2021/08/19

Thursday, August 19, 2021 2:35 PM

V. **COURSE REQUIREMENTS, ASSESSMENT AND EVALUATION METHODS:**

- Exam(s) (subject to change): 8%
- Assignments: Homework assignments take up 50% of the grade. Assignments typically involve a computational part that requires writing/modifying small computer codes (Matlab, C++) or using commercial packages such as COMSOL. The assignments include challenge problems" that can add up to 5-10% to the final grade. Percentage can be subject to change. 60%
- Term project(s): Computer FEM code (13%)& ~~commercial~~ FEM software (19%) → your implementation 32%
- Absences and excused grades: Excuses will be given only under the following circumstances:
  - illness
  - personal crisis (e.g. automobile accident, death of a close relative)

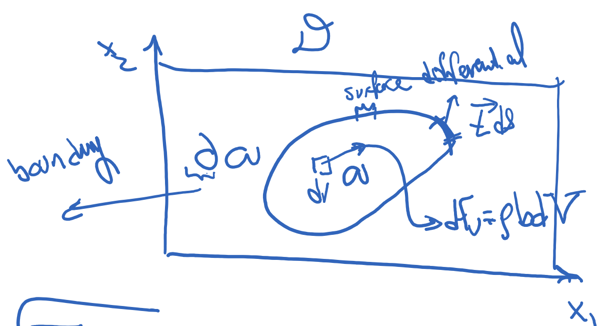
  otherwise there is a 15% penalty per day for late assignments.

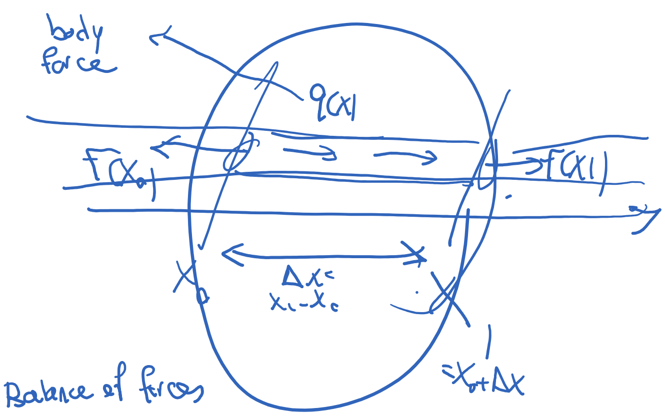
Course outline:

A. Finite Element mathematical formulation (40% of the course, it's more mathematical, and you can skip several parts in the course notes)
   a. Balance laws
   b. Strong form (Differential Equation)
   c. Weighted Residual Statement (WRS)
   d. Weak Statement (WK)
   e. Energy Formulation that replaces the steps from a -> d
B. 1D and 2D finite element formulation:
   a. bars, beams, trusses, frames, plates
   b. Some issues related to 2D elements (numerical integration, element shapes)
C. Implementation of a FEM code

Short overview of part A, without going to details

A.a Balance law





$\boxed{\sum F = 0}$

$$\int_{\partial\omega} \vec{t}\, ds + \underbrace{\int_{\omega} \rho b\, dV}_{dF_v} = 0$$

$$\int_{\partial\omega} \sigma \cdot n\, ds + \int_{\omega} \rho b\, dV = 0$$

Balance of forces

$\boxed{\sum F = 0}$ $\quad \underline{F(x_1) - F(x_0) + q(x)\Delta x = 0}$

$$\lim_{\Delta x \to 0} \frac{F(x_0+\Delta x) - F(x_0)}{\Delta x} + q(x) = 0$$

A.b Differential Equation

$\nabla \cdot \sigma + \rho b = 0$ ... Differential ... $\left[\frac{dF}{dx} + q(x) = 0\right]$

$$\boxed{\nabla \cdot \sigma + \rho b = 0} \quad \forall x \in D$$

2D: $\underbrace{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_y}}$

Partial Differential equation

$\Delta x \to 0$

Ordinary Differential Equation

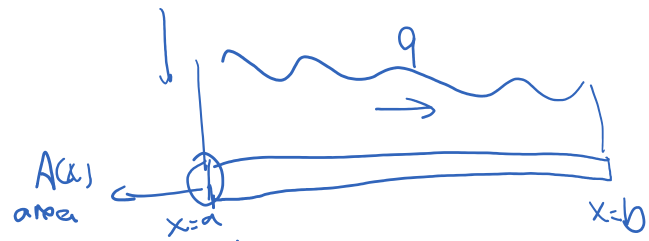
ODE

$$\boxed{\frac{dF}{dx} + q(x) = 0} \quad \text{for } \forall x \in (a,b)$$

This is also called strong form (as opposed to weak statements for anything that involves an integral) because Differential Equations are satisfied at (every) point.

Point -> strong

A.c Weighted Residual Statement



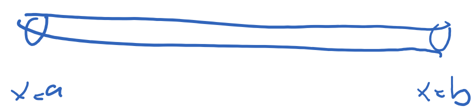
$$\frac{dF(x)}{dx} + q(x) = 0$$

$$F(x) = A(x) \underbrace{E(x)}_{\text{Elastic modulus}} \frac{du}{dx}$$

trial solution

$$\underline{R(u)} = \underbrace{\frac{d}{dx}\left(AE\frac{du}{dx}\right) + q}$$

Weighted



a function of x

$$\int_a^b w(x)\, R(u(x))\, dx = 0$$

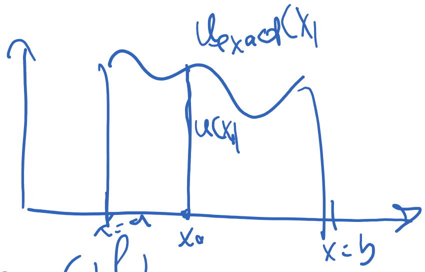
weighted residual statement

if we have the exact solution WR = 0

if WR = 0 for all possible weight functions $\Longrightarrow$

converse

U is the exact solution



Final step : Discretization

instead of ∞ unknowns $\rightarrow$

Finite # of unknowns / degrees of freedom (dof)

$$u^h = \phi_1(x)\, a_1 + \phi_2(x)\, a_2 + \phi_3(x)\, a_3 + \phi_4(x)\, a_4$$

4 dof approximation

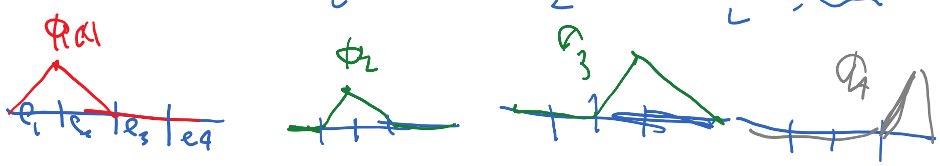
Discretized (numerical) solution

$a_1, \dots, a_4$ are unknowns

we'll chose $\phi_1, \dots, \phi_4$

$\phi_1 = 1$ $\quad$ $\phi_2 = x$ $\quad$ $\phi_3 = x^2$ $\quad$ $\phi_4 = x^3$

Fourier series $\quad 1, \quad \cos\frac{2\pi x}{L}, \quad \sin\frac{2\pi x}{L}, \quad \cos\frac{4\pi x}{L}, \dots$



chose this now

$$u^h(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3$$

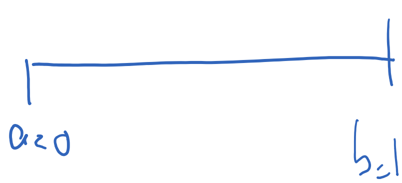
$$R(x) = \frac{d}{dx}\left(AE\frac{du}{dx}\right) + q(x)$$

$A = 1$
$E = 1$
$q(x) = \cos x$

$$R(x) = u'' + \cos x = 6a_4 x + 2a_3$$

$$WR = \int_{a=0}^{b=1} w(x) R(x)\, dx$$

weight #1



$\bar{a}$ ↓ $a=0$ $b=1$

weight #1

ω_1 $(WR)_1 = (WR_1)(\underbrace{a_1, a_2, a_3, a_4}_{4\text{ unknown}})$ e.g. $\omega=1$

ω_2

ω_3

ω_4

4 eqns

general nonlinear eqn $R(a) = 0$

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if the problem is linear like here you get

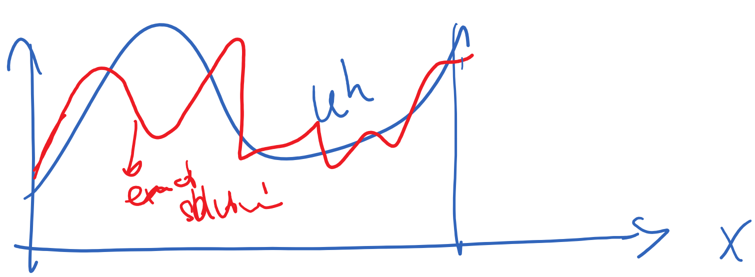
$$\underbrace{K_{4\times4}}_{\text{Stiffness matrix}} \; \underbrace{a_{4\times1}}_{\text{unknown vector}} = \underbrace{F_{4\times1}}_{\text{force vector}} \longrightarrow a = K^{-1}F$$

our choice of ω could have been

$$\omega = \{1, x, x^2, x^3\} \quad \text{as well}$$

e.g. $a = [1.2, 3.2, -1.7, 4.2]$

$u^h = 1.2 + 3.2x - 1.7x^2 + 4.2x^3$



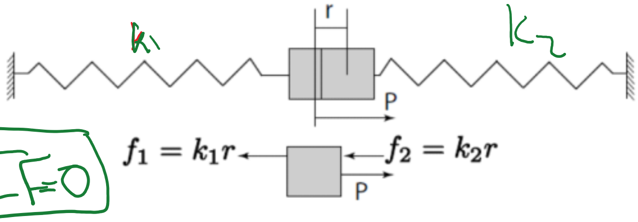
As we add more terms to the expansion of the solution, we get closer to the exact solution.

Balance laws:

- Why start with a balance law?
  - They are the actual physics laws.
  - They contain more *information* than their corresponding PDEs.
  - Larger solution space than the PDEs.
- Can we directly start the FE formulation from a PDE?
  - Yes, FE formulation starts from a differential equation.
  - A PDE may not be derived from a balance law.

Balance of forces in discrete setting:

Discrete problem



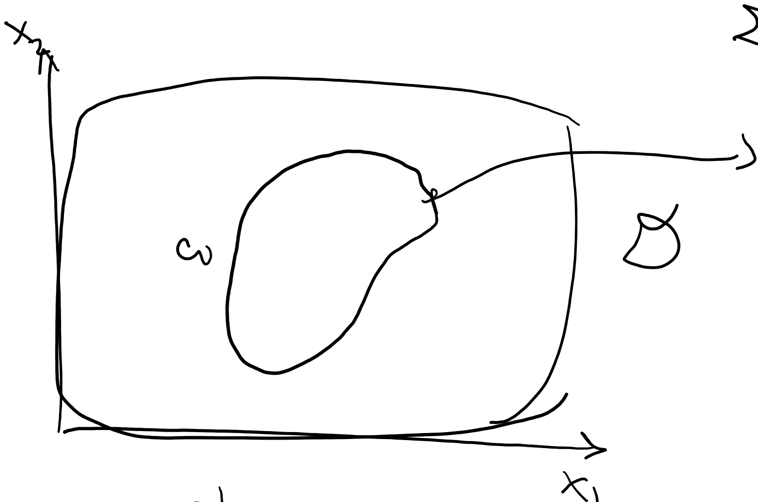
$\boxed{\sum F = 0}$

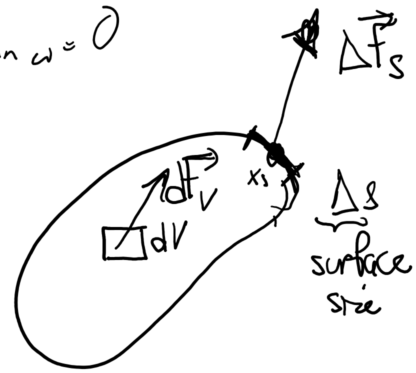
$-k_1 r \; - k_2 r + P = 0 \qquad (k_1 + k_2) r = P$

$\rightarrow \boxed{r = \dfrac{P}{k_1 + k_2}}$

Continuum problem 2D elasticity

$\sum F_{\text{on } \omega} = 0$





$\boxed{d\vec{F}_V = \underbrace{\rho \vec{b}}_{\text{intensity of force}}\, dV}$

$\omega \subseteq \mathcal{B}$
set / control volume

$\Delta \vec{F}_s$ force acting on Δs surface

$\Delta \vec{F}_s = \underbrace{\left(\dfrac{\Delta \vec{F}_s}{\Delta s}\right)} \Delta s$ let $\Delta s \to 0$
ds

weight : $\vec{b} = -g \underset{\text{vertical unit vector}}{\vec{e}_2}$

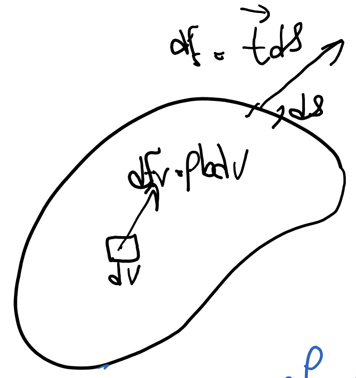
traction vector limit as $\Delta S \to 0$

"surface force intensity" $\vec{t} = \lim_{\Delta S \to 0} \dfrac{\Delta \vec{F}_s}{\Delta S}$

$$\boxed{d\vec{F}_s = \vec{t}\, dS}$$

$$\sum_{\text{on } \omega} F = 0 \rightarrow F_V + F_S = 0$$

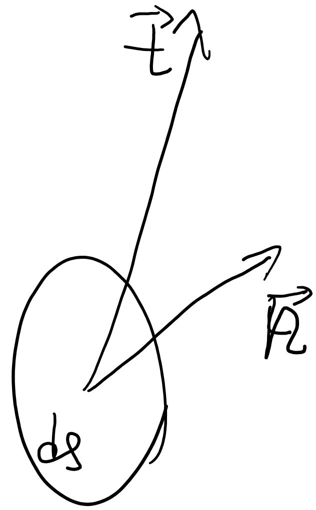
$$\int_{\omega} dF_V + \int_{\partial\omega} dF_S = 0$$



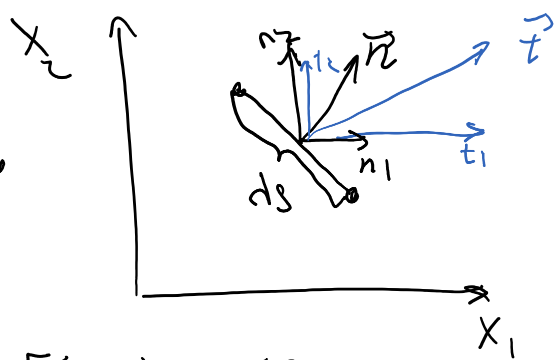
$\partial\omega$ boundary of ω

$$\boxed{\int_{\omega} \rho b\, dV + \int_{\partial\omega} \vec{t}\, dS = 0}$$

$$\vec{t} = \underbrace{\sigma}_{\text{stress tensor}} \cdot \vec{n}$$



2D elasticity



$$\begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}}_{2\times 2 \text{ stress tensor}} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$