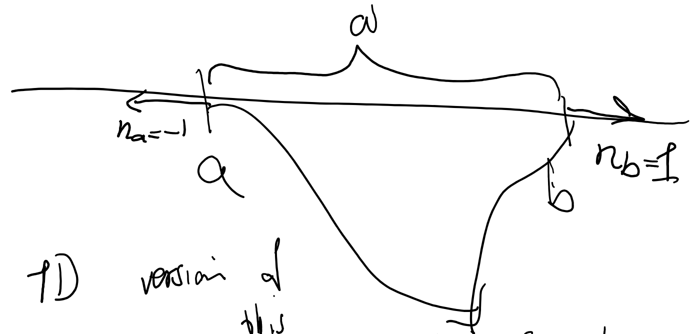
FEM2021/08/31

Tuesday, August 31, 2021 2:48 PM

Continue from Strong form in last session:

Divergence theorem $\int F(x)dx = f(b) - f(a)$ $\quad (F = f'$ or $f = \int F)$

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$
$$= f(b)n_b + f(a)n_a$$



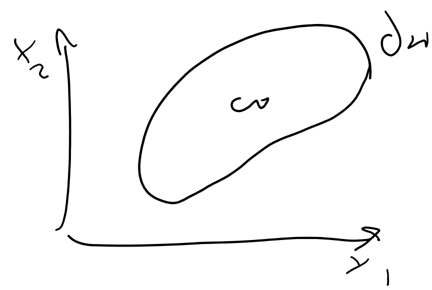
$$\int_\omega f'(x)\,dx = \int_{\partial\omega} f\cdot n\, ds$$ 1D version of this

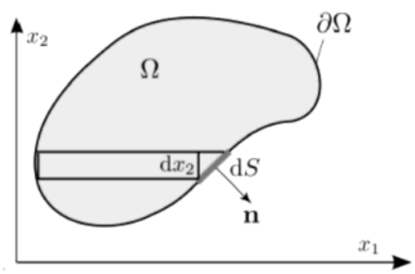
$\partial\omega = \{a,b\}$ → boundary of ω

$\omega = [a,b]$ in 1D $\quad \nabla\cdot f = f'$

this is the 1D version of divergence theorem

$$\int_\omega \nabla\cdot f\, dV = \int_{\partial\omega} f\cdot n\, ds$$





$$\underbrace{\int_{\partial\Omega} f\cdot n\, ds = \int \nabla\cdot f\, dV}_{}$$

only deal with the function. $\quad$ we need to be able to calculate divergence

this is more general

To use divergence theorem

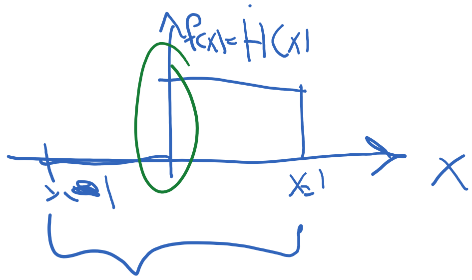
$\nabla\cdot f$ must exist &

$\nabla \cdot t$ must exist & be continuous

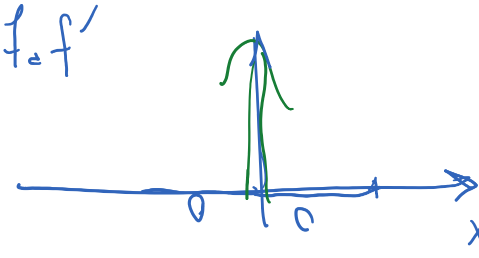
otherwise we cannot apply divergence theorem:

Example:

1D



$\nabla \cdot f = f'$

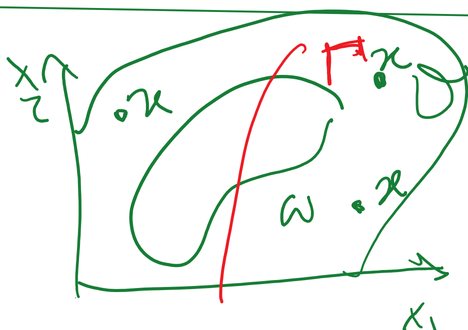


$$\int_{\partial\Omega} f(x)\, n\, ds = f(1) - f(-1) = 1-(0) = 1 \neq \int_{\partial\Omega} f'(x)\, dx \overset{?}{=} \int 0\, dx = 0$$

the problem is $f'(x)$ does not exist for all $x \in \Omega = [-1,1]$ and not continous

So we cannot apply the divergence theorem

Balance law:



$\forall \omega \subseteq \Omega$

$$\underline{\int_{\partial\omega} \sigma \cdot n\, ds} + \int_\omega \rho b\, dv = 0 \quad (*)$$

PDE (Strong form)

☆ Divergence theorem

$$\forall \omega \subseteq \Omega \quad \int_\omega (\nabla \cdot \sigma + \rho b)\, dv = 0$$

localization theorem

$$\underbrace{\nabla \cdot \sigma} + \rho b = 0$$

$\forall x \in \Omega$

Γ: jump manifold

on Γ $\nabla \cdot \sigma$ is not defined

the key is to be able to evaluate $\nabla \cdot \sigma$ (which is not always the case on Γ $\nabla\cdot\sigma$ is not defined

because similar to $H(x)$ above

σ jumps across Γ

evaluate $\nabla\cdot\sigma$

which is not always the case

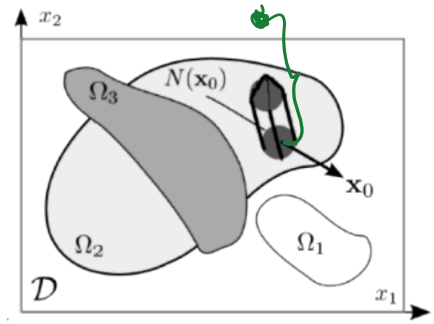
We also used localization theorem:
Just a brief overview of its proof

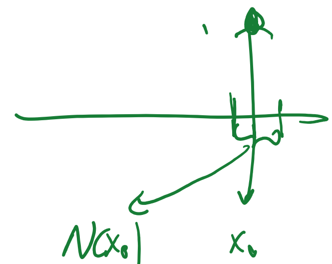
$\forall \Omega \subseteq D$

$$\int_\Omega g(x)\,dx = 0 \rightarrow$$

then $g(x) = 0$

Must be positive





where $g(x) > 0$

$$\int_{N(x_0)} \overbrace{g(x)}^{>0} dx > 0$$

Balance law always holds

However to get to the strong form (say for elastostatics) we need to:

1. Divergence of stress $\nabla\cdot\sigma$ should exist
2. Be continuous

$$\rightarrow \quad \underline{\nabla\cdot\sigma + \rho b = 0}$$

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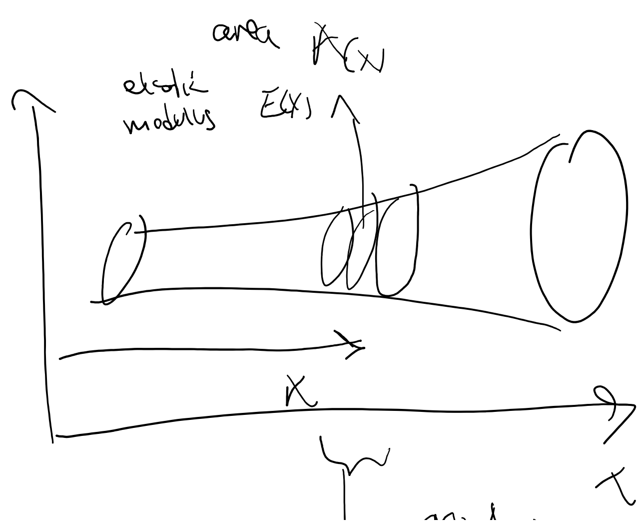
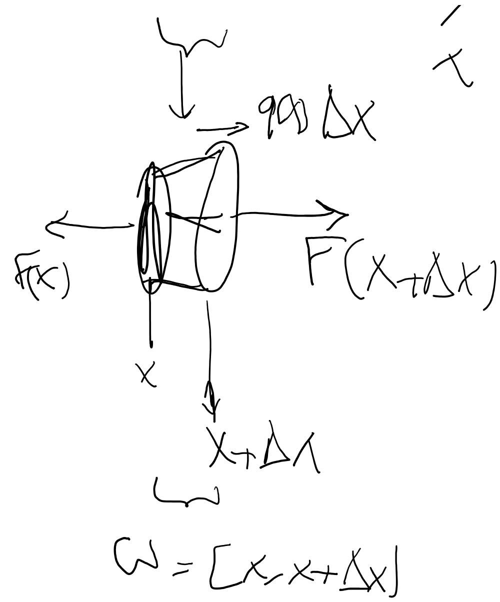
2 more examples

1D elasticity

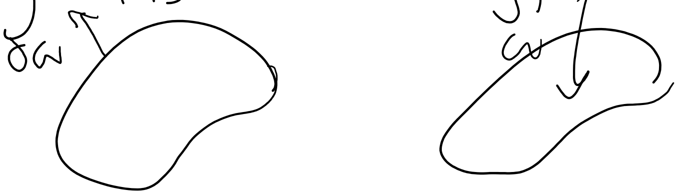


$\sum F_x = 0$



$$\underbrace{F(x+\Delta x) - F(x)}_{\text{similar to traction contribution in 2D}} + \underbrace{q(x)\Delta x}_{\text{similar to body force}} = 0$$

$\int_{\partial\omega} \sigma \cdot n \, ds + \int_{\omega} \rho b \, dv$



balance law for the set $\omega = [x, x+\Delta x]$

divide by Δx

$$\lim_{\Delta x \to 0} \left(\frac{F(x+\Delta x) - F(x)}{\Delta x} + q(x) \right) = 0$$

let $\Delta x \to 0$

$$\boxed{F'(x) + q(x) = 0}$$

Differential equation (Strong form)

in 1D $\quad \boxed{\nabla \cdot F + q(x) = 0}$ D=1

similar to 2D/3D $\quad \nabla \cdot \sigma + \rho b = 0$

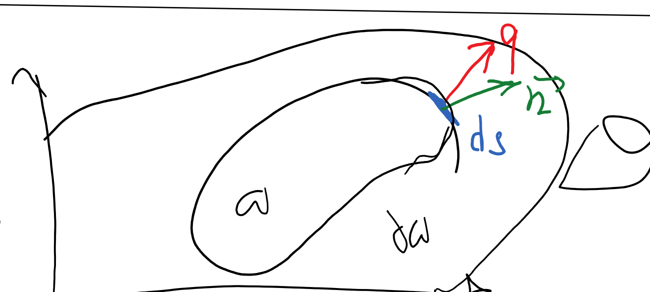
Example 2:

Heat conduction:

Steady state heat conduction

q = heat flux density vector

energy added to ω through ds = $-\vec{q} \cdot \vec{n} \, ds$



$T(x, \cancel{t})$ Balance of energy

$\cancel{I(x,t)}$ Balance of energy

Q : volumetric heat source (energy / volume)

Energy from Q

$$\int_{\omega} Q \, dV$$

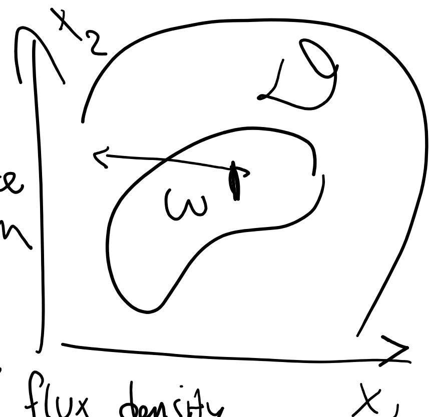
(similar to $\int \rho b \, dV$ for bal. of lin momentum)

$$-\int_{\partial\omega} q \cdot n \, ds \qquad \int_{\partial\omega} \sigma n \, ds$$

General Balance law

Source term		surface contribution
	$r \rightarrow \int_{\omega} r \, dV$	$\int_{\partial\omega} f_x \cdot n \, ds$ (f_x: Outward spatial flux density)
Solid Mech.	$\int_{\omega} \rho b \, dV$	$\int (\underbrace{-\sigma}_{f_x}) \cdot n \, ds$
Heat conduction	$\int_{\omega} Q \, dV$	$\int \underbrace{q}_{f_x} \cdot n \, ds$

source term



Balance law is

$$\int_{\omega} r \, dV - \int_{\partial\omega} f_x \cdot n \, ds = 0 \quad \xrightarrow{\forall \omega} \quad \int_{\omega} r - \nabla \cdot f_x = 0 \rightarrow \boxed{+\nabla \cdot f_x = r}$$

$\nabla \cdot q$ exists & its smooth

Solid $\quad \int_{\omega} \rho b \, dV + \int_{\partial\omega} \sigma \cdot n \, ds = 0$

heat conduction $\quad \int_{\omega} Q \, dV - \int_{\partial\omega} q \cdot n \, ds = 0 \xrightarrow{div} \int_{\omega} Q \, dV - \int_{\omega} \nabla \cdot q \, dV = 0$

$$\forall\, \omega \subset \Omega \quad \underbrace{\int_\omega (Q - \nabla \cdot q)\, dV = 0}$$

Localization

$$Q - \nabla \cdot q = 0$$

Closing the system:

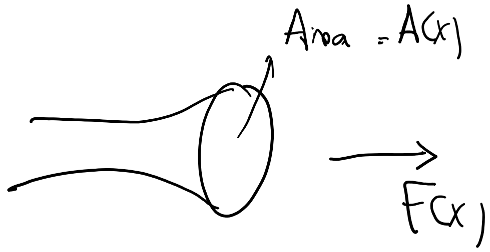
Example 1D elasticity

Balance law

$$\sum F_x = 0 \longrightarrow \frac{d F(x)}{dx} + q(x) = 0$$

$$\sigma(x) = \frac{F(x)}{A(x)}$$

stress



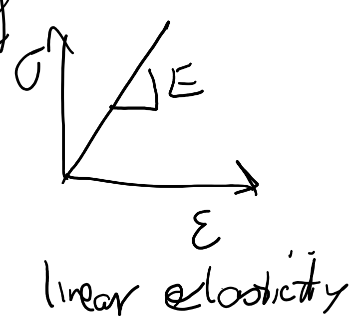
$$F(x) = A(x)\, \sigma(x)$$

DE $$\boxed{\frac{d\, A(x)\sigma(x)}{dx} + q(x) = 0}$$

$$\sigma(x) = E(x)\, \varepsilon(x)$$

Elastic modulus

constitutive equation



linear elasticity

linear elasticity

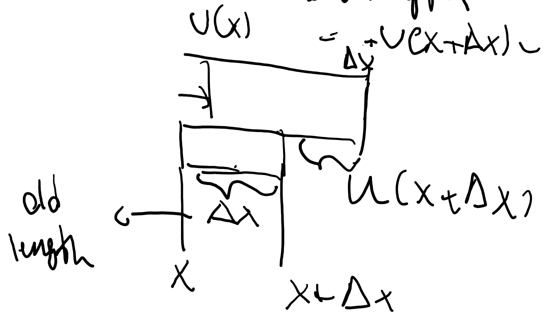
$$\frac{d\, A(x) E(x) \varepsilon(x)}{dx} + q(x) = 0$$

$$\underbrace{\varepsilon(x) = \frac{du(x)}{dx}}_{\text{Compatibility eqn}}$$

$$\boxed{\left(E(x) A(x) u'(x)\right)' + q(x) = 0}$$

DE in terms of u

new length $= \Delta x + U(x+\Delta x) - U(x)$



Change of length $= \dfrac{\text{new length} - \text{old length}}{\text{old length}}$

$$= \frac{U(x+\Delta x) - u(x)}{\Delta x}$$

$\lim_{\Delta x \to 0}$

$$= \frac{du}{dx}$$