FEM2021/09/21

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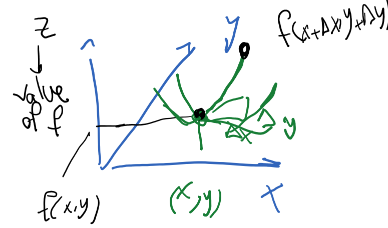
An easier way to calculate first variation of a functional

Motivation consider a function from R2 -> R (a function of two parameters)



$$f(x+\Delta x,\, y+\Delta y) = f(x,y) + \underbrace{\left[f_{,x}(x,y)\,\Delta x + f_{,y}(x,y)\,\Delta y \right]}_{\delta f \text{ first variation}}$$

$$+ \underbrace{\tfrac{1}{2} f_{,xx}(x,y)\,\Delta x^2 + f_{,xy}(x,y)\,\Delta x \Delta y + \tfrac{1}{2} f_{,yy}\,\Delta y^2}_{\text{2nd variation}} + \cdots$$

for minimum or max @ (x,y)

$$\boxed{\delta f = 0}$$

$$\equiv \nabla f = \begin{bmatrix} f_{,x} \\ f_{,y} \end{bmatrix} = 0$$

Hessian matrix

$$\tfrac{1}{2} \begin{bmatrix} \Delta x & \Delta y \end{bmatrix} \begin{bmatrix} f_{,xx} & f_{,xy} \\ f_{,xy} & f_{,yy} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

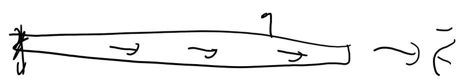
$$f_{,x} = \frac{\partial f}{\partial x}, \quad f_{,y} = \frac{\partial f}{\partial y}$$

$$* \quad \boxed{\delta f = \underbrace{f_{,x}}_{\frac{\partial f}{\partial x}} \Delta x + \underbrace{f_{,y}}_{\frac{\partial f}{\partial y}} \Delta y = 0 \quad \text{at an extremum point}}$$

Back to functionals

$$\Pi(u, u') = \frac{1}{2} \int_0^L EA\, u'^2\, dx - \int_0^L q\, u\, dx - \bar{F} u \Big|_{x=L}$$

in your mind treat this (u) as x in $*$; (u') as y



U(x) is a trial function

$$\delta^1 \Pi = \frac{\partial \Pi}{\partial u} \delta u + \frac{\partial \Pi}{\partial u'} \delta(u')$$

$$\delta^1 \Pi = \int_0^L \left[\cancel{\frac{\partial \frac{1}{2} EA u'^2}{\partial u}}\, \delta u + \frac{\partial \frac{1}{2} EA u'^2}{\partial u'}\, \delta(u') \right] dx$$

$$\text{Similar to asking} \quad \frac{\partial \frac{1}{2} EA y^2}{\partial y} = \tfrac{1}{2} EA\, 2y\, \delta y$$

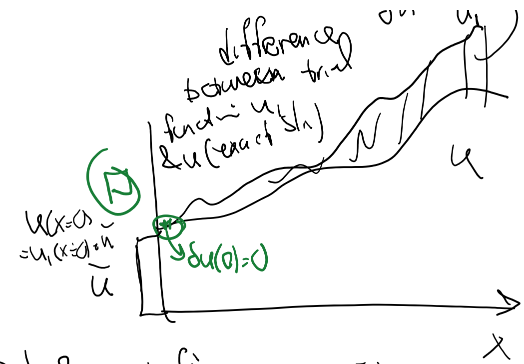
$y \rightarrow u'$, $\delta y \rightarrow \delta(u')$

$$- \int_0^L q\, \underbrace{\frac{\partial u}{\partial u}}_{1}\, \delta u\, dx - \bar{F} \frac{\partial u}{\partial u} \delta u \Big|_{x=L}$$

difference between trial

$\delta u \qquad \delta u \; |_{x=L}$

$$\boxed{\delta\Pi = \int_0^L \underbrace{\delta u'}_{(\delta u)'} EA\, u'\, dx - \int_0^L q\, \delta u\, dx - \bar{F}\, \delta u \Big|_{x=L}}$$



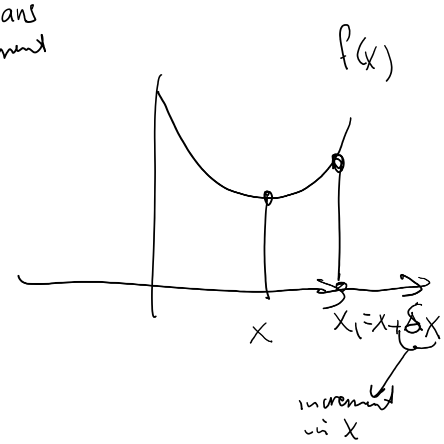
if u_1 is a trial function (in this case it's C^1 & satisfies $u_1(x=0) = \bar{u}$)

$$\delta u = \underbrace{u_1}_{\text{trial function}} - \underbrace{u}_{\text{exact solution}}$$

δ means increment

take derivative

$$\underbrace{(\delta u)'}_{\text{derivative of increment in } u} = u_1' - u' = \underbrace{\delta(u')}$$



Basically we have

$$\bigstar \int_0^L (\delta u)'\, EA\, u'\, dx = \int_0^L \delta u\, q\, dx + \bar{F}\, \delta u \Big|_{x=0}$$

Find $u \in \mathcal{V} = \{ f \in C^1([0,L]) \mid f(0) = \bar{u} \}$

$\ni \forall\ \delta u \in \mathcal{W} = \{ f \in C^1([0,L]) \mid f(0) = 0 \}$ see figure $\square$

This is our weak statement ($\delta u = w$)

Functionals Optimality condition

- For a function $f : \mathbb{R} \to \mathbb{R}$ we observed that a necessary condition for optimality of f at x_0 was, $\delta f = \frac{\mathrm{d}f}{\mathrm{d}x}(x_0)\Delta x = 0$.
- What do we expect a necessary optimality condition for a functional Π be?
  a necessary extremum condition for Π at y is

$$\delta \Pi = 0, \text{ where } \delta\Pi \text{ is a shorthand for } \delta\Pi(y, \delta y) \tag{92}$$

- How to evaluate $\delta\Pi$?

$$\Pi = \Pi(y, \frac{\mathrm{d}y}{\mathrm{d}x}, \dots, \frac{\mathrm{d}^n y}{\mathrm{d}x^n}) \quad \Rightarrow \quad \delta\Pi = \frac{\partial \Pi}{\partial y}\delta y + \frac{\partial \Pi}{\partial \frac{\mathrm{d}y}{\mathrm{d}x}}\delta(\frac{\mathrm{d}y}{\mathrm{d}x}) + \dots + \frac{\partial^n \Pi}{\partial \frac{\mathrm{d}^n y}{\mathrm{d}x^n}}\delta(\frac{\mathrm{d}^n y}{\mathrm{d}x^n}) \tag{93}$$

Note the similarity to the corresponding conditions for a function $f(x)$: $\delta f = \frac{\mathrm{d}f}{\mathrm{d}x}\Delta x = 0$.

$$\text{Having a } \delta y \Rightarrow \tilde{y} = y + \delta y \Rightarrow \frac{\mathrm{d}^n \tilde{y}}{\mathrm{d}x^n} = \frac{\mathrm{d}^n y}{\mathrm{d}x^n} + \frac{\mathrm{d}^n \delta y}{\mathrm{d}x^n} \Rightarrow \boxed{\delta(\frac{\mathrm{d}^n y}{\mathrm{d}x^n}) = (\frac{\mathrm{d}^n \delta y}{\mathrm{d}x^n}) := \delta y^{(n)}} \tag{94}$$

Thus, noting that $y^{(n)} := \frac{\mathrm{d}^n y}{\mathrm{d}x^n}$, (93) can be rewritten as,

$$\Pi = \Pi(y, y', \dots, y^{(n)}) \quad \Rightarrow \quad \boxed{\delta\Pi = \frac{\partial \Pi}{\partial y}\delta y + \frac{\partial \Pi}{\partial y'}\delta y' + \dots + \frac{\partial^n \Pi}{\partial y^{(n)}}\delta y^{(n)}.} \tag{95}$$

$f(x, y, z, \dots)$

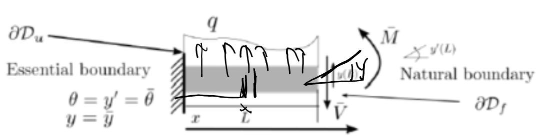
$\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \dots$

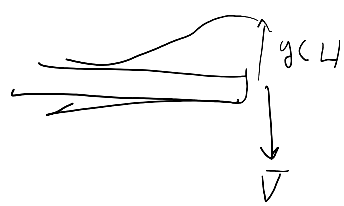
You can refer to the links on the course website that prove eqn (95) above and the optimality condition for functional

1. Useful links for energy method (not necessary to apply energy approach in the derivation of weak statement) – link Functional optimization: How an equation for first variation of a functional (e.g. equations 93, 95 on slide 78) can be derived. You clearly do not need to read this document for this course and this is only provided as a related material for students that want to understand the logic behind the derivation of equations 93, 95. – link Exact calculation of total, first, and second variations for a simple example: In this document the total variation of the energy functional for the bar problem is directly calculated. The first and second variations are directly obtained and higher variations are zero for this simple functional. It is observed that the first variation is exactly the same as what we would have obtained by equation 96 on slide 78.

From <http://rezaabedi.com/teaching/me-517-finite-elements/>

Example: Euler Bernoulli beam





We determined the internal energy of the beam to be (*cf.* (85c)),

$$V = \int_{\mathcal{D}} \frac{1}{2}\epsilon\sigma \,\mathrm{d}v = \int_0^L \left(\int_A \frac{1}{2}\epsilon^2 E \,\mathrm{d}A\right)\mathrm{d}x = \int_0^L \left(\int_A \frac{1}{2}(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}z)^2 E \,\mathrm{d}A\right)\mathrm{d}x$$

$$= \int_0^L \frac{1}{2}E(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2})^2 (\underbrace{\int_A z^2 \,\mathrm{d}A}_{I})\,\mathrm{d}x \Rightarrow$$

potential energy

internal energy $\leftarrow$ $\boxed{V = \int_0^L \frac{1}{2}EI(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2})^2 \,\mathrm{d}x}$ (100)

external work

$$\Pi = V - W = \int_0^L \frac{1}{2}EI\,y''^2 \,dx - \left(\bar{M}\,y'|_L - \bar{V}\,y|_{x=L} + \int_0^L y(x)\,q\,dx\right)$$

$\Pi(y, y', y'')$

$$\delta\Pi = \int_0^L \underbrace{\frac{\partial \frac{1}{2}EIy''^2}{\partial y''}}_{EI\,y''}\underbrace{\delta(y'')}_{(\delta y)''}dx - \left[\bar{M}\,\delta(y')|_L - \bar{V}\,\delta y|_{x=L} + \int_0^L \delta y(x)\,q\,dx\right] = 0$$

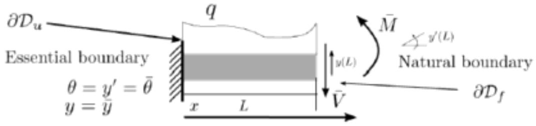
$EI y''$ $(\delta y)''$ $\int_0^L \delta y(x) q \, dx \Big] = 0$

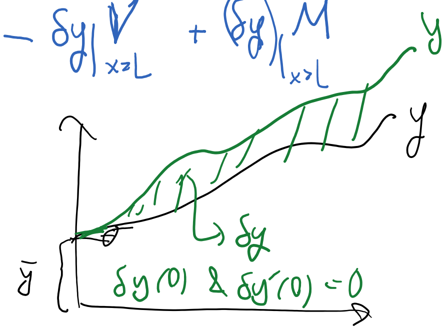
So

Find $y \in \mathcal{V} = \{ f \in C^2([0,L]) \mid f(0) = \bar{y},\ f'(0) = \bar{\theta} \}$

$\ni$ (such that) $\delta y \in \mathcal{W} = \{ f \in C^2([0,L]) \mid f(0) = \underline{0}\ \ f'(0) = 0 \}$

$$\int \delta y'' EI y'' \, dx = \int_0^L \delta y \, q \, dx - \delta y \big|_{x=L} \bar{V} + (\delta y)' \big|_{x=L} \bar{M}$$

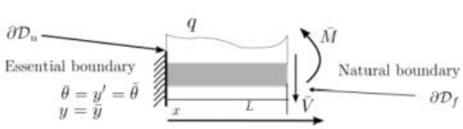




$y_1'(0) = \bar{\theta}$
$y'(0) = \bar{\theta}$ $\rightarrow (y_1' - y')(0) = 0$

$(\delta y)'(0) = 0$

Weak Statement (WS)



The weak statement for the Euler Bernoulli problem and the BCs in the figure are:

$$\text{Find } y \in \mathcal{V} = \{ u \in C^2(\mathcal{D}) \mid u(0) = \bar{y},\ \frac{\mathrm{d}u}{\mathrm{d}x}(0) = \bar{\theta} \}, \text{ such that,} \tag{62a}$$

$$\forall w \in \mathcal{W} = \{ u \in C^2(\mathcal{D}) \mid u(0) = 0,\ \frac{\mathrm{d}u}{\mathrm{d}x}(0) = 0 \} \tag{62b}$$

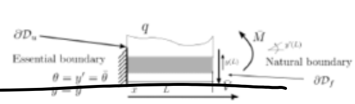
$$0 = \int_0^L \left[\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} EI \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - wq \right] \mathrm{d}x + \left\{ -\frac{\mathrm{d}w}{\mathrm{d}x} \bar{M} + w \bar{V} \right\}_{x=L} \tag{62c}$$

Two more points:

Q1) we used PDE & natural BCs before, multiplied them by weights and after IBP obtained the weak statement. Energy method, directly gives us the weak statement. Can we obtain the PDE and natural BCs from energy methods?

Energy method to Strong Form and Boundary Conditions

We realized the convenience of energy methods in deriving the weak form in one step. They can also be used to strong form and boundary conditions by the common approach of integration by part (divergence theorem in $D > 1$).



The weak form from (107) is:

$$\boxed{\delta \Pi = \int_0^L \delta y''(x) EI y''(x) \, \mathrm{d}x - \int_0^L \delta y(x) q(x) \, \mathrm{d}x + \delta y(L) \bar{V} - \delta y'(L) \bar{M} = 0}$$

Two consecutive integration by parts yield:

$$\int_0^L -\frac{\mathrm{d}\delta y}{\mathrm{d}x} \frac{\mathrm{d}}{\mathrm{d}x}\left(EI \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) \mathrm{d}x - \int_0^L \delta y(x) q(x) \, \mathrm{d}x + \delta y(L) \bar{V} - \delta y'(L) \bar{M} + \delta y' \left(EI \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) \Big|_0^L = 0 \Rightarrow$$

$$\int_0^L \delta y \left(\frac{\mathrm{d}^2 EI}{\mathrm{d}x^2} \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - q \right) \mathrm{d}x - \delta y'(L) \left(\bar{M} - EI \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}(L) \right) - \delta y'(0) EI \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}(0)$$

2 IBPs

$$\int_0^L -\frac{\mathrm{d}\delta y}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}(EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2})\,\mathrm{d}x - \int_0^L \delta y(x)q(x)\,\mathrm{d}x + \delta y(L)\bar{V} - \delta y'(L)\bar{M} + \delta y'(EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2})\Big|_0^L = 0 \Rightarrow$$

$$\int_0^L \delta y\left(\frac{\mathrm{d}^2 EI}{\mathrm{d}x^2}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - q\right)\mathrm{d}x - \delta y'(L)\left(\bar{M} - EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}(L)\right) - \delta y'(0)EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}(0)$$

$$+ \delta y(L)\bar{V} - \{\delta y\frac{\mathrm{d}}{\mathrm{d}x}(EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2})\}\Big|_0^L = 0$$

$$\int_0^L \delta y\left(\boxed{\frac{\mathrm{d}^2 EI}{\mathrm{d}x^2}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - q}\right)\mathrm{d}x - \delta y'(L)\left(\bar{M} - EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}(L)\right) + \delta y(L)\left(\bar{V} - \frac{\mathrm{d}}{\mathrm{d}x}(EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2})\right)$$

$$+ \delta y(0)\frac{\mathrm{d}}{\mathrm{d}x}(EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2})(0) = 0$$

DE

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Natural BCs

BCS

Energy Method vs. Principle of Virtual Work

- Principle of virtual work or virtual displacement in solid mechanics states that if $\mathbf{u}$ is the solution to a boundary value problem, the virtual internal and external works produces by admissible virtual displacements $\delta\mathbf{u}$ are equal.
- Virtual displacements $\delta\mathbf{u}$ refer to displacements that are zero at essential boundary values (so that solution displacement plus virtual displacement $\tilde{\mathbf{u}} = \mathbf{u} + \delta\mathbf{u}$ (*cf.* (79)) as another admissible trial function also satisfies essential boundary conditions).
- Virtual Displacement/Virtual work is basically the equation we obtain by minimizing the energy function $\delta\Pi = 0$.
- Similar principles (virtual temperature for heat flow in solids and virtual velocities for fluid flow) are also directly derived from $\delta\Pi = 0$.
- While principle of virtual work can be obtained from $\delta\Pi = 0$, it is often quite easy to directly write and equate internal and external works for a given problem.

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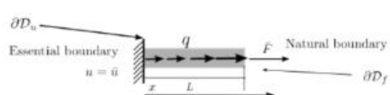
Virtual work: 1D solid bar



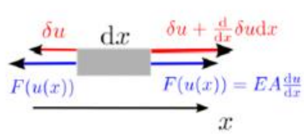
Equation (98) can be written as,

Find $u \in \mathcal{V} = \{v \in C^1([0,L]) \mid v(0) = \bar{u}\}$, such that,
$\forall\delta u \in \mathcal{W} = \{v \in C^1([0,L]) \mid v(0) = 0\}$

$$\underbrace{\int_0^L \overbrace{\delta u'(x)}^{\frac{\mathrm{d}}{\mathrm{d}x}\delta u}\overbrace{EAu'(x)}^{F(u(x))}\,\mathrm{d}x}_{\text{Virtual Internal Work}} = \underbrace{\int_0^L \delta u(x)q(x)\,\mathrm{d}x + \delta u(L)\bar{F}}_{\text{Virtual External Work}} \tag{109}$$

Note that the internal work differential is:

$$\mathrm{d}V = F(u(x)).(\delta u + \frac{\mathrm{d}}{\mathrm{d}x}\delta u\mathrm{d}x) - F(u(x)).\delta u$$
$$= \frac{\mathrm{d}\delta u}{\mathrm{d}x}.F(u(x))\mathrm{d}x \tag{110}$$



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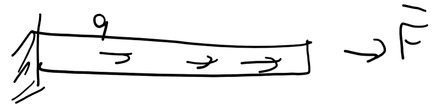
We'll do some examples of virtual work next time

Discretization and numerical examples:

Discretization: Going from continuum solution to "discrete" solution, where we have a finite number of unknowns

$$u^h(x) = \phi_p(x) + \sum_{i=1}^{n} a_i \, \phi_i(x)$$



$a_i \quad i=1,\dots,n$ are unknowns

ϕ_i's are called basis functions (they are known beforehand)

ϕ_p particular solution

$$\begin{cases} \phi_p(\partial\Omega_u) = \bar{u}(\partial\Omega_u) \\ \phi_i(\partial\Omega_u) = 0 \end{cases}$$

$$\rightarrow u^h(\partial\Omega_u) = \bar{u}(\partial\Omega_u)$$

this is the way ensure that the discrete solution satisfies all essential BCs

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we have two statements

WRS
$$\int_0^L w\Big[\underbrace{(EAu')' + q}_{R_i}\Big]dx + w\big|_{x=L}\underbrace{(\bar{F}-F)}_{R_f} = 0$$

we may have a different function here

or in general setting

General
$$\int_\Omega w\Big[\underbrace{L_M(u) - r}_{R_i}\Big]dv + \int_{\partial\Omega_f} w_f\underbrace{(\bar{F} - f_x \cdot n)}_{R_f}\,ds = 0$$

source term

spatial flux

Bar $\quad L_M(\cdot) = [EA(\;)']'$ $\qquad$ Beam $\quad L_M(\cdot) = (EI(\;)'')''$

IBP / Gauss theorem $\longrightarrow$ WK (weak statement)

Bar
$$\int_0^L w' \, EA \, u' \, dx = \int_0^L w \, q \, dx + \bar{F} w\big|_{x=L}$$

General
$$\int_\Omega L_m(w) \, \mathbf{D} \, L_m(u) \, dv = \int_\Omega w \, q \, dv + \int_{\partial\Omega_f} w \bar{F} \, ds$$

section & constitutive eqn

bar

L_m here is ...

L_m here is $\quad L_m(\ ) = (\ )'$ (bar) $\quad$ beam $\quad L_m(\ ) = (\ )$

$D = EA \qquad = EI$