FEM2021/09/07

Tuesday, September 7, 2021 2:51 PM

Voigt notation

$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \longrightarrow$ Voigt stress $S = \begin{bmatrix} S_{11} \\ S_{22} \\ S_{12} \end{bmatrix}$

$\sigma_{21} = \sigma_{12}$

3D 3x3

$S \to 6\times 1$

$\varepsilon = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{bmatrix} \longrightarrow$ Voigt (engineering) strain $\gamma = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix}$

$\varepsilon_{21} = \varepsilon_{12}$

$\tilde{\sigma} = \overset{\equiv}{C} \tilde{\varepsilon} \longrightarrow$ Voigt notation

$S = \tilde{C}\gamma$

Voigt stress, stiffness, strain

$\tilde{C}$ is symmetric in 2D it's 3x3 matrix

" 3D " 6x6

2D)

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{13} \\ \tilde{C}_{12} & \tilde{C}_{22} & \tilde{C}_{23} \\ \tilde{C}_{13} & \tilde{C}_{23} & \tilde{C}_{33} \end{bmatrix}$$

if the material is isotropic

$\tilde{C}_{13} = \tilde{C}_{23} = 0$

$\tilde{C}_{22} = \tilde{C}_{11}$

$\tilde{C}_{33} = \dfrac{\tilde{C}_{11} - \tilde{C}_{12}}{2}$

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last time we had

Balance of linear momentum (statics)

(PDE) $\nabla\cdot\sigma + \rho b = 0$

(Eq1) $\sigma_{11,1} + \sigma_{12,2} + \rho b_1 = 0$

(Eq2) $\sigma_{12,1} + \sigma_{22,2} + \rho b_2 = 0$

need 1 more eqn

3 unknowns $\sigma_{11}, \sigma_{12}, \sigma_{22}$

$S = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$

$f_{,x} + q = 0$

$(A\sigma)_{,x} + q = 0$

Const. eqn

$S = \tilde{C}\gamma$

$\begin{bmatrix} S_{11} \\ S_{22} \\ S_{12} \end{bmatrix} = \tilde{C} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix}$

3 more eqns

" = unknowns $\gamma = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix}$

$\sigma = E\varepsilon$

we still need 3 more eqns

Compatibility Equation

$\varepsilon \approx u$

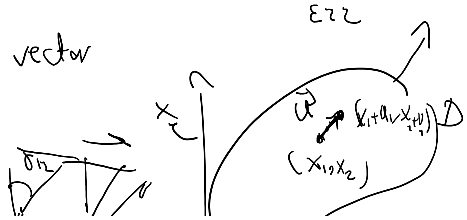
ε_{11} $\varepsilon_{12} = \varepsilon_{21}$

$$\varepsilon = \frac{(\nabla u + \nabla u^T)}{2} = \begin{bmatrix} u_{1,1} & \dfrac{u_{1,2} + u_{2,1}}{2} \\ \dfrac{u_{1,2} + u_{2,1}}{2} & u_{2,2} \end{bmatrix}$$

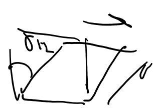
ε_{22}

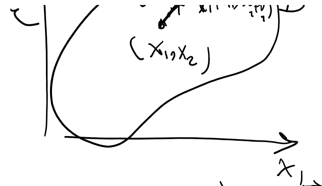
$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is the displacement vector

$\gamma = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \end{bmatrix} = \begin{bmatrix} u_{1,1} \\ u_{2,2} \end{bmatrix}$



$$\gamma = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{bmatrix} = \begin{bmatrix} u_{1,1} \\ u_{2,2} \\ u_{1,2}+u_{2,1} \end{bmatrix}$$





3 eqns added $(\epsilon_{11}, \epsilon_{22}, 2\epsilon_{12})$

2 unknowns added $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$$u_{i,j} = \frac{\partial u_i(x)}{\partial x_j}$$

BC's

$$R_i(u) = \nabla \cdot \sigma(\varepsilon(u)) + \rho b$$ must be zero for exact solution

inside

$$\bar{\bar{\sigma}} = \bar{\bar{\bar{\bar{C}}}}\,\bar{\bar{\varepsilon}}$$

$$\bar{\bar{\varepsilon}} = \frac{\nabla u + \nabla u^T}{2}$$

$$\sigma = C\left(\frac{\nabla u + \nabla u^T}{2}\right) = C\,\nabla u$$ symmetry of C

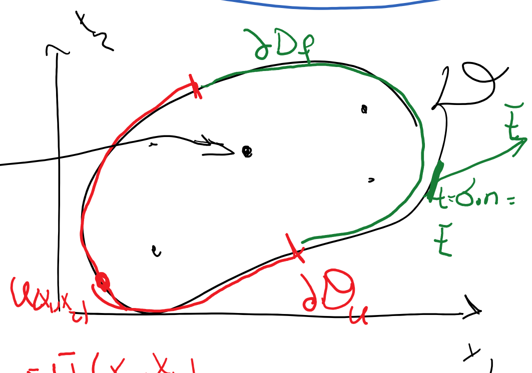
unknown is $\begin{bmatrix} u_1(x_1, x_2) \\ u_2(x_1, x_2) \end{bmatrix}$

$$\boxed{\begin{aligned} R_i(u) &= \nabla \cdot \sigma(\varepsilon(u)) + \rho b \\ &= \underbrace{\nabla \cdot}_{1}\, C\, \underbrace{\nabla u}_{1} + \rho b \quad 1+1=2 \end{aligned}}$$

inside

$$\boxed{\nabla u = \begin{bmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{bmatrix}}$$

$\forall x \in D$



2nd order PDE

$$M = 2 \rightarrow m = \frac{M}{2} = 1$$

Dirichlet BC

$\forall x \in \partial D_u \quad u(x_1, x_2) = \bar{u}(x_1, x_2) \rightarrow R_u(u) = \bar{u} - u$ order $= 0$

Dirichlet

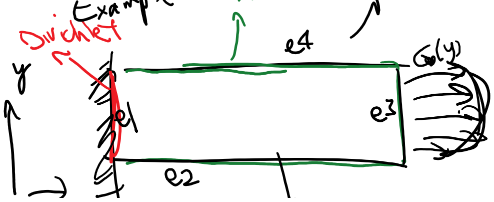
Neumann BC:

$\forall x \in \partial D_f \quad t(x_1, x_2) = \bar{t}(x_1, x_2) \quad R_f = \bar{t} - t = \bar{t} - \sigma \cdot n$ on ∂D_f

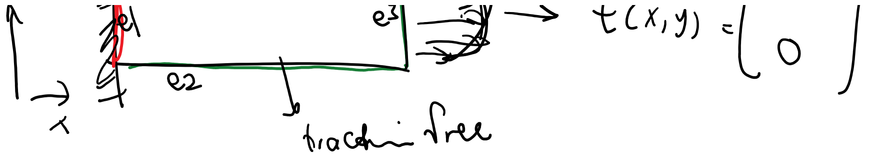
$= \bar{t} - C\,\nabla u \cdot n$

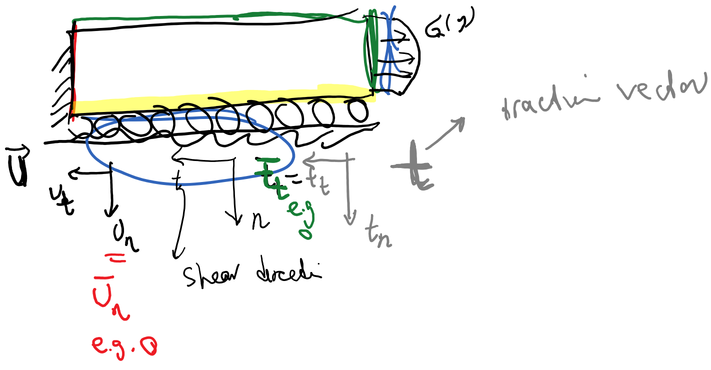
order $= 1$

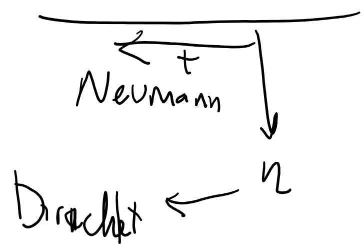
Example Neumann traction free $(\bar{t} = 0)$



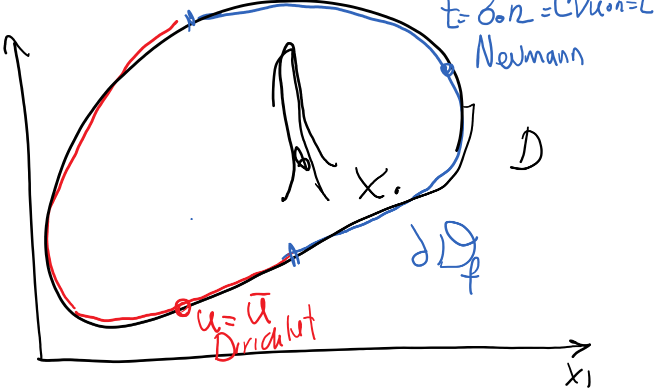
$$\rightarrow \bar{t}(x, y) = \begin{bmatrix} \sigma_0(y) \\ 0 \end{bmatrix}$$







Weighted Residual Method (WRM)



$$\begin{cases} R_i = \nabla \cdot \sigma + \rho b = \nabla \cdot C \nabla u + \rho b & \text{in } D \\ R_t = \bar{t} - t = \bar{t} - \sigma n = \bar{t} - C\nabla u \cdot n & \text{on } \partial D_t \\ R_u = \bar{u} - u & \text{on } \partial D_u \end{cases}$$

residuals

Weighted: we multiply these by weights

Find $u \ni$ such that $\forall\, \omega(x_1, x_2)$ weight function, we have

$$\text{WRS}(\omega(x)) = \underbrace{\int_D \omega(x)\, R_i(u(x))\, dV}_{} + \int_{\partial D_t} \omega(x)\, R_t(u)\, ds + \cancel{\int_{\partial D_u} \omega(x) R_u(u)\, ds} = 0$$

Generally we don't add this term

I'll show next times how we can drop this term by looking in a solution space wherein Ru(u) is strongly zero.

Going from WRS to ~~"Weak Statement"~~

$$(\star)\quad \int_{D} \underbrace{\omega(x)}\left(\underbrace{\nabla\cdot\sigma}_{\nabla\cdot C\nabla u} + \rho b \right) dV + \int_{D_f} \omega(x)\left(\bar{t} - \sigma\cdot n\right) dS + \cancel{\int \omega_f(x)\left(\bar{u} - u\right) dx}$$

order 1   order $2 = M$

WRS:

Find $u \in \mathcal{V} = \left\{ f(x_1, x_2) \,\middle|\, f \in C^2(D) \quad f(x_1,x_2) = u(x_1,x_2) \text{ on } \partial D_u \right\}$

$u,\ u_{,i},\ u_{,ij}$'s exist & are continuous

$\ni \forall\, \omega \in \mathcal{W} = \left\{ f(x_1,x_2) \,\middle|\, f \in C^0(D) \quad \right\}$

continuous function on ω

$(\star)$ is satisfied

"we'll make the space bigger later"