FEM2021/10/05

Tuesday, October 5, 2021 2:54 PM

Ritz method, the idea here is that we compute the discrete form of the potential energy and minimize it then.

Energy method

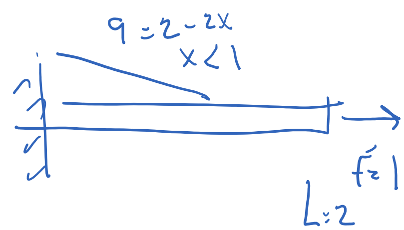
static $\Pi(u) = \underset{\text{internal energy}}{V(u)} - \underset{\text{external work}}{W(u)}$

Recall
minimized Π
weak statement

Ritz Method

Discretize energy <u>then</u> minimize it

$$\Pi(u) = \int_0^{L \to 2} \frac{1}{2} \underbrace{EA}_{1} u'^2 \, dx - u(x=L) \underbrace{\bar{F}}_{1} - \int_0^L q(x) u(x) \, dx$$



For this numerical example

$$\boxed{\Pi(u) = \int_0^2 \frac{1}{2} u'^2 \, dx - \int_0^1 u(x)(2-2x)\, dx - u(x=2) \cdot 1}$$

E. cont.

Continuum Energy statement

Discretize

$$u^h = \phi_p(x) + a_1 \phi_1(x) + a_2 \phi_2(x)$$

$\left.\begin{aligned} \phi_1 &= x \\ \phi_2 &= x^2 \end{aligned}\right\}$ spectral basis

$n = 2$
unknown

$\phi_p = 1$

$$u^h = 1 + a_1 x + a_2 x^2$$

$$\Pi(u^h) = \int_0^2 \frac{1}{2} \Big[\big(\overbrace{1 + a_1 x + a_2 x^2}^{u^h} \big)' \Big]^2 dx - \int_0^1 \big(\overbrace{1 + a_1 x + a_2 x^2}^{u^h} \big) \overbrace{(2-2x)}^{q} \, dx$$

$$- \left. (1 + a_1 x + a_2 x^2) \right|_{x=2} \cdot \overbrace{1}^{\bar{F}}$$

$$\Pi(u^h) = \int^2 \frac{1}{2} (a_1 + 2a_2 x)^2 \, dx \quad \int^1$$

$$\underbrace{\Pi(u^h)}_{\substack{\text{function} \\ \text{of} \\ a_1 \,\&\, a_2}} = \int_0^1 \frac{1}{2}\left(a_1 + 2a_2 x\right)^2 dx - \int_0^1 (1 + a_1 x + a_2 x^2)(2 - 2x)\, dx - (1 + 2a_1 + 4a_2)$$

$$\Pi(a_1, a_2) = \left(a_1^2 + 4a_1 a_2 + \frac{16}{3} a_2^2\right) - \left(\frac{7}{3} a_1 + \frac{25}{6} a_2 + 2\right)$$

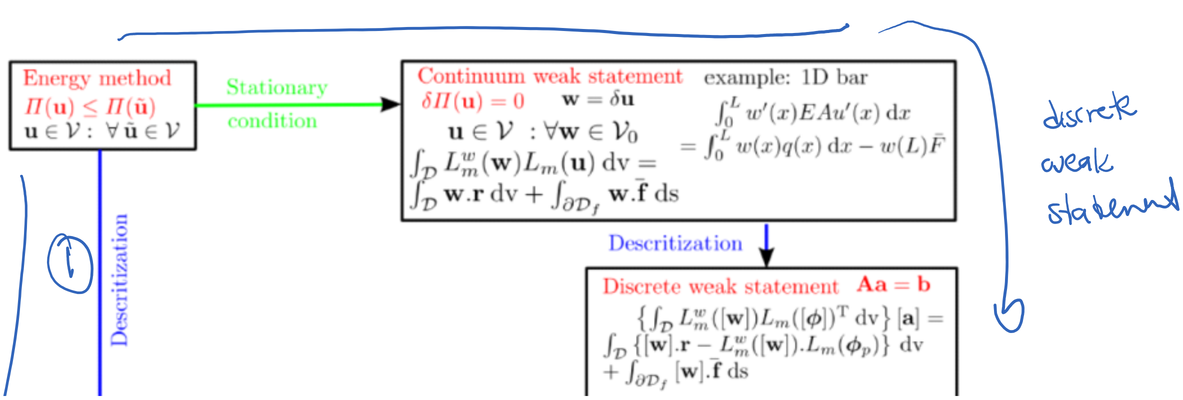
we minimize the energy :

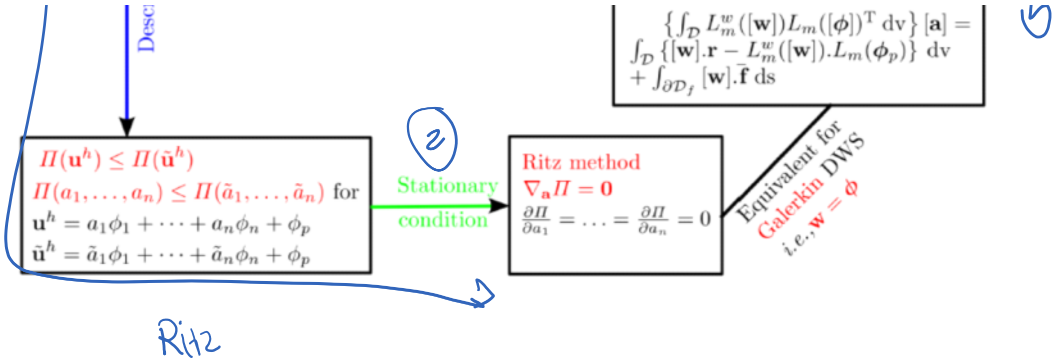
$$\underbrace{\nabla}_{\text{gradient}} \Pi = \begin{bmatrix} \frac{\partial \Pi}{\partial a_1} \\ \frac{\partial \Pi}{\partial a_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2a_1 + 4a_2 - \frac{7}{3} \\ 4a_1 + \frac{32}{3} a_2 - \frac{25}{6} \end{bmatrix}$$

$$\Longleftrightarrow \quad Ka = f \qquad K = \begin{bmatrix} 2 & 4 \\ 4 & 32/3 \end{bmatrix} \qquad F = \begin{bmatrix} 7/3 \\ 25/6 \end{bmatrix}$$

$$\rightarrow \quad a = \begin{bmatrix} 37/24 \\ -3/16 \end{bmatrix} \quad \longrightarrow \quad u^h = 1 + \overbrace{\frac{37}{24}}^{a_1} x \overbrace{- \frac{3}{16}}^{a_2} x^2$$

K, F, a are identical with what we had with weak statement solution + spectral Galerkin ($w = \phi = \begin{Bmatrix} x \\ x^2 \end{Bmatrix}$)

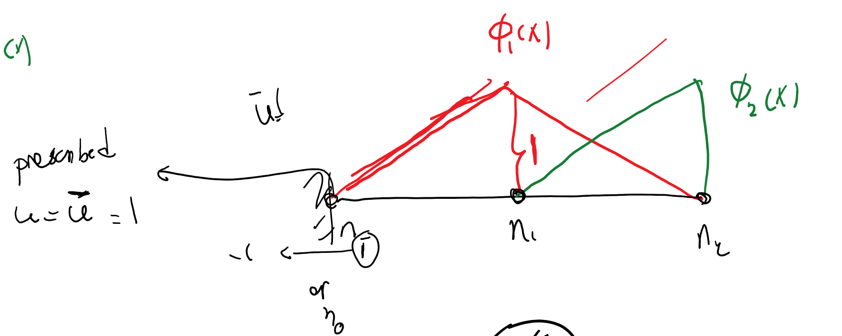




Finite Element Galerkin

FE is a Galerkin method ($w=\phi$) with specific basis functions

$$u^h = \phi_p(x) + a_1 \phi_1(x) + a_2 \phi_2(x)$$

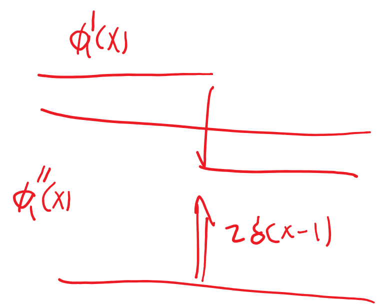


We cannot use Weighted Residual Statement (WRS) for finite element basis functions because they are not smooth enough

$$\int w (u'')$$

bar problem

We must use the weak statement

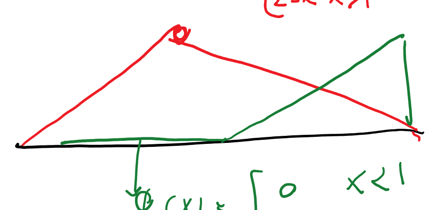


$$K = \int_0^2 \begin{bmatrix} w_1' \\ w_2' \end{bmatrix} EA \, [\phi_1' \ \ \phi_2'] \, dx$$

$w_1 = \phi_1$, $w_2 = \phi_2$ Galerkin

$$F = \int_0^1 \begin{bmatrix} w_1(x) \\ w_2(x) \end{bmatrix} q(x) \, dx + \begin{bmatrix} \phi_1(x) \\ \phi_2(x) \end{bmatrix} \bar{F} \Big|_{x=2}$$

$$\phi_1(x) = \begin{cases} x & x<1 \\ 2-x & x>1 \end{cases}$$



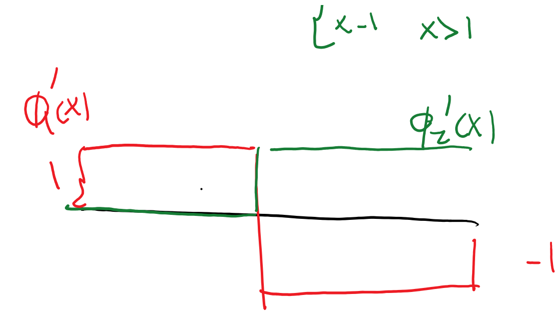
$$\phi_2(x) = \begin{cases} 0 & x<1 \\ x-1 & x>1 \end{cases}$$

$$K = \int_0^1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1 \ \ 0] \, dx + \int_1^2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} [-1 \ \ 1] \, dx$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ & \end{bmatrix}$$

ϕ'

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & +1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$



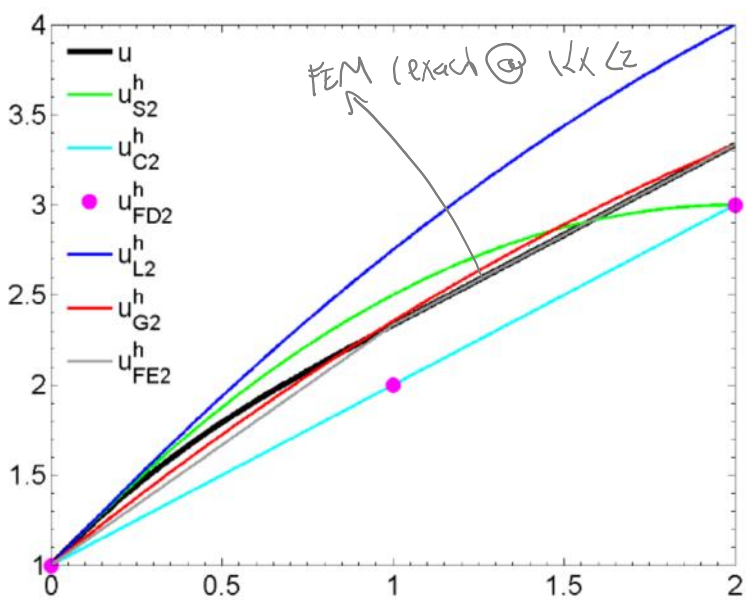
$$F = \int_0^1 \begin{bmatrix} x \\ 0 \end{bmatrix} \underbrace{(2-2x)}_{q(x),\, x<1} dx + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{w|_{x=2}} \cdot 1$$

$w_1 = \varphi_1 = x$ $\quad$ $w_2 = \varphi_2 = 0$ for $x<1$

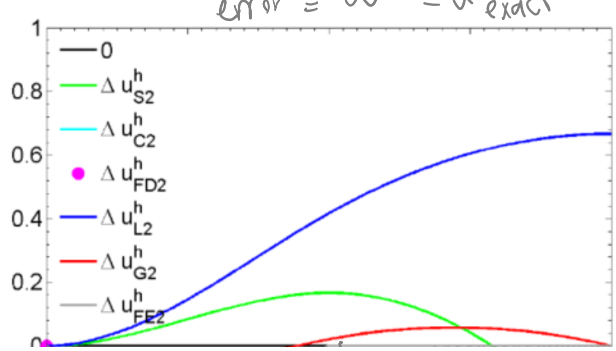
$$F = \begin{bmatrix} 1/3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} \quad , Ka = F$$

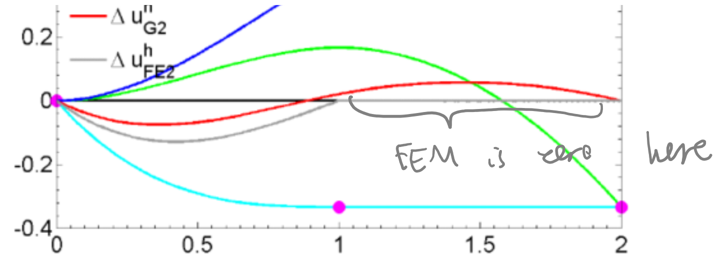
$$a = \begin{bmatrix} 4/3 \\ 7/3 \end{bmatrix} \longrightarrow u^h_{FE2} = \varphi_p + a_1 \varphi_1(x) + a_2 \varphi_2(x)$$

$$\cancel{= 1 + \frac{4}{3} x + \frac{7}{3} x^2}$$



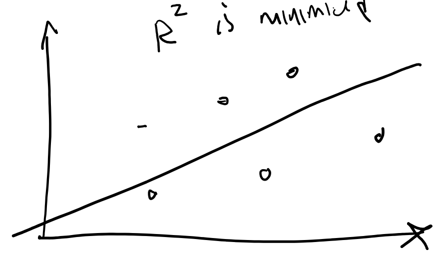
error = $u^h - u_{exact}$





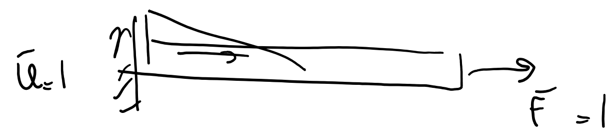
FEM error is zero at finite element nodes for 1D problems under certain conditions that often hold (Hughes reference book has the proof of this)

Least square (LS) method
LS method is used in many different settings, for example in (linear) regression, etc.



in our context

$q(x) = 2 - 2x \quad x < 1$



$$R_i := \underbrace{L_M}_{PDE}(u^h) = u_h'' + q(x) \qquad 0 < x < 2$$

$$R_f = L_f = \bar{F} - F\big|_{x=L} = \left(\bar{F} - EA\, u_h'\right)\Big|_{x=2} \qquad x = 2$$

these are our residuals (errors)

$$R^2 = \int_0^2 R_i^2(x)\, dx + R_f^2$$

Zero iff we have the exact solution

The exact solution MINIMIZES R2

Now I'm going to 1st discretize

$$u^h = \varphi_0 + a_1 \varphi_1 + a_2 \varphi_2 = 1 + a_1 x + a_2 x^2$$ spectral basis $\varphi = \{x, x^2\}$

$$\boxed{R_i = (u^h)'' + q(x) = 2a_2 + q(x)} \qquad 0 < x < 2$$

$$R_f = \bar{F} - F\big|_{x=2} = 1 - u^{h\prime}\big|_{x=2} = 1 - (a_1 + 2a_2 x)\big|_{x=2}$$

$$\boxed{R_f = 1 - (a_1 + 4a_2)} \quad @ \ x=2$$

$$R^2 = \int_0^2 R_i^2(x)\,dx + R_f^2 = \int_0^2 (2a_2 + q(x))^2\,dx + (1 - (a_1 + 4a_2))^2$$

$x<1$

$$\int_0^1 (2a_2 + \overbrace{(2-2x)}^{q(x)})^2 + \int_1^2 (2a_2 + 0)^2\,dx$$

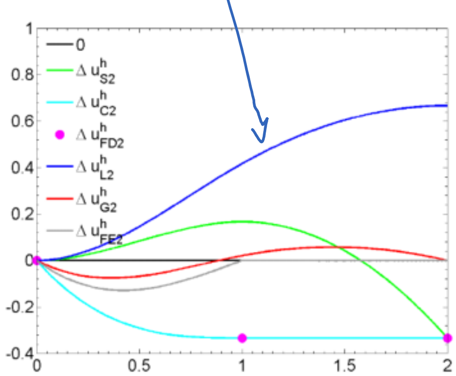
$$R^2_{(a_1,a_2)} = 1 + a_1^2 + 20a_2^2 - 2a_1 - 8a_2 + 8a_1 a_2$$

$$\nabla_{a_1,a_2} R^2 = 0 \quad \text{minimize } R^2$$

$$\begin{bmatrix} 2a_1 + 8a_2 - 2 \\ 8a_1 + 4a_2 - 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$K = \begin{bmatrix} 2 & 8 \\ 8 & 48 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \rightarrow a = \begin{bmatrix} 2 \\ -\frac{1}{4} \end{bmatrix}$$

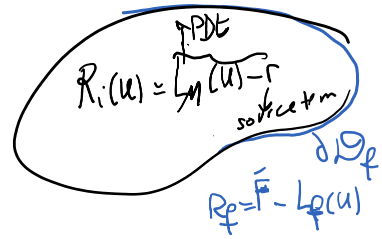
$$\boxed{u^h_{L2} = 1 + 2x - \frac{1}{4}x^2}$$



Solution process was
(1) Discretize
(2) Minimize

$$R^2(u) = \int_{\mathcal{D}} R_i^2(u(x))\,dx + \int_{\partial\mathcal{D}_f} R_f^2(u(x))\,ds$$

continuum setting



$u = u^h = \phi_p + a_i \phi_i$ discretization

$L_M(u^h) = L_M(\phi_p + a_i\phi_i) = L_M(\phi_p) + a_i L_M(\phi_i)$ Differential operator is linear

$L_f(u^h) = L_f(\phi_p) + a_i L_f(\phi_i)$

↓ linear

e.g. bar problem

$L_M(\;) = (\;)''$

$$R^2(a_1,\dots,a_n) = \int_{\mathcal{D}} R_i^2\,dv + \int_{\partial\mathcal{D}_f} R_f^2\,ds \rightarrow$$

$$\nabla R^2\Big|_{a_1,\dots,a_n} = 0 \rightarrow \int_{\mathcal{D}} \underbrace{\frac{\partial R_i^2}{\partial a_j}}_{2R_i\frac{\partial R_i}{\partial a_j}} dv + \int \underbrace{\frac{\partial R_f^2}{\partial a_j}}_{2R_f\frac{\partial R_f}{\partial a_j}} ds = 0 \quad \forall j = 1,\dots,n$$

$$R_i = L_M(u^h) - r = L_M(\phi_p) + a_j L_M(\phi_j) - r$$

$$R_f = \bar{F} - L_f(u^h) = \bar{F} - L_f(\phi_p) - a_j L_f(\phi_j)$$

$$\frac{\partial R_i}{\partial a_j} = L_M(\phi_j)$$

$$\frac{\partial R_f}{\partial a_j} = -L_f(\phi_j)$$

$$\nabla R^2 = 0 \rightarrow \int_{\mathcal{D}} \underbrace{L_M(\phi_j)}_{\omega_j} R_i\,dv + \int_{\partial\mathcal{D}_f} \underbrace{(-L_f(\phi_j))}_{(\omega_f)_j} R_f\,ds = 0$$

LS

WRS in general

WRS in general

for ω_j $\int_{\Omega} \omega_{j} R_i \, dV + \int_{\partial\Omega_f} \omega_{f_j} R_f \, dS = 0$