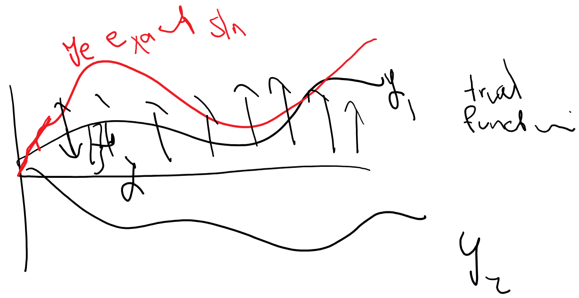
Why we need to exactly satisfy essential BC when using energy methods

$$\Pi(y_e) \leq \Pi(y)$$

y is a trial solution

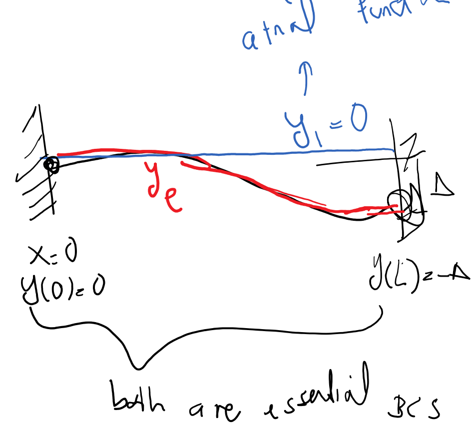


$$\Pi(y) = \underbrace{\int_0^L \frac{1}{2} EI y''^2 \, dx}_{V_{\text{internal energy}}} - \underbrace{\int y q \, dx}_{W_{\text{external work}}}$$

from here a trial solution should have 2 derivatives

Example $q = 0$

$$\Pi(y_1) = \int_0^L \frac{1}{2} EI y_1''^2 - \int y_1 \overset{0}{q} \, dx$$
$$= \overset{0}{0}$$



Exact solution

$$\Pi(y_e) = \int \frac{1}{2} EI y_e''^2 \, dx > 0$$

$$\Pi(y_e) > \Pi(y_1)$$

$\Pi(y_e)$ → exact solution

whereas we should have had

$$\Pi(y_e) \leq \Pi(y) \quad \forall y$$

For energy method, we need to exactly satisfy all essential BCs. Unlike different forms of WRS, here we have not choice. We must satisfy them for trial functions.

-> "Essential"

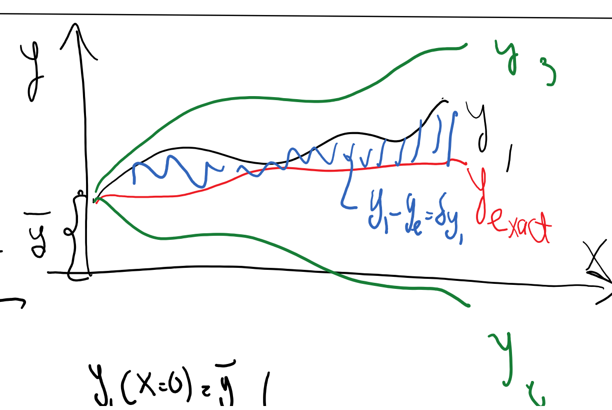
Let's look at energy statement again:

Essential BC $y(0) = \bar{y}$

Find y_e such that $\forall y \in V = \{ f \in C^2(\Omega) \mid f(0) = \bar{y} \}$

$$\Pi(y_e) \leq \Pi(y)$$

where



$\Pi(y_e) \leq \Pi(\tilde{y})$

where

$$\Pi(y) = \int_0^L \frac{1}{2} EI y''^2 \, dx - \int_0^L q y \, dx$$

$y_1(x=0) = \bar{y}$

$y_e(x=0) = \bar{y}$

$\delta y_1(0) = (y_1 - y_e)(0) = \bar{y} - \bar{y} = 0$

y_1

another way to write (*) is by looking at the increment of y_e

So the increment of a trial solution w.r.t. the exact solution is zero at all essential BCs (similar to the weight function condition in weak statement)

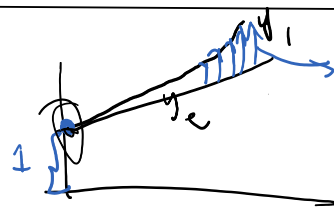
Find y_e such that for all increments $\delta y \in W = \{ f \in C^2(\Omega) \mid f(0) = 0 \}$

$$\Pi(y_e) \leq \Pi(\underbrace{y_e + \delta y}_{\tilde{y}})$$

$\bar{y} = 1$

$y_e = 1 + x$

$y_1 = 1 + 2x^2$



$\delta y_1 = y_1 - y_e = 2x^2 - x$

$\delta y_1(x=0) = 0$

Energy Method for Solid Mechanics

The total energy in solid mechanics is,

$$\Pi = (V - W) - T = \text{Total energy} \quad (85a)$$

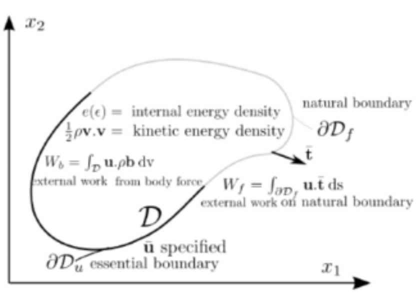
$$T = \int_{\mathcal{D}} \frac{1}{2} \rho \mathbf{v}.\mathbf{v} \, dv = \text{Kinetic energy} \quad (85b)$$

$$V = \int_{\mathcal{D}} e(\epsilon) \, dv = \text{Internal energy} \quad (85c)$$

$$W = W_b + W_f = \text{External work} \quad (85d)$$

$$W_b = \int_{\mathcal{D}} \mathbf{u}.\rho \mathbf{b} \, dv \quad (85e)$$

$$W_f = \int_{\partial \mathcal{D}_f} \mathbf{u}.\bar{\mathbf{t}} \, ds \quad (85f)$$



- For static problems $T = 0$.
- Internal energy density, $e(\epsilon) = \frac{1}{2}\epsilon : \sigma(\epsilon) = \frac{1}{2} C_{ijkl} \epsilon_{ij} \epsilon_{kl}$ for linear solid.
- Natural boundary forces are naturally incorporated into the energy (W_f).
- Essential boundary conditions are incorporated into function space:

$$\mathbf{u} \in \mathcal{V} = \{ \mathbf{v} \mid \mathbf{v} \in C^1(\mathcal{D}) : \forall \mathbf{x} \in \partial\mathcal{D}_u \ \mathbf{v}(\mathbf{x}) = \bar{\mathbf{u}}(\mathbf{x}) \}, \text{ is a solution if}$$
$$\forall \tilde{\mathbf{u}} \in \mathcal{V}, \quad \Pi(\mathbf{u}) \leq \Pi(\tilde{\mathbf{u}}). \quad (86)$$

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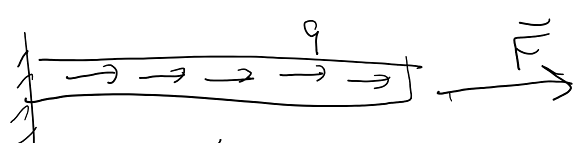
Derive the energy statement for a bar problem:

$$V = \int \underbrace{e(\epsilon)}_{\text{internal energy per unit volume}} dV$$

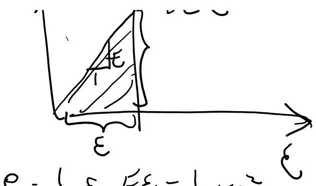


1D

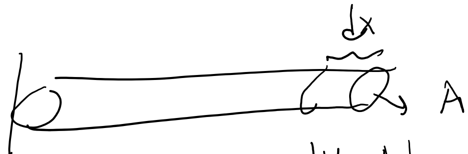
1D linear elasticity

1D

1D linear elasticity



$e = \frac{1}{2}\varepsilon \cdot E\varepsilon = \frac{1}{2}E\varepsilon^2$



$dV = A\,dx$

$e(\varepsilon) = \frac{1}{2}E\varepsilon^2 , \quad \varepsilon = \frac{du}{dx}$

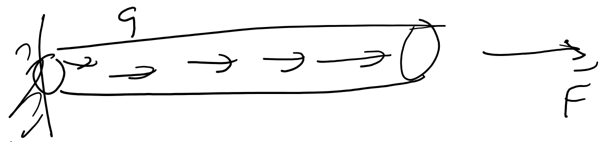
$\Psi = \int_{\Omega} e(\varepsilon)\,dV = \int_0^L \underbrace{\frac{1}{2}Eu'^2}_{e(\varepsilon)} \cdot \underbrace{A\,dx}_{dV} = \boxed{\int \frac{1}{2}EA\,u'^2\,dx}$ (1)

internal energy

volume

External work

$W = \underbrace{W_b}_{\text{bulk}} + W_f$ (2)



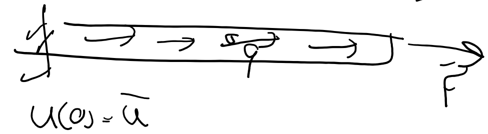
force per unit length

$W_b = \int_{\Omega} u \cdot dF_v = \int_0^L u\,(q\,dx) = \int u q\,dx$ (3)

$W_f = u\,F\big|_{x=L}$ (4)

1 → 4

$$\boxed{\Pi(u) = \int_0^L \frac{1}{2}EA\,u'^2\,dx - \int_0^L q\,u\,dx - u\,F\big|_{x=L}}$$



$u(0) = \bar{u}$

Let's find the exact solution for this problem

Assume u is the exact solution

$$\Pi(u) \leq \Pi(u+\delta u) \quad \forall \delta u \in \mathcal{W} = \{ f \in C^1([0,L]) \mid f(0) = 0 \}$$

$\Downarrow$

$\Pi(u+\delta u) - \Pi(u) \geq 0$ we must have this condition

$\Pi(u+\delta u) - \Pi(u) \geq 0$ we must have this condition

$$\Pi(u+\delta u) - \Pi(u) = \int_0^L \frac{1}{2} EA[(u+\delta u)']^2 - \int_0^L (u+\delta u)\, q\, dx - (u+\delta u)\Big|_{x=L} \bar{F}$$

$$- \left\{ \int \frac{1}{2} EA\, u'^2 - \int_0^L u\, q\, dx - u\Big|_{x=L} \bar{F} \right\}$$

$$= \cancel{\int \frac{1}{2} EA u'^2 dx} + \int_0^L \frac{1}{2} EA(2u'\delta u')dx + \int \frac{1}{2} EA(\delta u')^2 dx - \cancel{\int u q dx} - \int q \delta u dx - \cancel{u\Big|_{x=L} \bar{F}} - \delta u\Big|_{x=L} \bar{F}$$

$$\cancel{- \int \frac{1}{2} EA u'^2 dx} \qquad \cancel{+ \int u q dx} \qquad \cancel{+ u \bar{F}\Big|_{x=L}}$$

$\Rightarrow$

Note this for ⊛

$(u+\delta u)' = u' + (\delta u)'$

$((u+\delta u)')^2 = u'^2 + 2u'(\delta u)' + (\delta u')^2$

$$\Delta\Pi = \Pi(u+\delta u) - \Pi(u) = \underbrace{\int_0^L \delta u' EA\, u'\, dx - \int_0^L q\, \delta u\, dx - \delta u\Big|_{x=L} \bar{F}}_{\delta\Pi \text{ (1st order in } \delta u)} + \underbrace{\int_0^L \frac{1}{2} EA[(\delta u)']^2}_{\delta^2 \Pi} \geq 0$$

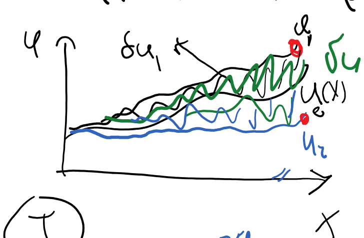
1st order in δu

$\delta\Pi = 0$ necessary

2nd order in increment

$\delta^2\Pi$ if, in addition, $\delta^2\Pi > 0$ we have a local minimum

(I)



$2u_e - u_1$

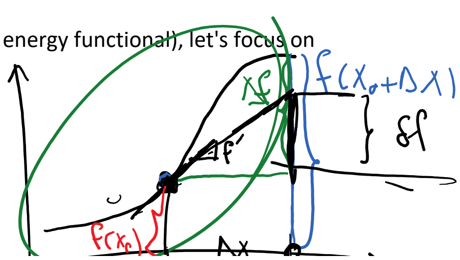
$\boxed{\delta u_2 = -\delta u_1}$

Let's discuss what we can conclude from the inequality in equation (I)

Instead of functionals (a scalar function whose argument is a function as in the case of energy functional), let's focus on functions from real numbers to real numbers

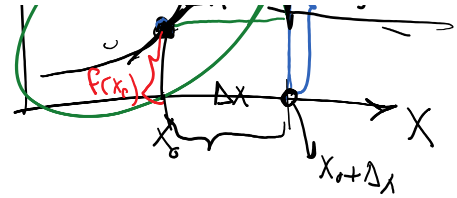
Δf total variation $= f(x_0 + \Delta x) - f(x_0)$ Taylor's expansion

$f(x_0) + \Delta x\, f'(x_0) + \frac{1}{2}\Delta x^2 f''(x_0) \ldots$



total variation $= f(x_0+\Delta x) - f(x_0)$

$f(x_0) + \Delta x\, f'(x_0) + \frac{1}{2}\Delta x^2 f''(x_0) + \dots$
$- f(x_0)$



$$\Delta f = \underbrace{\cancel{f'(x_0)\Delta x}}_{\substack{\text{first variation} \\ \delta f}} + \underbrace{\frac{1}{2} f''(x_0)\Delta x^2}_{\substack{\delta^2 f \\ \text{2nd variation}}} + \cancel{\frac{1}{6} f'''(x_0)\Delta x^3} \dots$$

if $\Delta x \to 0$ we can ignore $\delta^2 f, \delta^3 f, \dots$ relative to δf

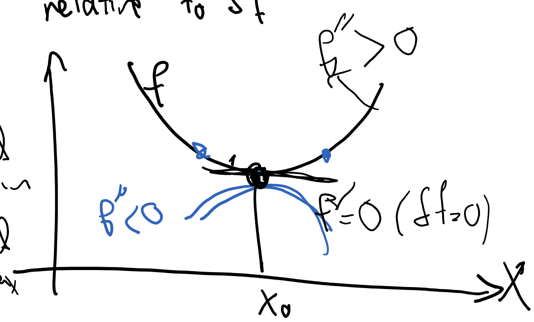
if f has a minimum @ x_0 what can we say about $\boxed{\delta f = 0}$, $\delta^2 f$

necessary condition for min/max

$\delta^2 f > 0$ local min

< 0 local max

$= 0$ ... need to go to H.O.T



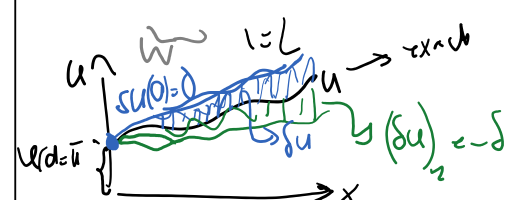
$\boxed{\text{if we have } \delta f = 0 \;\&\; \delta^2 f > 0 \text{ we have a local minimum}}$

$$\Delta\Pi = \Pi(u+\delta u) - \Pi(u) = \underbrace{\int_0^L \delta u'\, EA\, u'\, dx - \int_0^L q\, \delta u\, dx - \delta u\big|_{x=L} F}_{\text{1st order in } \delta u \quad \delta\Pi = 0 \text{ necessary}}$$

$$+ \underbrace{\int_0^L \frac{1}{2} EA\left[(\delta u)'\right]^2}_{\delta^2\Pi} \geq 0$$

2nd order in increment

if, $\delta^2 \Pi > 0$ we have



$\delta^2\Pi$ in addition, if $\delta^2\Pi > 0$ we have a local

For o to be the exact sln, we should have

$$\Pi(u) \leq \Pi(u+\delta u) \rightarrow \delta\Pi = \Pi(u+\delta u) - \Pi(u) \geq 0$$

$$\Longrightarrow \delta\Pi = \int_0^L (\delta u)' EA u' \, dx - \int_0^L \delta u \cdot q \, dx - \delta u|_{x=L} \bar{F} = 0$$

$$\delta^2\Pi = \int_0^L \underbrace{\tfrac{1}{2} EA (\delta u')^2}_{\delta u' \neq 0} > 0$$

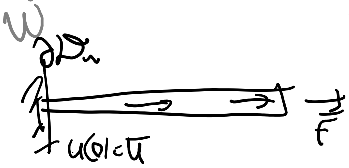
for exact sln all we need to do is to satisfy

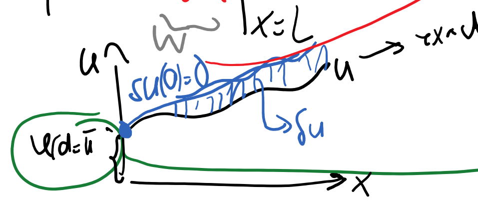
$\delta\Pi = 0$

Find $u \in \mathcal{V} = \{ f \in C^1 \mid f(0) = \bar{u} \}$

$\ni \forall \, \delta u \in \mathcal{W} = \{ f \in C^1 \mid f(0) = 0 \}$

$$\delta\Pi = 0 \rightarrow \int_0^L \underbrace{(\delta u)'}_{\mathcal{W}} EA u' \, dx = \int_0^L \underbrace{\delta u}_{\mathcal{W}} q + \underbrace{\delta u}_{\mathcal{W}}\Big|_{x=L} \bar{F}$$





Weak Statement :)