2020/11/23

Monday, November 23, 2020 1:05 PM

From 11/18/20:

general

$$K^e = \int_{\xi_1=-1}^{1}\int_{\xi_2=-1}^{1} \underbrace{\underbrace{\begin{bmatrix} -\frac{(1-\xi_2)}{4} & -\frac{(1-\xi_1)}{4} \\ \frac{(1-\xi_2)}{4} & -\frac{(1+\xi_1)}{4} \\ \frac{1+\xi_2}{4} & \frac{1+\xi_1}{4} \\ -\frac{(1+\xi_2)}{4} & \frac{1-\xi_1}{4} \end{bmatrix}}_{B_\xi^t} \left(J^{-t} \underset{2\times2}{k} J^{-1} \det J\right)_{2\times2} \underbrace{\begin{bmatrix} -\frac{(1-\xi_2)}{4} & \frac{1-\xi_2}{4} & \frac{1+\xi_2}{4} & -\frac{(1+\xi_2)}{4} \\ -\frac{(1-\xi_1)}{4} & -\frac{1+\xi_1}{4} & \frac{1+\xi_1}{4} & \frac{1-\xi_1}{4} \end{bmatrix}}_{B_\xi}}_{I(\xi_1,\xi_2)} d\xi_1 d\xi_2$$

$$J^{-t} K_{2\times2} J^{-1} \det J = \frac{8k}{4+\xi_1+\xi_2} \begin{bmatrix} \frac{3+\xi_1}{4} & -\frac{(1+\xi_1)}{4} \\ -\frac{(1-\xi_2)}{4} & \frac{3+\xi_2}{4} \end{bmatrix} \begin{bmatrix} \frac{3+\xi_1}{4} & -\frac{(1-\xi_2)}{4} \\ -\frac{(1+\xi_1)}{4} & \frac{3+\xi_1}{4} \end{bmatrix}$$

$K_{2\times2} = k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ isotropic

for ONLY

Last time full integration order = 2 in ξ_1 & ξ_2

$o_{\xi_1} = 2$, $o_{\xi_2} = 2$

ξ_2 is const for ξ_1 integral

$$K^e_{4\times4} = \int_{\xi_2=-1}^{\xi_2=1} d\xi_2 \int_{\xi_1=-1}^{\xi_1=1} \underbrace{I(\xi_1,\xi_2)}_{\text{order } o_{\xi_1}=2} d\xi_1$$

Gauss Quad $o_{\xi_1} = 2 \rightarrow n_{\xi_1} = \text{ceil}\left(\frac{o_{\xi_1}+1}{2}\right) = \text{ceil}(1.5) = 2$

$$K^e_{4\times4} = \int_{\xi_2=-1}^{1} d\xi_2 \underbrace{\left(\omega_1 \cdot I(\xi_{qp_1},\xi_2) + \omega_2 I(\xi_{qp_2},\xi_2)\right)}_{J(\xi_2)}$$

$J(\xi_2)$ is 2nd order in ξ_2

$\xi_{qp_1} = \frac{-1}{\sqrt{3}}$

$\xi_{qp_2} = \frac{1}{\sqrt{3}}$

TABLE 5.6 *Sampling points and weights in Gauss-Legendre numerical integration (interval −1 to +1)*

n	r_i			α_i		
1	0.	(15 zeros)		2.	(15 zeros)	
2	±0.57735	02691	89626	1.00000	00000	00000
3	±0.77459	66692	41483	0.55555	55555	55556
	0.00000	00000	00000	0.88888	88888	88889
4	±0.86113	63115	94053	0.34785	48451	37454
	±0.33998	10435	84856	0.65214	51548	62546
5	±0.90617	98459	38664	0.23692	68850	56189
	±0.53846	93101	05683	0.47862	86704	99366
	0.00000	00000	00000	0.56888	88888	88889
6	±0.93246	95142	03152	0.17132	44923	79170
	±0.66120	93864	66265	0.36076	15730	48139
	±0.23861	91860	83197	0.46791	39345	72691

$o_{\xi_2} = 2 \rightarrow n_{\xi_2} = \text{ceil}\left(\frac{1+2}{2}\right) = 2$

$= \omega_1 \, \mathcal{J}(\xi_2 = \xi_{qp_1}) + \omega_2 \, \mathcal{J}(\xi_2 = \xi_{qp_2})$

replace for $\mathcal{J}$

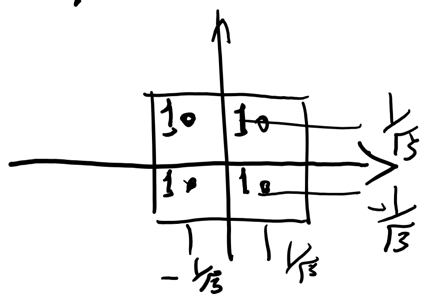
$\longrightarrow$

$$\boxed{\begin{aligned} K &= \omega_1^2 \, I(\xi_{qp_1}, \xi_{qp_1}) + \omega_1 \omega_2 \, I(\xi_{qp_1}, \xi_{qp_2}) \\ &\quad + \omega_2 \omega_1 \, I(\xi_{qp_2}, \xi_{qp_1}) + \omega_2^2 \, I(\xi_{qp_2}, \xi_{qp_2}) \end{aligned}}$$

$\omega_1 = \omega_2 = 1$

$\xi_{qp_1} = \frac{-1}{\sqrt{3}}, \quad \xi_{qp_2} = \frac{1}{\sqrt{3}}$

4pt full
Quad scheme



What if we had used NC?

$o_{\xi_1} = 2 \longrightarrow n_{\xi_1} = 2 + 1 = 3$

$\xi_{qp_1} = -1 \quad \xi_{qp_2} = 0 \quad \xi_{qp_3} = 1$

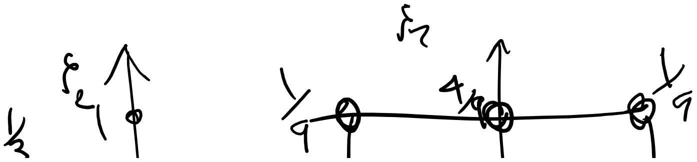
$\omega_1 = 2 \times \frac{1}{6} \quad \omega_2 = 2 \frac{4}{6} \quad \omega_3 = 2 \frac{1}{6} \qquad L_{\xi} = 2$

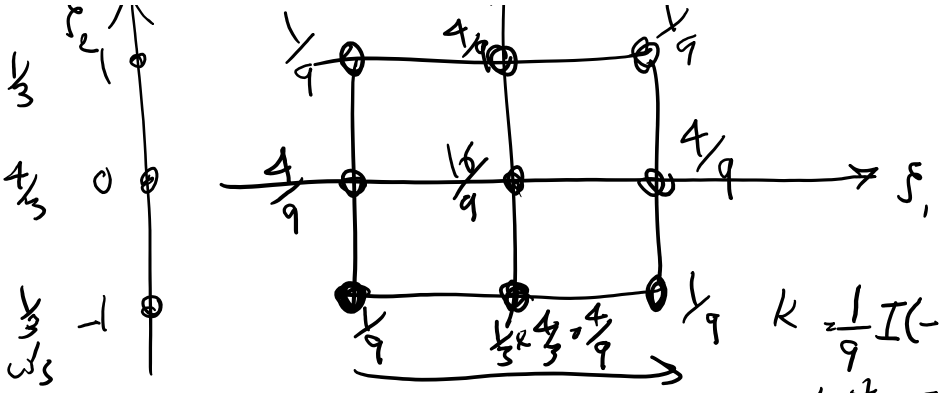
Same applies in ξ_2 direction

$\xi_{qp_1} = -1 \quad \xi_{qp_2} = 0 \quad \xi_{qp_3} = 1$

$\omega_1 = \frac{1}{3} \quad \omega_{\xi_2} = \frac{4}{3} \quad \omega_3 = \frac{1}{3}$

very much like Gauss Quad example
Quad schemes "multiply each other"





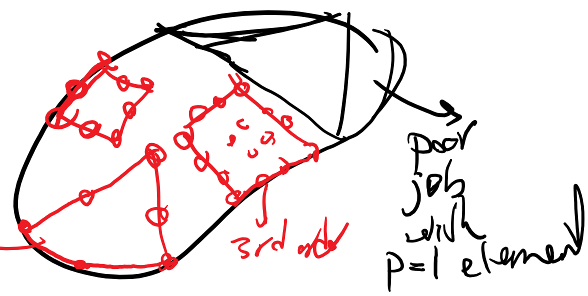
$$K = \frac{1}{9} I(-1,-1) + \frac{4}{9} I(0,-1) + \frac{1}{9} I(1,-1) + \dots + \frac{1}{9} I(1,1)$$

ξ_1	-1	0	1
w_ξ	$1/3$	$4/3$	$1/3$

Higher order elements in 2D and 3D. Motivation:

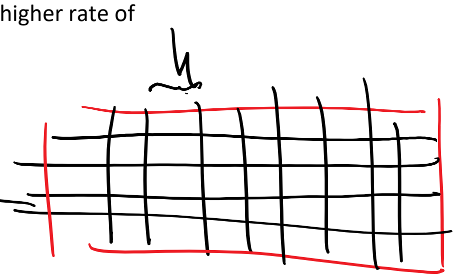
Reason 1: better approximation of geometry

2nd order curve
parabola
for $p=2$



Reason 2: Higher order elements have a higher rate of convergence

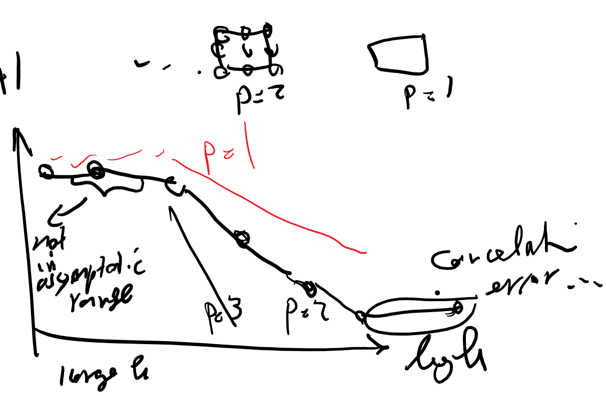
uniform



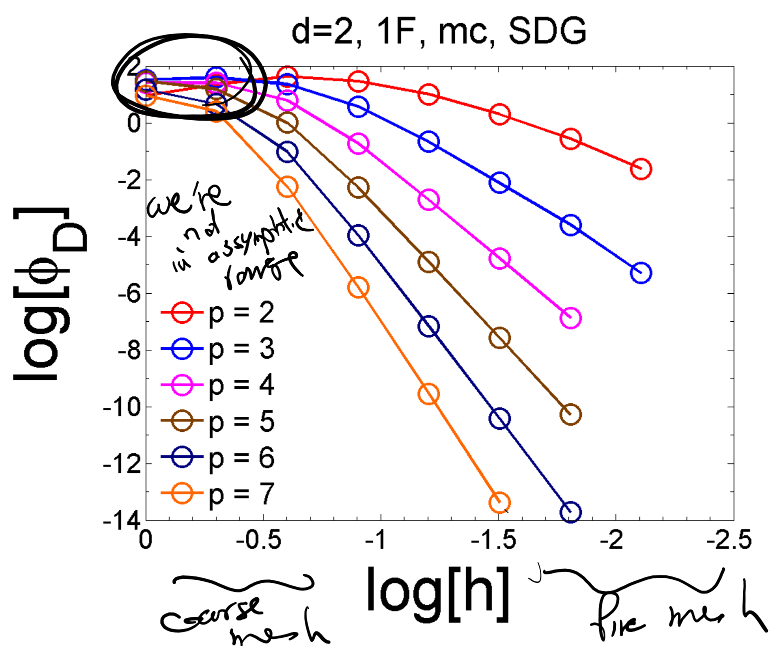
use elements of order p

$$\|u_h - u_{exact}\|_2 = C h^{p+1}$$

$$\log \text{error} = c + \underbrace{(p+1)}_{\text{slope}} \log h$$



DG method



If the solution is sufficiently smooth (more on this below) for low accuracy solutions low order FEM can be more efficient (lowers wall clock time for the same accuracy) whereas for high accuracy solutions often higher order FEMs are more economical.

Lohner_2011_Error_and_work_estimates_for_high_order_elements

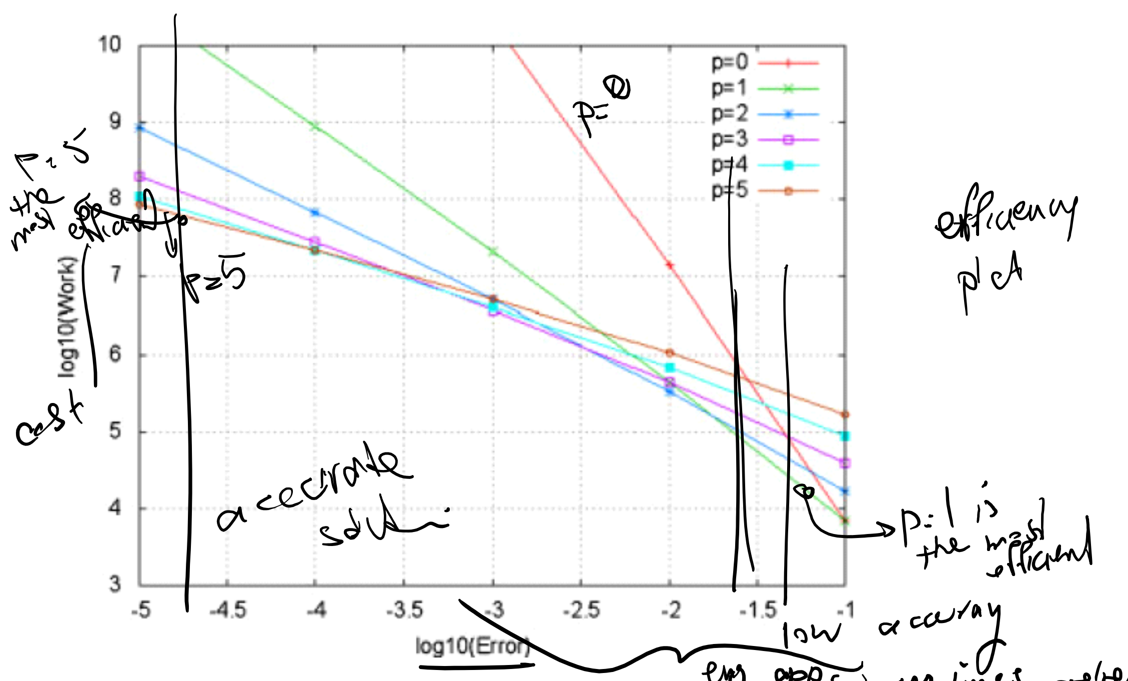


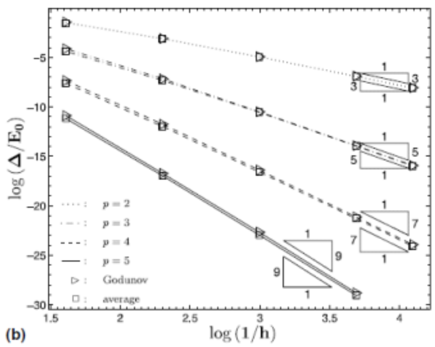
Figure 1. Asymptotic work estimates for 3-D linear problems.

2nd point about efficiency is the regularity of solution

$$\|u^h - u_{exact}\|_h = C\, h^{\min(p,s)+1}$$

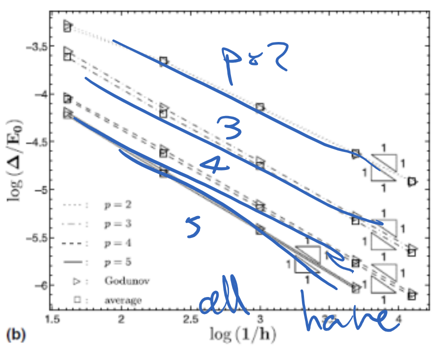
$$\|u^h - u_{exact}\|_2 = C\, h^{\min(p,s)}$$

regularity of the exact soln

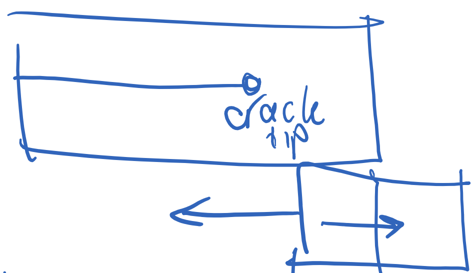


(b)

Smooth exact sln
$\cos x$ $\sin x$



(b)

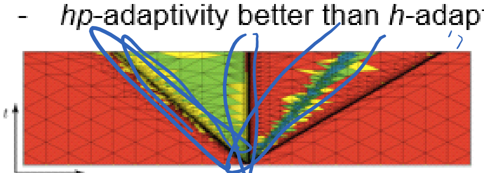


all have the same slope! displacement C^0

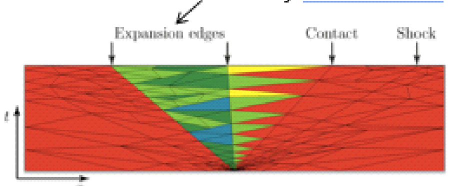
- Adaptive operations in spacetime:
  - Front-tracking better than shock capturing
  - *hp*-adaptivity better than *h*-adaptivity

Sod's shock tube problem

Results by Scott Miller



Shock capturing: 473K elements

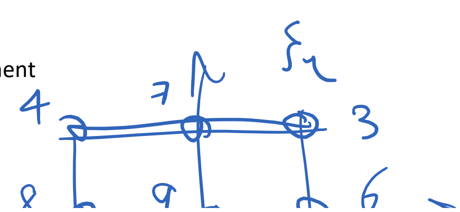


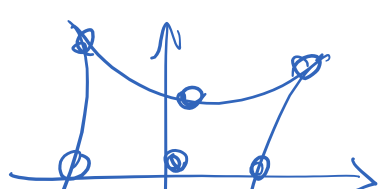
Shock tracking: 446 elements

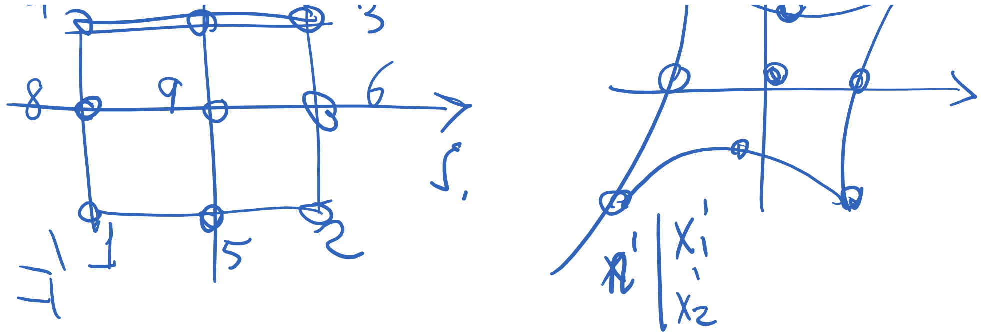
Message: when the solution is nonsmooth (shocks, crack tip fields, etc) higher order elements do not result in higher convergence rate ->

- The method may be more efficient with lower order element there.
- In nonlinear problems (e.g. fluids) the method may not even work if high order elements are used in the shock, …

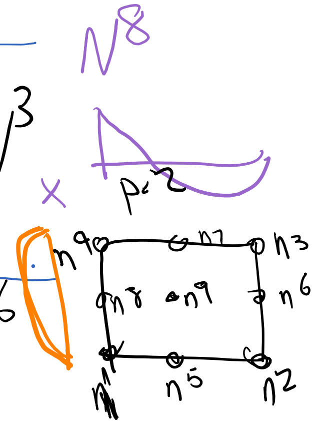
P = 2 element





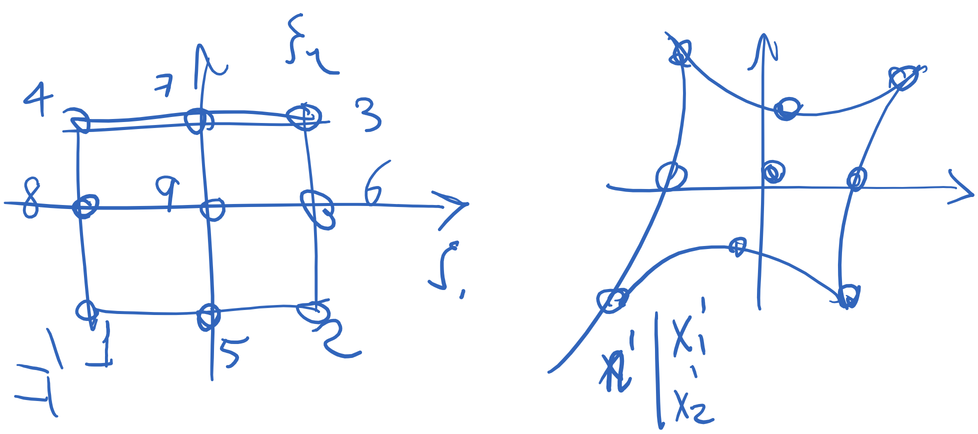


ξ_2 \ ξ_1	$N_1(\xi_1) = \xi_1(\xi_1-1)/2$	$N_2(\xi_1) = \frac{\xi_1(\xi_1+1)}{2}$	$N_3(\xi_1) = 1-\xi_1^2$
$N_1(\xi_2) = \frac{\xi_2(\xi_2-1)}{2}$	N^1	N^2	N^5
$N_2(\xi_2) = \frac{\xi_2(\xi_2+1)}{2}$	N^4	N^7_9	N^3
$N_3(\xi_2) = 1-\xi_2^2$	$N^8 = \xi_1(\xi_1-1)/2 * (1-\xi_2^2)$	N^9	N^6



Geometry

Isoparametric formulation:



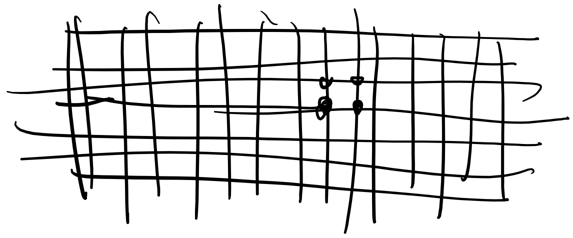
$$T(\xi_1,\xi_2) = T^1 N^1(\xi_1,\xi_2) + \cdots + T^9 N^9(\xi_1,\xi_2)$$

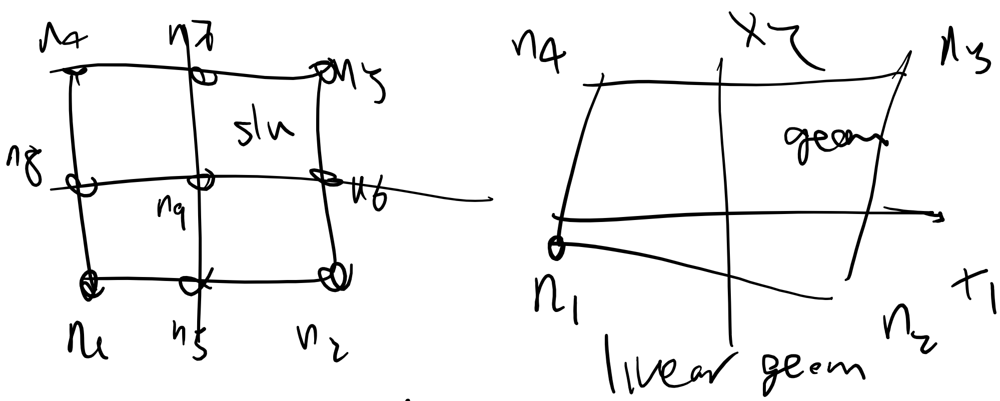
Iso
geometry is the same

$$x_1(\xi_1,\xi_2) = x_1^1 N^1(\xi_1,\xi_2) + \dots x_1^9 N^9(\xi_1,\xi_2)$$

same for x_2

x_2 " x_2^1 " x_2^9 "





$$T = \sum T^i N^i(\xi_1,\xi_2)$$

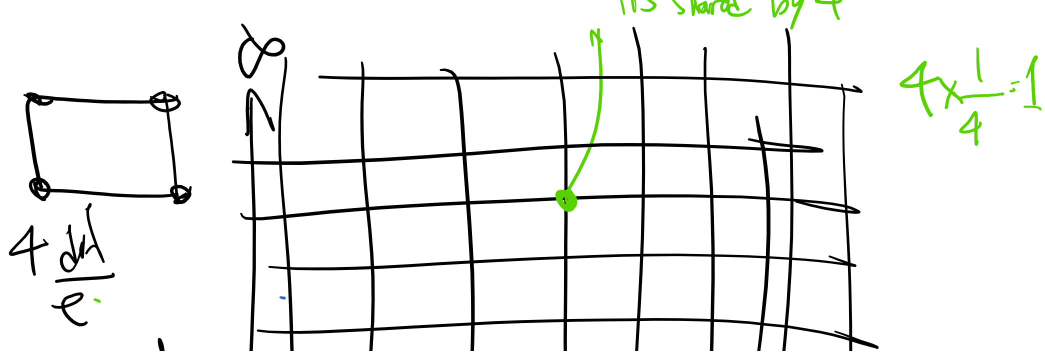
$$x_1 = \sum_{j=1}^{4} X_1^j \left(N^{p=1}\right)^j(\xi_1,\xi_2)$$

Sub parametric formulation

HW

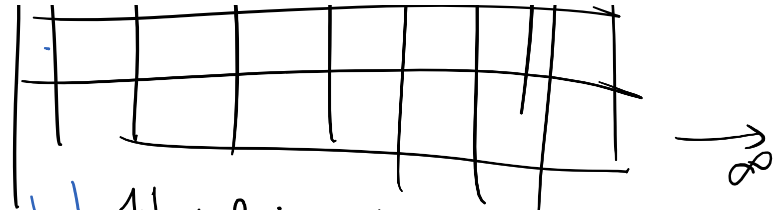
I'll ask you to compute average dof/element

it's shared by 4

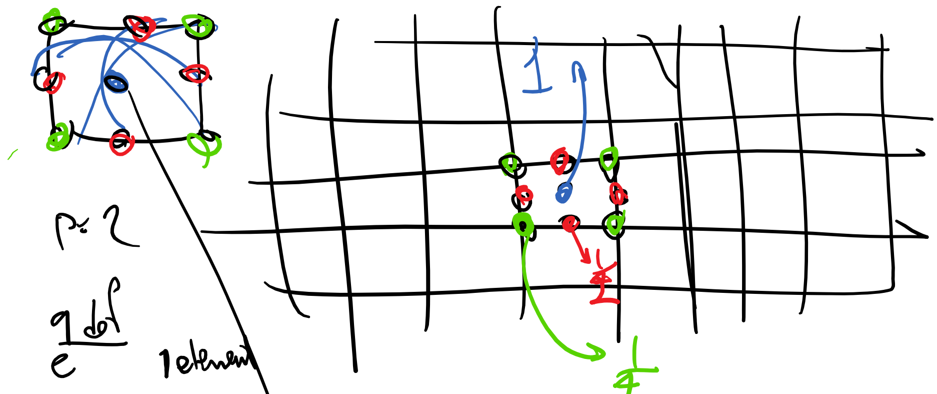


$\frac{1\ dof}{e}$

1 element



$\underline{4\times\frac{1}{4} + 4\times\frac{1}{2} + 1\times 1 = 4}$ dof/element in average



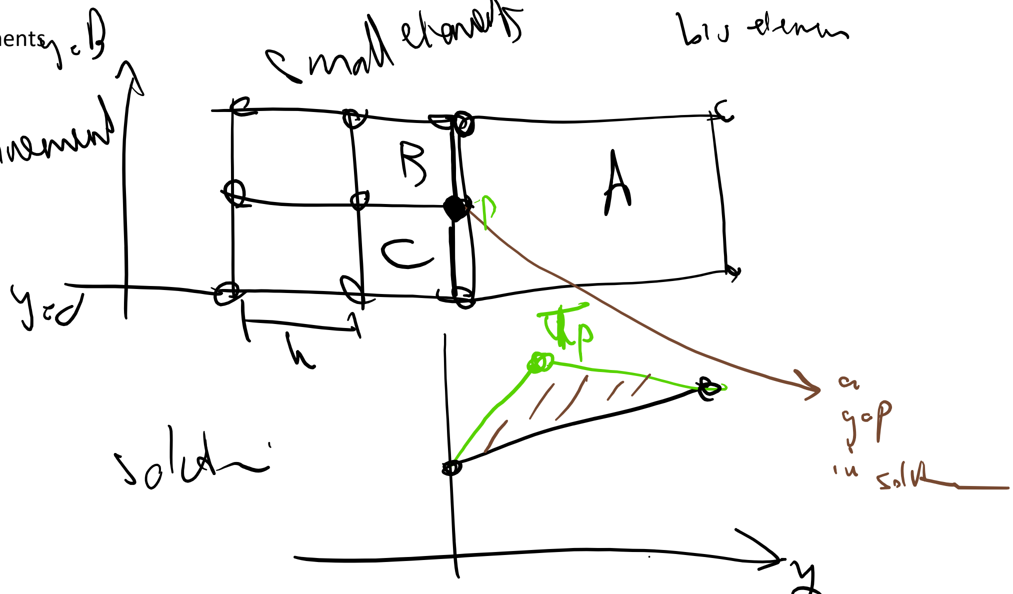
p=2

$\frac{9\ dof}{e}$ 1 element

we can get rid of this by static condensation

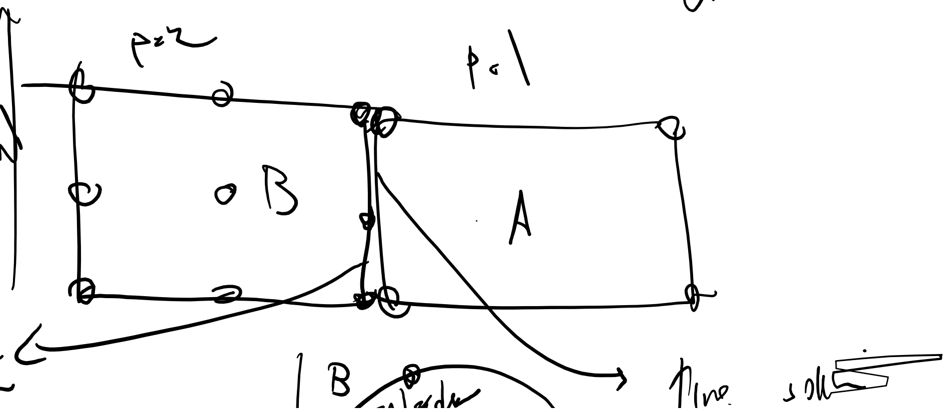
Connection elements

h_refinement



solution

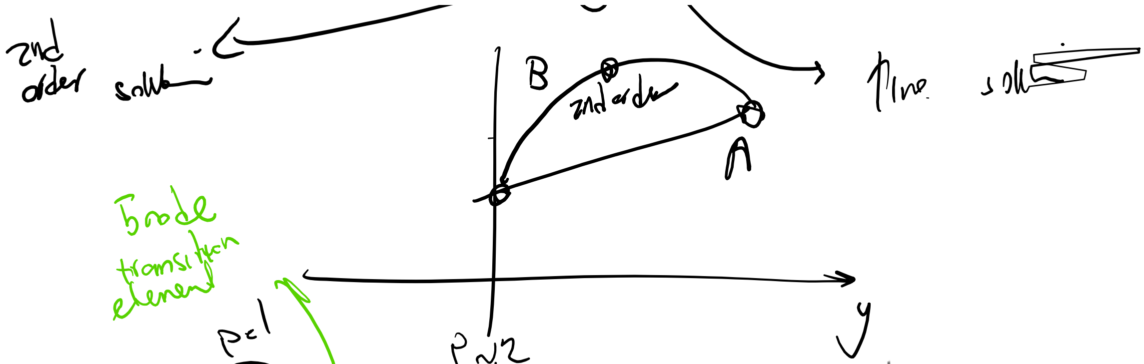
P enrichment

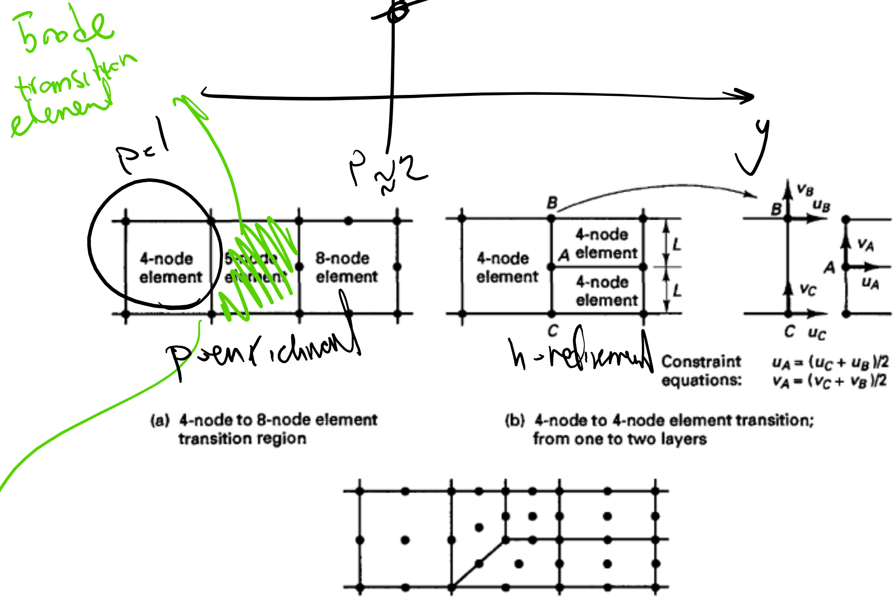


2nd order solution

line solution

B 2nd order



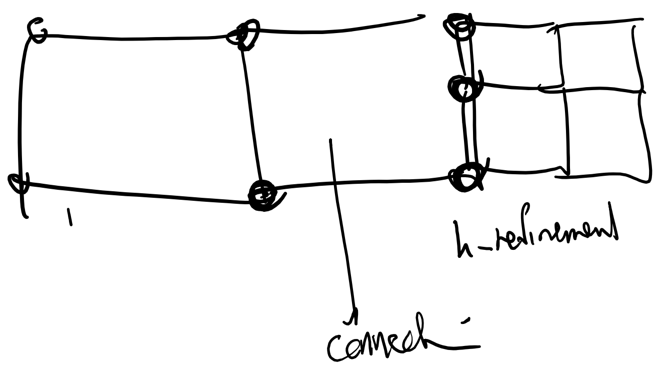


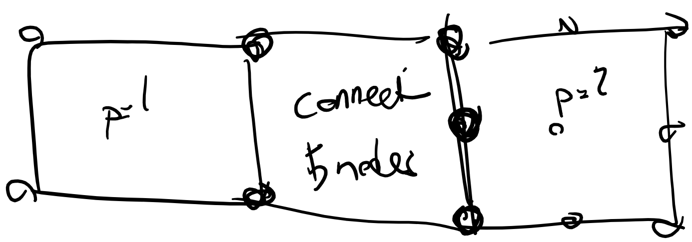
(a) 4-node to 8-node element transition region

(b) 4-node to 4-node element transition; from one to two layers

(c) 9-node to 9-node element transition region; from one to two layers

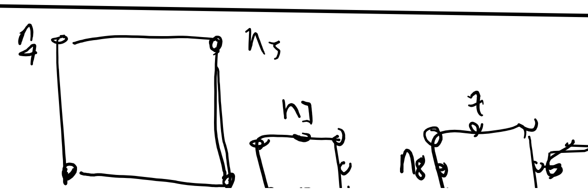
in CFEM to change h or p we generally need connection elements



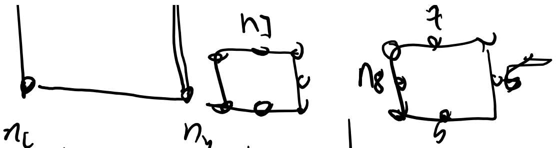


How do we form the shape functions for these connective elements?

Idea: start from linear elements



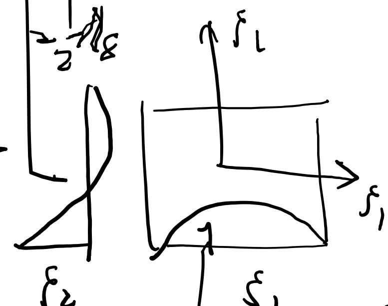
Idea: start from linear elements



		5 node element	6 node element	7 node	
N^1	$\frac{(1-\xi_1)(1-\xi_2)}{4}$	$-\frac{1}{2}N_5$	—	—	$-\frac{1}{2}N_8$
N_2	$\frac{(1+\xi_1)(1-\xi_2)}{4}$	$-\frac{1}{2}N_5$	$-\frac{1}{2}N_6$	—	—
N_3	$\frac{(1+\xi_1)(1+\xi_2)}{4}$	—	$-\frac{1}{2}N_6$	$-\frac{1}{2}N_7$	—
	$\frac{(1-\xi_1)(1+\xi_2)}{4}$	—	—	$-\frac{1}{2}N_7$	$-\frac{1}{2}N_8$
	$\frac{(1-\xi_1^2)\,\xi_2(\xi_2-1)}{2}$				
	$\frac{(1-\xi_2^2)\,\xi_1(\xi_1+1)}{2}$				

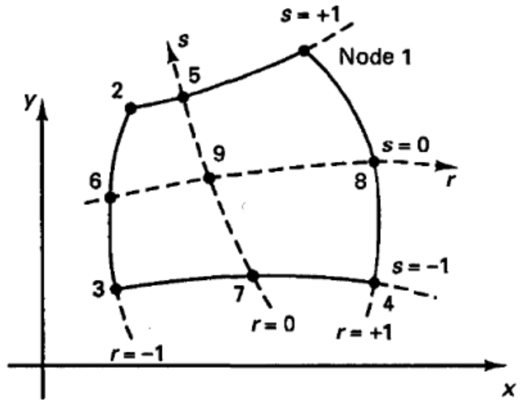
$1-\xi_2^2$

$\frac{\xi_1(\xi_1+1)}{2}$



$N^5 = (1-\xi_1^2)\left[\frac{\xi_2(-1+\xi_2)}{2}\right]$

bathe_arbOrder2Delement_358



(a) 4 to 9 variable-number-nodes two-dimensional element

Include only if node i is defined

		$i=5$	$i=6$	$i=7$	$i=8$	$i=9$
$h_1 =$	$\frac{1}{4}(1+r)(1+s)$	$-\frac{1}{2}h_5$			$-\frac{1}{2}h_8$	$-\frac{1}{4}h_9$
$h_2 =$	$\frac{1}{4}(1-r)(1+s)$	$-\frac{1}{2}h_5$	$-\frac{1}{2}h_6$			$-\frac{1}{4}h_9$
$h_3 =$	$\frac{1}{4}(1-r)(1-s)$		$-\frac{1}{2}h_6$	$-\frac{1}{2}h_7$		$-\frac{1}{4}h_9$
$h_4 =$	$\frac{1}{4}(1+r)(1-s)$			$-\frac{1}{2}h_7$	$-\frac{1}{2}h_8$	$-\frac{1}{4}h_9$
$h_5 =$	$\frac{1}{2}(1-r^2)(1+s)$					$-\frac{1}{2}h_9$
$h_6 =$	$\frac{1}{2}(1-s^2)(1-r)$					$-\frac{1}{2}h_9$
$h_7 =$	$\frac{1}{2}(1-r^2)(1-s)$					$-\frac{1}{2}h_9$
$h_8 =$	$\frac{1}{2}(1-s^2)(1+r)$					$-\frac{1}{2}h_9$
$h_9 =$	$(1-r^2)(1-s^2)$					

(b) Interpolation functions