FEM2021/09/16

Thursday, September 16, 2021 2:52 PM

Please read the derivation of the weak statement for the beam problem at home

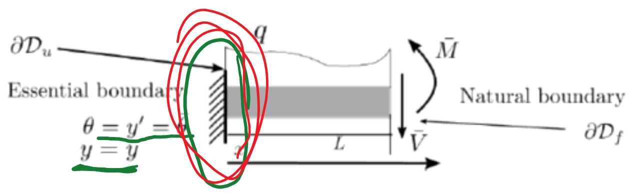
Find $y \in \mathcal{V}^{\mathrm{WRS}} = \{u \in C^4(\mathcal{D}) \mid u(0) = \bar{y},\ \frac{\mathrm{d}u}{\mathrm{d}x}(0) = \bar{\theta}\}$, such that, (61a)

$\forall w \in \mathcal{W}^{\mathrm{WRS}} = C^1(\mathcal{D})$ no need to enforce the homogeneous essential BCs for WRS (61b)

$$0 = \int_{\mathcal{D}} w \mathcal{R}_i(y)\,\mathrm{d}v + \int_{\partial\mathcal{D}_f} \mathrm{w}_f \mathcal{R}_f(y)\,\mathrm{d}s$$

$$= \int_0^L w\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) - q\right)\mathrm{d}x - \frac{\mathrm{d}w}{\mathrm{d}x}(\bar{M} - M(y))|_{x=L} + w(\bar{V} - V(y))|_{x=L} \quad (61c)$$

WRS :(



$$0 = \int_0^L \left[\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} EI \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - wq\right]\mathrm{d}x + \left\{-\frac{\mathrm{d}w}{\mathrm{d}x}\bar{M} + w\bar{V}\right\}_{x=L}$$

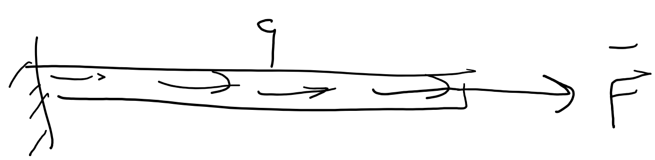
:)

$y \in \mathcal{V} = \{u \in C^2(\mathcal{D}) \mid u(0) = \bar{y},\ u' = \bar{\theta}\}$

$w \in \mathcal{W} = \{u \in C^2(\mathcal{D}) \mid u(0) = 0,\ u' = 0\}$ WK weight satisfy homogeneous Essential BC

0

Dicretization of the solution



$\partial\mathcal{D}_u$

$u(0) = \bar{u}$

$\partial\mathcal{D}_f$

$F = AEu'|_{x=L} = \bar{F}$

wk last time

$$\int_0^L w' EAu'\,dx = \int_0^L w q\,dx + Fw|_{x=L}$$

Find $u \in \mathcal{V} = \{f \in C^1 \mid f(0) = \bar{u}\}$

for $w \in \mathcal{W} = \{f \in C^1 \mid f(0) = 0\}$

let's say we want to approximate the exact solution by a 3rd order polynomial

discrete

$$u^h(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

We must satisfy essential BC

$u^h(0) = a_0 + a_1 \cdot 0 + a_2 \cdot 0^2 + a_3 0^3 = a_1 = \bar{u}$

$u^h(x) = \bar{u} + a_1 x + a_2 x^2 + a_3 x^3$

possible weight functions

$1 \quad x \quad x^2 \quad x^3 \quad \cos x \quad \sin x \quad e^x$

$1 @ x=0 \qquad 0 @ x=0$

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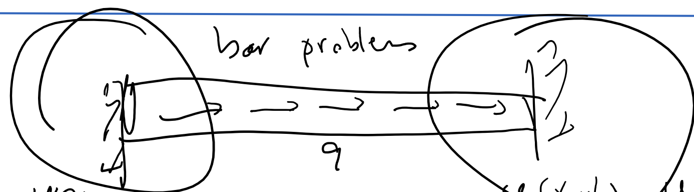
bar problem



$u(0) = u_0 \qquad u(x=L) = u_1$

$\partial\Omega_u = \{0, L\} \qquad \partial\Omega_f = \emptyset$

WRS $\int_0^L w[(AEu')' + q]\,dx = 0$

No natural BC

Find $u \in \mathcal{V} = \{f \in C^2 \mid \ni f(0) = u_0, \; f(L) = u_1\}$

IBP

WF $\int_0^L w' AE u' \,dx = \int_0^L q\,dx$

Find $u \in \mathcal{V} = \{f \in C^1 \mid \ni f(0) = u_0, \; f(1) = u_1\}$

$w \in \mathcal{W} = \{f \in C^1 \mid \ni f(0) = 0, \; f(1) = 0\}$

$u^h(x) = \phi_p(x) + \sum_{i=1}^{n} a_i \phi_i(x)$

$\phi_i(\partial\Omega_u) = 0$

$\phi_p(\partial\Omega_u) = \bar{u}(x) \;\; \forall x \in \partial\Omega_u$

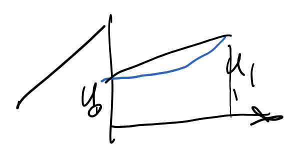
here $\partial\Omega_u = \{0, L=1\}$

$\phi_p(x=0) = u_0, \quad \phi_p(x=1) = u_1$

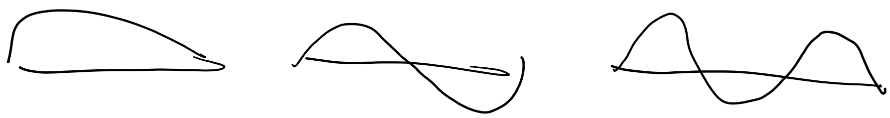
$\phi_i(x=0) = 0 \qquad \phi_i(x=1) = 0$

$\phi_p(x) = (u_1 - u_0)x + u_0$

$u_0 + (u_1 - u_0) \ldots x^2$



$\sin x\pi, \; \sin 2\pi x, \; \sin 3\pi x$ all good ϕ_i's



$u^h(x) = \phi_p(x) + \sum_{i=1}^{n} a_i \, \phi_i(x)$ n # unknowns a's

$u^h(x=0) = \underbrace{\phi_p(x=0)}_{u_0} + \sum a_i \, \underbrace{\phi_i(x=0)}_{0} = u_0$

$u^h(x=1) = \underbrace{\phi_p(x=1)}_{u_1} + \sum a_i \, \underbrace{\phi_i(x=1)}_{0} = u_1$

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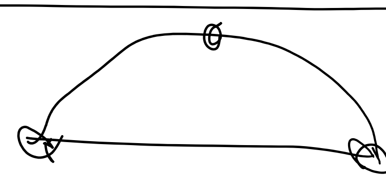
as we see ϕ_i's are excellent choices for weight functions too in (WK) [they are always zero there]

$\phi_1(x) = \cancel{b_0} + b_1 x + b_2 x^2$

$\phi_1(0) = 0$, $\phi_1(1) = 0 \longrightarrow \phi_1(x) = x(1-x)$



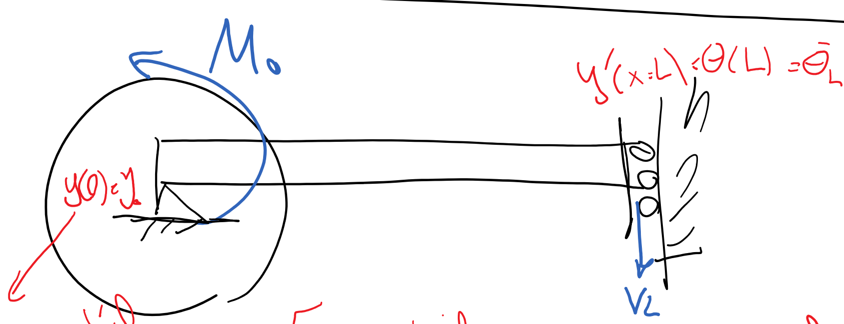
$\phi_1(0) = \phi_1(1) = 0$

Example 1

Essential y, $y' = \theta$

Natural $M = EIy''$, $V = M' = (EIy'')'$

pair



[Essential BC $y(0) = y_0$ & $y'(x=L) = \theta_L$]

Natural BC $EIy''(0) = M(x=0) = M_0$

$(EIy'')'(x=L) = V(x=L) = V_L$

$y^h(x) = \phi_p(x) + \sum a_i \, \phi_i(x)$

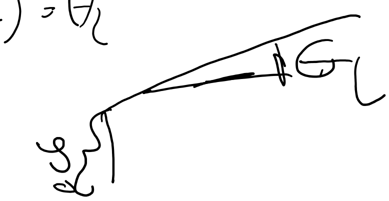
conditions on ϕ_p : satisfies all essential BC's

$\phi_p(0) = y_0$, $\phi_p'(x=L) = \theta_L$

$$\phi_p(0) = y_0, \quad \phi_p'(x=L) = \theta_L$$

$$\phi_p(x) = y_0 + x\,\theta_L$$

$$\phi_p'(x) = \theta_L$$



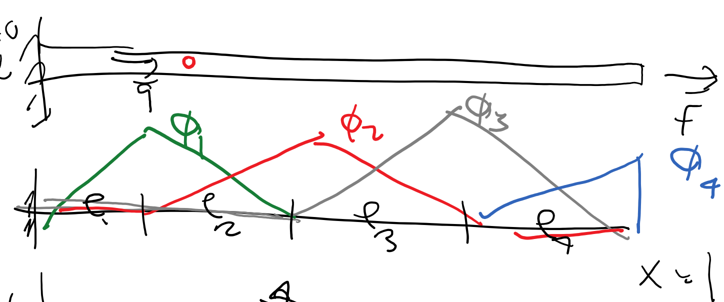
ϕ_i must satisfy $\quad \phi_i(0) = 0 \qquad \phi_i'(L) = 0$

Why we cannot use FEMs in the weighted residual form

$$\int_0^L w\underbrace{\left((EAu')' + q\right)}_{R_i} dx + \underbrace{w(F - F)\big|_{x=L}}_{R_f} = 0$$

u'' is needed

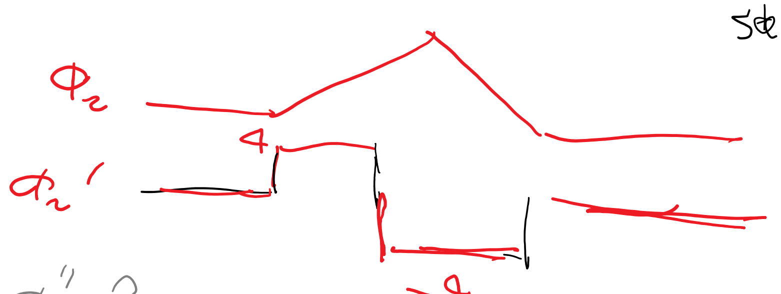
$u(0) = \bar{u}$



$$u^h = \phi_p(x) + \sum_{i=1}^{4} a_i\,\phi_i(x) \qquad \phi_p(x) = \bar{u}$$

$\phi_i(x=0) = 0$:)

satisfy the essential BC



ϕ'' ?

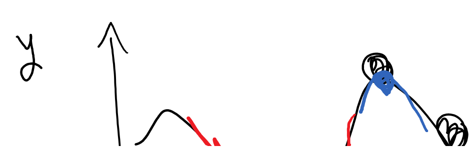
Cannot use WRS

$$\int_0^L w' EA u'\,dx = \int_0^L w\,q\,dx \qquad (WF)$$

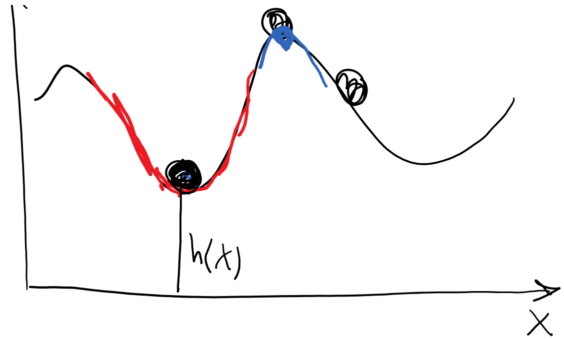
FEM works for weak statement

Energy Methods

$$\Pi(x) = mgh(x)$$



$\Pi(x) = mgh(x)$



Equilibrium

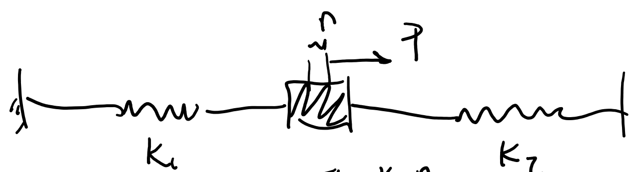
$$\frac{d\,\Pi(x)}{dx} = 0$$

energy is an extremum

It's stability is determined by higher order derivatives

$$\begin{cases} \dfrac{\partial^2 \Pi}{\partial x^2} > 0 & \text{stable} \\ \dfrac{\partial^2 \Pi}{\partial x^2} < 0 & \text{unstable} \end{cases}$$

Example 1



$$\sum F_x = 0 : \; -k_1 r - k_2 r + P = 0$$

$$\boxed{r = \frac{P}{k_1 + k_2}}$$

Energy Statement

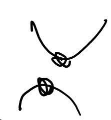
$$\Pi = \underbrace{V}_{\text{internal energy}} - \underbrace{W}_{\text{external work}}$$

$$V = \frac{1}{2} k_1 (r)^2 + \frac{1}{2} k_2 (-r)^2$$

$$W = Pr$$

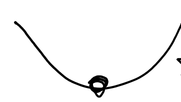
$$\boxed{\Pi(r) = \frac{1}{2}(k_1 + k_2) r^2 - Pr}$$

$$\frac{d\,\Pi(r)}{dr} = (k_1 + k_2) r - P$$



if $0 \rightarrow \boxed{r = \dfrac{P}{k_1 + k_2}}$

$$\frac{d^2 \Pi}{dr^2} = k_1 + k_2 > 0$$



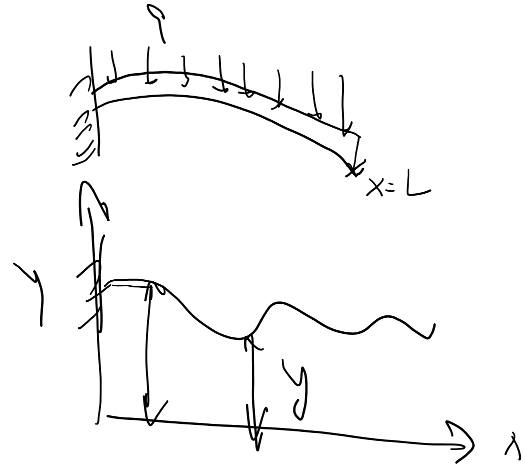
stable

Matches the force approach

Energy statement for a beam

$$\Pi(y) = V - W$$

$$= \frac{1}{2}\int_0^L EI\, y''^2\, dx - \int_0^L q\, y\, dx$$



This is called a functional

a function whose argument is a function & returns a real value