = 2 \frac{\partial \bar{G}}{\partial C}\Big|_{C=1} \rightarrow \mathcal{C} = 4\rho_0 \frac{\partial^2 \bar{e}(C)}{\partial C\, \partial C}$$

$$\mathcal{C}_{ijkl} = 4\rho_0 \frac{\partial^2 \bar{e}}{\partial C_{ij} \partial C_{kl}} = \mathcal{C}_{klij}$$

$$\longrightarrow \bar{G}(C) = \frac{2\rho_0}{\det F} \frac{\partial \bar{e}}{\partial C}$$

$$\boxed{\bar{G}(C) = \frac{2\rho_0}{\sqrt{\det C}} \frac{\partial \bar{e}}{\partial C}} \quad (1)$$

$\bar{e} \longrightarrow \bar{G}$

energy function — stress constitutive eqn

I want to derive $\mathcal{C}_{ijkl}$ for hyperelastic materials

$$\bar{G}(C=I) = \frac{2\rho_0}{\sqrt{1}} \frac{\partial \bar{e}}{\partial C}(C=I) = 0$$

stress free initial condition

$$\rightarrow \boxed{\frac{\partial \bar{e}}{\partial C}(C=I) = 0} \quad (2)$$

Recall $\mathcal{C} = 2 \dfrac{\partial \bar{G}}{\partial C}\Big|_{C=I} \rightarrow$

$$\mathcal{C}_{ijkl} = 2 \frac{\partial \bar{G}_{ij}}{\partial C_{kl}}(C=I) \rightarrow$$

$$\bar{G}_{ij} = \frac{2\rho_0}{\quad} \frac{\partial \bar{e}}{\quad}$$

∂C_{kl}

$$\tilde{G}_{ij} = \frac{2\rho_0}{\sqrt{\det C}} \frac{\partial \bar{e}}{\partial C_{ij}}$$

$= 0$ because of 2

$$\mathcal{C}_{ijkl} = \cancel{4}\rho_0 \frac{\partial \frac{1}{\sqrt{\det C}}}{\partial C_{kl}} \frac{\partial \bar{e}}{\partial C_{ij}}(C = I) + \frac{4\rho_0}{\underbrace{\sqrt{\det C}}_{C=I}} \frac{\partial^2 e}{\partial C_{kl}\partial C_{ij}}(C=I)$$

$$\underbrace{\mathcal{C}_{ijkl} = 4\rho_0 \frac{\partial^2 \bar{e}}{\partial C_{kl}\partial C_{ij}}}$$

major symmetry

$$= 4\rho_0 \frac{\partial^2 \bar{e}}{\partial C_{ij}\partial C_{kl}} = \mathcal{C}_{klij}$$

Only for hyperelastic

$$\begin{bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{12} \\ S_{23} \\ S_{31} \end{bmatrix} = \tilde{\mathcal{C}}_{6\times 6} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{12} \\ 2E_{23} \\ 2E_{31} \end{bmatrix}$$

sym with 21 components

Note 1

How is $\frac{\partial \bar{e}}{\partial C}$ computed?

$$\bar{e}(C) = C_{11} + C_{22} + C_{12} + 3C_{21} - 2 \qquad \bar{e}(I) = 0$$

$$\frac{\partial \bar{e}}{\partial C} = ? \qquad \left(\frac{\partial \bar{e}}{\partial C}\right)_{ij} := \frac{\partial \bar{e}}{\partial C_{ij}}$$

$\frac{\partial \bar{e}}{\partial C} = ?$ $\quad \left(\frac{\partial \bar{e}}{\partial C}\right)_{ij} := \frac{\partial \bar{e}}{\partial C_{ij}}$

$\frac{\partial \bar{e}}{\partial C_{11}} = 1 \quad \frac{\partial \bar{e}}{\partial C_{22}} = 1 \quad \frac{\partial \bar{e}}{\partial C_{12}} = 1 \quad \frac{\partial \bar{e}}{\partial C_{21}} = 3 \longrightarrow \frac{\partial \bar{e}}{\partial C} = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ~~X~~ ☹

we know that $C_{21} = C_{12}$

should be writen in terms of its independent components

$$\bar{e}(C) = \bar{e}(\underbrace{C_{11}, C_{22}, C_{33}, C_{12}, C_{23}, C_{31}}) \qquad C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ & C_{22} & C_{23} \\ \text{sym} & & C_{33} \end{bmatrix}$$

$\frac{\partial \bar{e}}{\partial C_{12}}$ ✓

$$\frac{\partial \bar{e}}{\partial C_{21}} = \cancel{\frac{\partial \bar{e}}{\partial C_{11}} \frac{\partial C_{11}}{\partial C_{21}}}^{0} + \cancel{\dots} + \frac{\partial \bar{e}}{\partial C_{12}} \overbrace{\frac{\partial C_{12}}{\partial C_{21}}}^{1} + \dots \cancel{}$$

$\frac{1}{2}$

$$\frac{\partial \bar{e}}{\partial C_{21}} = \frac{\partial \bar{e}}{\partial C_{12}} = \frac{1}{2} \frac{\partial e}{\partial C_{12}} \qquad \frac{\partial \bar{e}}{\partial C} \text{ is sym.}$$

$$\bar{e}(C) = C_{11} + C_{22} + \cancel{C_{12} + 3C_{21}} - 2 \quad \dots$$

$4C_{12}$

write it in a sym. fashion $C_{12} = C_{21}$

$$\bar{e}(C) = C_{11} + C_{22} + 2C_{12} + 2C_{21} - 2$$

non physical

✓

$$\frac{\partial e}{\partial C} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbb{C} = \frac{\partial^2 e}{\partial C \partial C}$$