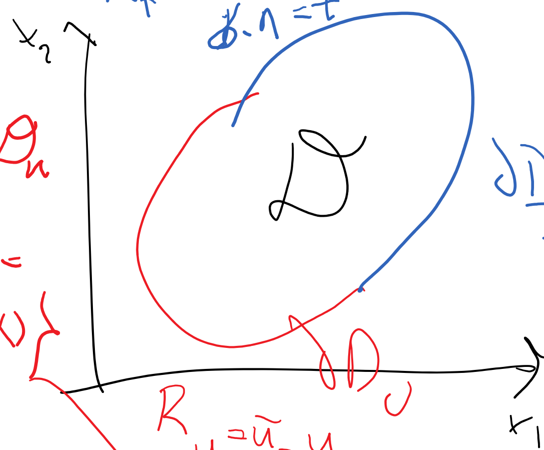
FEM2021/09/14

Tuesday, September 14, 2021 2:55 PM

WRS → " Weak Statement "

Find u Ω such that $u \in V = \{ f \in C^2(\Omega) \mid \forall x \in \partial\Omega_u \ f(\vec{x}) = \bar{u}(x) \}$



$\forall w \in W = \{ f \in C^0(\Omega) \mid \cdots \}$

$$\int_\Omega \underbrace{w(\overbrace{\nabla\cdot\sigma + \rho b}^{R_i})}\, dV + \int_{\partial\Omega_f} w(\bar{t} - \sigma\cdot n)\, dS \cancel{+ \int_{\partial\Omega_u} w_f R_u \, dS} = 0$$

$\underbrace{w\cdot\nabla\cdot\sigma}$, $\sigma = C\varepsilon(u) = C\nabla u$

$w_i \quad \sigma_{ij,j}$

$$= (w_i \sigma_{ij})_{,j} - w_{i,j}\sigma_{ij}$$

$[(fg)' = f'g + fg' \quad fg' = (fg)' - f'g]$

I redo this

$$\frac{\partial w_i \sigma_{ij}}{\partial x_j} = \frac{\partial w_i}{\partial x_j}\sigma_{ij} + \underbrace{w_i \frac{\partial \sigma_{ij}}{\partial x_j}}_{w_i \sigma_{ij,j}}$$

$$w_i \sigma_{ij,j} = (w_i \sigma_{ij})_{,j} - w_{i,j}\sigma_{ij}$$

$$\boxed{w\cdot\nabla\cdot\sigma = \nabla\cdot(w\sigma) - \nabla w : \sigma}$$

$$\int w\cdot(\nabla\cdot\sigma)\,dV = \int [\nabla\cdot(w\sigma) - \nabla w : \sigma]\, dV$$

2D $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ $\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$

$\nabla\cdot\sigma = \begin{bmatrix} \sigma_{11,1} + \sigma_{12,2} \\ \sigma_{21,1} + \sigma_{22,2} \end{bmatrix}$ $\sigma_{11,1} = \frac{\partial \sigma_{11}}{\partial x_1}$

$w\cdot(\nabla\cdot\sigma) = w_1(\sigma_{11,1} + \sigma_{12,2}) + w_2(\sigma_{21,1} + \sigma_{22,2})$

$$= \sum_{i=1}^{2}\sum_{j=1}^{2} w_i \sigma_{ij,j}$$

just drop the summation symbol when there are two repeated indices

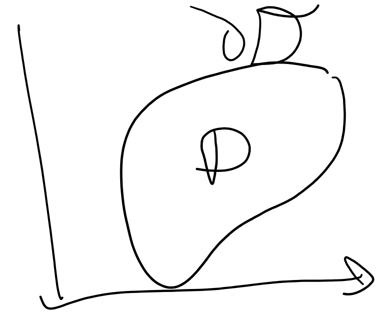
$A : B = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$

$A_{11}B_{11} + A_{12}B_{12} + A_{21}B_{21} + A_{22}B_{22} = \underbrace{\sum_{i=1}^{2}\sum_{j=1}^{2}}_{\text{don't write this}} A_{ij}B_{ij}$

$$\hookrightarrow \int_{\mathcal{D}} w\cdot(\nabla\cdot\sigma)\,dV = \int_{\mathcal{D}}\left[\nabla\cdot(w\sigma) - \nabla w : \sigma\right] dV$$

don't write this

$$= \underbrace{\int_{\mathcal{D}} \nabla\cdot(\underbrace{w\,\sigma}_{f})\,dV}_{\text{in } \star} - \int_{\mathcal{D}} (\nabla w : \sigma)\,dV$$



$$\star \quad \int_{\mathcal{D}} \nabla\cdot f\,dV = \int_{\partial\mathcal{D}} f\,n\cdot dS$$

divergence theorem

$$\Rightarrow \boxed{\begin{aligned} &\int_{\mathcal{D}} w(\nabla\cdot\sigma)\,dV = \underbrace{\int_{\partial\mathcal{D}} w\sigma\cdot n\,dS - \int_{\mathcal{D}} (\nabla w : \sigma)\,dV} \\ &\text{WRS} \\ &\int_{\mathcal{D}} w(\nabla\cdot\sigma + \rho b)\,dV + \underbrace{\int_{\partial\mathcal{D}_f} w(\bar{t} - \sigma\cdot n)\,dS} = 0 \end{aligned}}$$

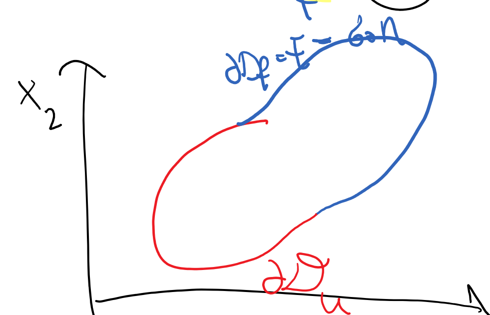
Replace $\int w\,\nabla\cdot\sigma\,dV$ with top line expression

$$\underset{(3)}{\int_{\mathcal{D}} w\rho b\,dV} + \underbrace{\underset{(1)}{\int_{\partial\mathcal{D}} w\,\sigma\cdot n\,ds} - \int_{\mathcal{D}} \nabla w : \sigma\,dV}_{(2)} + \underset{(4)}{\int_{\partial\mathcal{D}_f} w\bar{t}\,ds} - \underset{(5)}{\int_{\partial\mathcal{D}_f} w\,\sigma\cdot n\,ds} = 0$$

$$\partial\mathcal{D} = \partial\mathcal{D}_f \cup \partial\mathcal{D}_u$$

$$\underset{(3)}{\int_{\mathcal{D}} w\rho b\,dV} + \underbrace{\int_{\partial\mathcal{D}_u} w\,\sigma\cdot n\,dS + \cancel{\int_{\partial\mathcal{D}_f} w\,\sigma\cdot n\,dS}}_{\text{breakdown of } (1)} \underset{(2)}{- \int_{\mathcal{D}} \nabla w : \sigma\,dV}$$

$$+ \int_{\partial\mathcal{D}_f} w\bar{t}\,dV - \cancel{\int_{\partial\mathcal{D}_f} w\,\sigma\cdot n\,dS} = 0$$



$$\partial\mathcal{D} = \partial\mathcal{D}_u \cup \partial\mathcal{D}_f$$

$$+\int_{\partial D_f} w\,t\,ds + \int_{\partial D_p} w\,\sigma n\,ds = 0$$

(4) (5)

$$\int_D \nabla w\,\sigma\,dV = \int_{\partial D_f} w\bar{t}\,ds + \int_D w\rho b\,dV + \cancel{\int_{\partial D_u} w\,\sigma n\,ds}$$

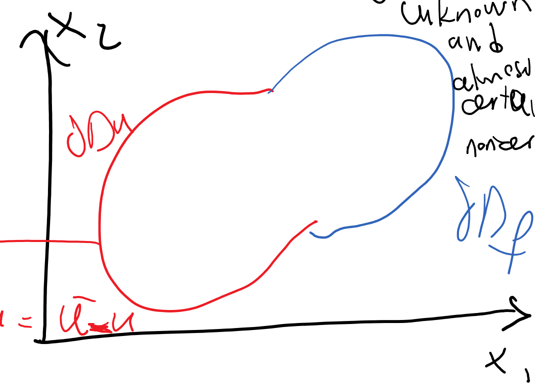
$\sigma = C\varepsilon(u) = C\left(\frac{\nabla u + \nabla u^T}{2}\right) = C\nabla u$

t: unknown and almost certainly nonzero

$\neq 0$

We make the choice that

$w = 0$ on ∂D_u



Find $u \in V = \{f \in C^1(D) \mid \forall \vec{x} \in \partial D_u\ f(\vec{x}) = \bar{u}(x)\}$ satisfy essential BC

$\ni \forall w \in W = \{f \in C^1(D) \mid \forall x \in \partial D_u\ f(x) = 0\}$

$$\int_D \underbrace{\nabla w}_{\frac{\partial w_i}{\partial x_j}}\, C\, \underbrace{\nabla u}_{\frac{\partial u_i}{\partial x_j}}\, dV = \int_{\partial D_f} w\bar{t}\,ds + \int_D w\rho b\,dV$$

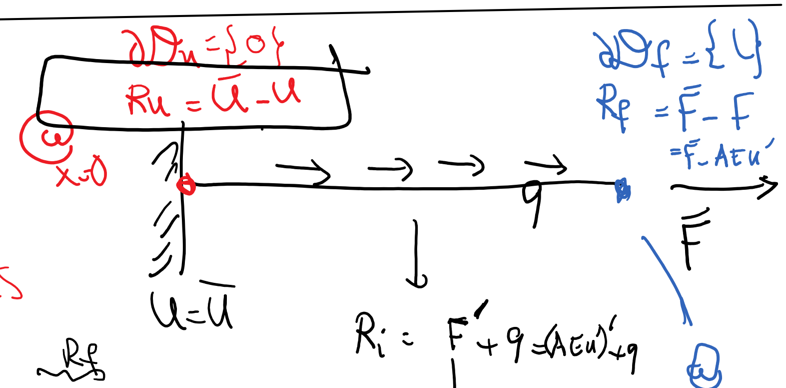
$\varepsilon(w) \cdot C\varepsilon(u)$

$\varepsilon(w) : \sigma(u)$

1D version

R_u will be exactly satisfied

→ Not added to WRS



$R_i = F' + q = (AEu')' + q$

$\frac{\partial F}{\partial x}$

$(AEu')' \to 2$

$$\int_0^L w(F' + q)\,dx + w\underbrace{(\bar{F} - F)}_{R_f}\Big|_{x=L} = 0$$

$$\int_0 \omega \underbrace{(F' + q)}_{R_i} dx + \omega(\bar{F} - F)\Big|_{\underbrace{x=L}_{\partial D_f}} = 0$$

$\frac{\partial F}{\partial x}$ $\quad x = L$

$D = [0, L]$

$F = A\sigma = AE\varepsilon = AEu'$

$F' = \underbrace{(AEu')'}_{\text{2 derivatives on } u}$

Find $u \in \mathcal{V} = \{ f \in C^2([0,L]) \mid \overbrace{f(x=0) = \bar{u}}^{\partial D_u} \}$ $\quad$ WRS

$\Rightarrow \forall \omega \in \mathcal{W} = \{ f \in C^0([0,L]) \mid \text{Nothing} \}$

$$\int_0^L \omega (\underbrace{F'}_{(AEu')'} + q)\, dx + \omega(\bar{F} - F)\Big|_{x=L} = 0$$

$$\int_0^L \omega F' dx = \omega F\Big|_0^L - \int_0^L \omega' F dx \qquad (\omega F)' = \omega' F + \omega F'$$

IBP

$$-\int_0^L \omega' F + \underbrace{\omega F\Big|_0^L}_{} + \int_0^L \omega q\, dx + \omega \bar{F}\Big|_{x=L} - \omega F\Big|_{x=L} = 0$$

$$-\int_0^L \omega' F + \cancel{\omega F\Big|_L} \underbrace{- \omega F\Big|_{x=0}}_{} + \int_0^L \omega q\, dx + \omega \bar{F}\Big|_{x=L} - \cancel{\omega F\Big|_{x=L}} = 0$$

If we require $\omega = 0$ on $x = 0$ (∂D_u) $\quad$ like $f \in G.n$ ∂D_f $(i = 1)$

then we get rid of highlighted term

$$\int_0^L \omega' \underbrace{F}_{F = AEu'} dx = \int_0 \omega q\, dx + \omega \bar{F}\Big|_{x=L}$$

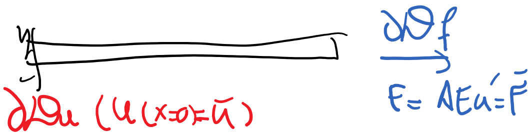
(WK)

Weak statement

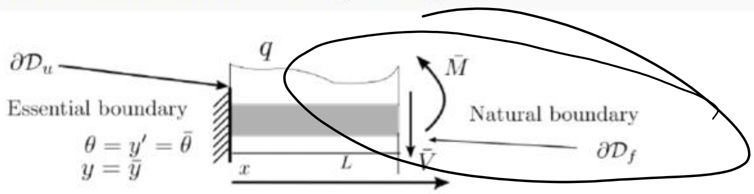
Find $u \in \mathcal{V} = \{ f \in C^1([0,L]) \mid f(x=0) = \bar{u} \}$

$\ni \forall w \in \mathcal{W} = \{ f \in C^1([\ \ ]) \mid f(x=0) = 0 \}$

$$\int_0^L w' AE u' \, dx = \int_0^L w q \, dx + w \bar{F}\big|_{x=L}$$



Weighted Residual Statement (WRS)



The weighted residual for the Euler Bernoulli problem and the boundary conditions in the figure are:

$$\text{Find } y \in \mathcal{V}^{\mathrm{WRS}} = \{ u \in C^4(\mathcal{D}) \mid u(0) = \bar{y},\ \frac{\mathrm{d}u}{\mathrm{d}x}(0) = \bar{\theta} \},\ \text{such that,} \tag{61a}$$

$$\forall w \in \mathcal{W}^{\mathrm{WRS}} = C^1(\mathcal{D}) \text{ no need to enforce the homogeneous essential BCs for WRS} \tag{61b}$$

$$\begin{aligned} 0 &= \int_{\mathcal{D}} w \mathcal{R}_i(y)\, \mathrm{d}v + \int_{\partial\mathcal{D}_f} \mathrm{w}_f \mathcal{R}_f(y)\, \mathrm{d}s \\ &= \int_0^L w \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \right) - q \right) \mathrm{d}x - \frac{\mathrm{d}w}{\mathrm{d}x}(\bar{M} - M(y))|_{x=L} + w(\bar{V} - V(y))|_{x=L} \end{aligned} \tag{61c}$$

So in this second version of the weighted residual statement, we no longer enforce essential boundary conditions weakly. The typical practice, like here, is to enforce the differential equation and natural boundary conditions weakly and the essential boundary conditions strongly.