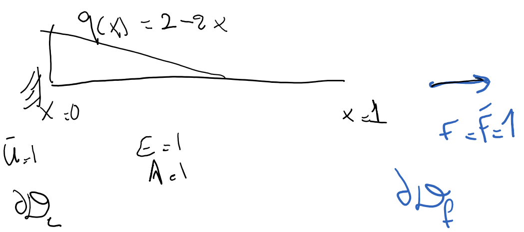
FEM2021/09/23

Thursday, September 23, 2021 2:51 PM

Numerical Example:



WRS Exactly satisfy Essential BC by $u^h = \phi_p + \sum \phi_i a_i$

$$\int_0^L \underbrace{\omega}\, \underbrace{R_i}_{\text{Interior residual}}\, dx + \underbrace{\omega_f}\, \underbrace{R_f}_{\text{Natural BC residual}} = 0$$

we'll see that for least square (R^2) we have a different weight for R_f

$$R_i = \underbrace{(EAu')'}_{L_\Omega(u)\ \text{Diff operator}} + \underbrace{q}_{-r\ \text{source term}} =$$

$$L_\Omega(\ ) = \underline{(EA(\ )')'} \qquad r = -q$$

$$R_f = \bar{F} - L_f(u)\cdot n = \bar{F} - EA(u)'\cdot n \qquad L_f(\ ) = EA(\ )'$$

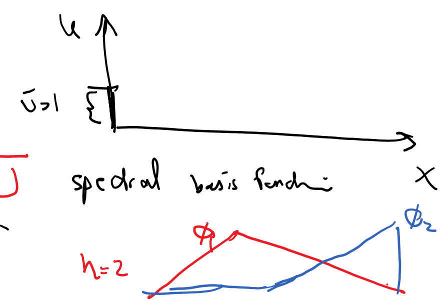
Solve the problem with 2 unknowns

$\phi_p = ?$ $\quad \phi_p(0) = 1 \longrightarrow \boxed{\phi_p = 1}$

$\phi_i' = ?$ $\quad \underline{\phi_i(0) = 0}$ / $\{x, x^2, \dots\}$ spectral basis functions

$\{\sin x, \sin 2x, \dots$

FEM basis



$n = 2$

$$u^h = \phi_p + \phi_1 a_1 + \phi_2 a_2 = \phi_p + [\phi_1 \ \ \phi_2] \underbrace{\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}}_{a}$$

Although, I know what will be used for phi_1, phi_2, phi_p, I'll keep them as general functions below.

we need $\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$ & potentially $\begin{bmatrix} \omega f_1 \\ \omega f_2 \end{bmatrix}$ to solve the problem.

I'll discuss the choice of ω's later.

$$W = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \qquad \omega_f = \begin{bmatrix} \omega f_1 \\ \omega f_2 \end{bmatrix}$$

$$\int_0^L \underbrace{\omega}_{\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}} \left[(EAu^{h\prime})' + q \right] dx + \omega_f \left(\bar{F} - \underbrace{EAu^{h\prime}}_{F} \right)\Big|_{x=L} = 0$$

$$\int \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \left[\left[EA \Big(\underbrace{\overbrace{\phi_p + [\phi_1 \ \ \phi_2] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}}^{u^h}}_{\phi_p' + [\phi_1' \ \ \phi_2'] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}} \Big)' \right]' + q \right] dx + \omega_f \left(\bar{F} - EA \Big(\overbrace{\phi_p + [\phi_1 \ \ \phi_2] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}}^{u^h} \Big)' \right)\Bigg|_{x=L} = 0$$

$$u^h = \phi_p + \phi_1 a_1 + \phi_2 a_2$$

$$u^{h\prime} = \phi_p' + \phi_1' a_1 + \phi_2' a_2 = \phi_p' + [\phi_1' \ \ \phi_2'] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

Assume E & A are constant

$$\int_0^L EA\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\left(\phi_p'' + [\phi_1'' \ \ \phi_2'']\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right)dx + \int_0^L \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} q \, dx + \begin{bmatrix} w_{f_1} \\ w_{f_2} \end{bmatrix}\left(\bar{F} - EA\phi_p' - [\phi_1' \ \ \phi_2']\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right)\Big|_{x=L} = 0$$

$$\boxed{\left(\int_0^L EA\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}[\phi_1'' \ \ \phi_2''] \, dx\right)\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - \left(\begin{bmatrix} w_{f_1} \\ w_{f_2} \end{bmatrix}[\phi_1' \ \ \phi_2']\Big|_{x=L}\right)\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}} \quad Ka$$

$$= \underbrace{-\int_0^L EA\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\phi_p'' + \begin{bmatrix} w_{f_1} \\ w_{f_2} \end{bmatrix}EA\phi_p'\Big|_{x=L}}_{F_D} \underbrace{- \int \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} q \, dx}_{F_r} \underbrace{- \begin{bmatrix} w_{f_1} \\ w_{f_2} \end{bmatrix}\bar{F}\Big|_{x=L}}_{F_N}$$

"contribution from Dirichlet BC because $\phi_p(0) = \bar{u}$"

Summary

stiffness matrix

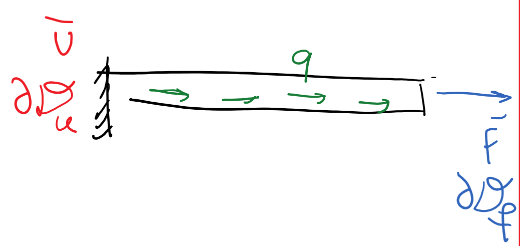
force vector

$$K_{2\times 2} \, a_{2\times 1} = F_{2\times 1}$$

unknown vector

$$K_{2\times 2} = \int_0^L EA\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}[\phi_1'' \ \ \phi_2''] \, dx - \begin{bmatrix} w_{f_1} \\ w_{f_2} \end{bmatrix}[\phi_1' \ \ \phi_2']\Big|_{x=L}$$

$$F_{2\times 1} = \underbrace{F_r}_{\text{force of source term}} + \underbrace{F_N}_{\text{Neumann BC force}} + F_D$$



$$F_r = -\int_0^L \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} q(x) \, dx$$

$$F_N = \begin{bmatrix} w_{f_1} \\ w_{f_2} \end{bmatrix}\Big|_{x=L} \bar{F}$$

$$F_D = -\int_0^L EA\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\phi_p'' + \begin{bmatrix} w_{f_1} \\ w_{f_2} \end{bmatrix}EA\phi_p'\Big|_{x=L}$$

Now we can solve this

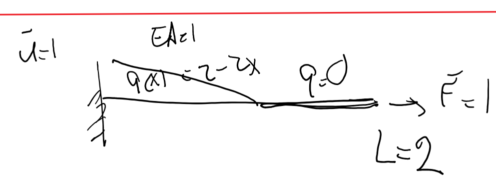
$\phi_p = 1 \rightarrow \phi_p' = 0, \ \phi_p'' = 0$

$\phi = [\phi_1 \ \ \phi_2] = [x \ \ x^2]$

$[\phi_1' \ \ \phi_2'] = [1 \ \ 2x]$

$[\phi_1'' \ \ \phi_2''] = [0 \ \ 2]$

$F_D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$



$$u^h = 1 + a_1 x + a_2 x^2$$

$\hookrightarrow F_D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$u^h = 1 + a_1 x + a_2 x^2$

plug all in $\circledast$ to get

$$K = \int_0^2 \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \underbrace{[0 \quad 2]}_{\phi''} dx - \begin{bmatrix} \omega_{f_1} \\ \omega_{f_2} \end{bmatrix}_{x=2} \underbrace{[1 \quad 4]}_{=\phi'|_{x=2}}$$

$\phi_1 = 0$ for $x > 1$

$$F = \underbrace{- \int_0^{\cancel{2}} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} (2 - 2x)\, dx}_{F_r} \underbrace{- \begin{bmatrix} \omega_{f_1} \\ \omega_{f_2} \end{bmatrix}_{x=2}}_{f_N}$$

$$Ka = F, \quad a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

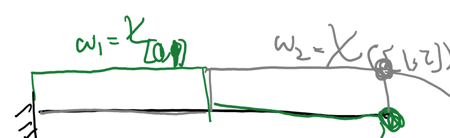
$\circledast\circledast$

Choices for ω:

① Subdomain Method

characteristic function of Ω $\quad \chi_{\Omega}(x) = \begin{cases} 1 & x \in \Omega_i \\ 0 & \notin \Omega \end{cases}$

Basically this choice of weight function satisfies the **balance law** on finite number of characteristic sets considered.



Row 1 of K & f

$\omega_1 = 0$ for $x > 1$

K_1: row 1 of K $\quad = \int_0^1 (1) [0 \quad 2] - (\omega_1)_{x=2} [1 \quad 4]$, with $(1) = \omega_1$ and $(\omega_1)_{x=2} = 0$

$= [0 \quad 2]$

$K_2: = \int_1^2 (1) [0 \quad 2]\, dx - (\omega_2)\big|_{x=2} [1 \quad 4]$, with $(\omega_2)_{x=2} = 1$

$\omega_2 = 0$ for $x < 1$

$[0 \quad 2] - [1 \quad 4] = [-1 \quad -2]$

$$K = \begin{bmatrix} 0 & 2 \\ -1 & -2 \end{bmatrix}$$

$$F_1 = -\int_0^1 (1)(2-2x)\,dx - \overbrace{\omega_1(x=2)}^{0} = -1$$

$$F_2 = -\int_0^1 \underbrace{\omega_2(x)}_{0}(2-2x) - \omega_2(x=2) = -1$$

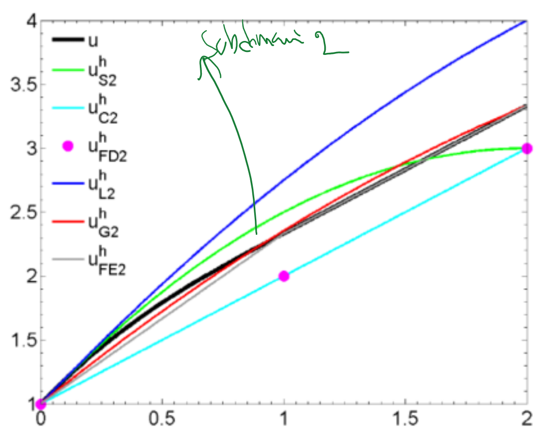
$$F = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \rightarrow Ka = f \qquad a = \begin{bmatrix} 2 \\ -1/2 \end{bmatrix}$$

discrete sln

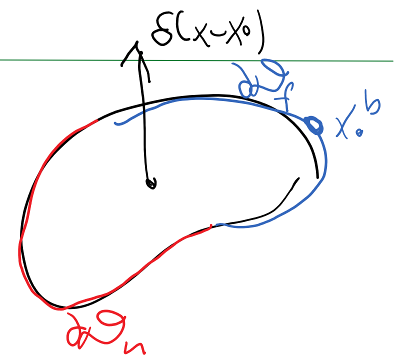
$$\boxed{u^h_{S2}(x) = \varphi_p + [\varphi_1 \;\; \varphi_2]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 1 + [x \;\; x^2]\begin{bmatrix} 2 \\ -\frac{1}{2} \end{bmatrix} = 1 + 2x - \frac{x^2}{2}}$$

subdomain

2 unknowns



2. Collocation method



$$\int_\Omega \omega R_i(x)\,dv + \int_{\partial\Omega_f} \omega R_f(x)\,ds = 0$$

$$\omega = \delta(x - x_0)$$

$$\rightarrow \quad R_i(x_0) = 0$$

$$\omega = \delta(x - x_0^b) \quad \rightarrow \quad R_f(x_0^b) = 0$$

$$\omega = \delta(x - x_0^b) \rightarrow R_f(x_0^b) = 0$$

$$R_f = F - F = 1 - u' = 1 - [\phi_1' \ \phi_2'] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$R_i = u'' + q(x) = [\phi_1'' \ \phi_2''] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + q$$

$x = 1$

Eq 1

$$[\phi_1'' \ \phi_2''] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + q \Big|_{x=1} = 0$$

Eq 2

$$1 - [\phi_1' \ \phi_2'] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \Big|_{x=2} = 0$$

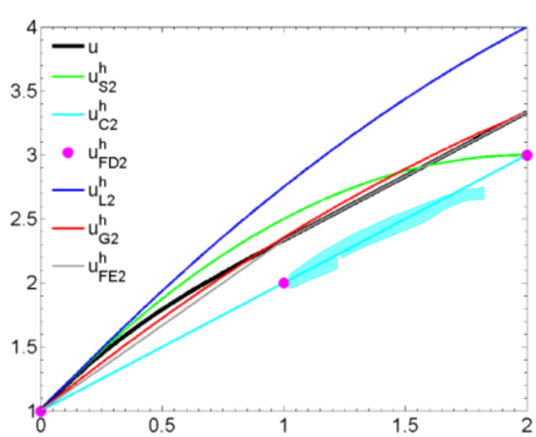
$$[0 \ \ 2] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + 0 = 0$$

$$1 - [1 \ \ 4] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$$

$$\rightarrow \begin{bmatrix} 0 & 2 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\boxed{u^h_{C2} = 1 + 1x + 0x^2 = 1 + x}$$



inside

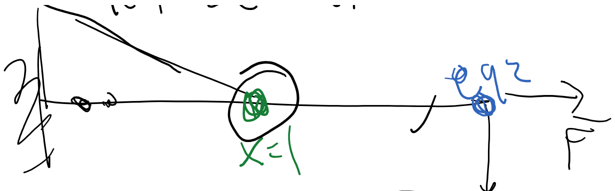
$q(x) = 2(1-x)$

$R_i = u'' + q(x)$

$\phi_1 q^2$

inside

$$R_i = u'' + q(x) \qquad 0 < x < 2$$



$$R_f = \bar{F} - F = 1 - u'$$

$$u^h = \overset{1}{\phi_p} + a_1 x + a_2 x^2 \Rightarrow u^{h''} = 2a_2$$

need two eqns

How about 1 eqn from R_i @ 1 pt and R_f @ $x=2$?

eq1) $R_i(x=1) = 0$: $u^{h''} + q(x=1) = 0$

eqn(1) $\boxed{2a_2 + 0 = 0}$

eq2) $R_f\big|_{x=2} = 0$

$$1 - u'\big|_{x=L} = 0$$

$$1 - (1 + a_1 x + a_2 x^2)'\big|_{x=2} = 0$$

$$1 - (a_1 + 2a_2 x)\big|_{x=2} = 0$$

eqn(2) $\boxed{-a_1 - 4a_2 = -1}$

$$\begin{aligned} 2a_2 &= 0 \\ -a_1 - 4a_2 &= -1 \end{aligned} \qquad \begin{bmatrix} 0 & 2 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$a = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow u^h_{C2} = \overset{1}{\phi_p} + \overset{x}{\phi_1} \underset{1}{a_1} + \overset{x^2}{\phi_2} \underset{0}{a_2} = 1 + x$$

$$\boxed{u^h_{C2} = 1 + x}$$

HW 4 unknowns

$$u^h = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$R_i = u'' + q(x)$$