FEM2021/10/07

Thursday, October 7, 2021 2:48 PM

From last time

$$R^2_{(a_1,a_2)} = 1 + a_1^2 + 20a_2^2 - 2a_1 - 8a_2 + 8a_1a_2$$

$$\nabla_{a_1,a_2} R^2 = 0 \qquad \text{minimize } R^2$$

$$\begin{bmatrix} 2a_1 + 8a_2 - 2 \\ 8a_1 + 4a_2 - 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$K = \begin{bmatrix} 2 & 8 \\ 8 & 48 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \rightarrow a = \begin{bmatrix} 2 \\ -\frac{1}{4} \end{bmatrix}$$

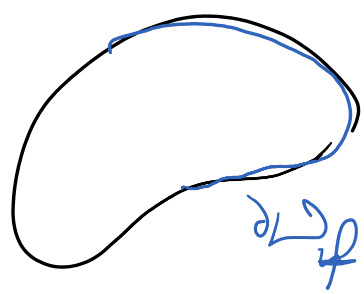
$$\boxed{u^h_{L2} = 1 + 2x - \frac{1}{4}x^2}$$

Much easier way to do the LS method

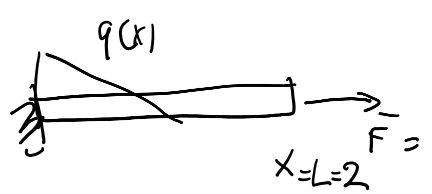
Basically, least square method is a WRS wherein the weights for Ri (inside) and Rf (natural boundary) are different

$$\int_{\Omega} w R_i \, dv + \int_{\partial\Omega_f} w_f R_f \, ds = 0$$

$$w \rightarrow L_M^t(\phi) \qquad w_f \rightarrow -L_f^t(\phi)$$



Ar example



$$R_i = (\underbrace{EA}_{1}u')' + q(x) = u'' + q(x)$$

$R_i = \underbrace{(EAu')'}_{L_M(u)} + q(x) = u'' + q(x)$ $\qquad x = L = 2 \quad F = 1$

$$\boxed{L_M = (\ )''}$$

$R_f = F - (\underbrace{EAu'}_{F = L_f(u)})\Big|_{x=L} = 1 - \underbrace{u'}_{L_f = (\ )'}\Big|_{x=2}$

$$\int_0^2 \underbrace{L_M^t(\phi)}_{\text{LS weight}} \overbrace{\left(\underbrace{\left(\phi_p + [\phi_1 \quad \phi_2]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right)}_{u^h}'' + q(x) \right)}^{R_i} dx$$

$\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$$+ \underbrace{\left(-L_f^t(\phi)\right)}_{\text{LS weight on } \partial\Omega_f} \overbrace{\left[1 - \underbrace{\left(\phi_p + [\phi_1 \quad \phi_2]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right)'}_{u^{h'}} \right]}^{Rf}\Bigg|_{x=L}$$

$\omega_f = \begin{bmatrix} -1 \\ -2x \end{bmatrix}$

$\phi_p = 1, \quad \phi_1 = x, \quad \phi_2 = x^2, \quad \phi = [x \quad x^2] \qquad \phi' = [1 \quad 2x] \qquad \phi'' = [0 \quad 2]$

$\omega = (L_M(\phi))^t = (\phi'')^t = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$\omega_f = -(L_f(\phi))^t = -\phi'^t = -\begin{bmatrix} 1 \\ 2x \end{bmatrix}$

$$K = \int_0^2 \underbrace{\begin{bmatrix} \phi_1'' \\ \phi_2'' \end{bmatrix}}_{\omega} [\phi_1'' \quad \phi_2''] \, dx - \underbrace{\left(-\begin{bmatrix} \phi_1' \\ \phi_2' \end{bmatrix}\right)}_{\omega_f} [\phi_1' \quad \phi_2'] \Big|_{\underbrace{x=2}_{\partial\Omega_f}}$$

$$= \int_0^2 \begin{bmatrix} 0 \\ 2 \end{bmatrix} [0 \quad 2] \, dx - \left(-\begin{bmatrix} 1 \\ 2x \end{bmatrix} [1 \quad 2x]\right)\Big|$$

$$= \int_0 \begin{bmatrix}0\\2\end{bmatrix}\begin{bmatrix}0 & 2\end{bmatrix} dx - \left(-\begin{bmatrix}1\\2x\end{bmatrix}\begin{bmatrix}1 & 2x\end{bmatrix}\right)\Bigg|_{x=2}$$

w_ϕ $\partial\Omega_f$

$$= \int_0^2 \begin{bmatrix}0 & 0\\0 & 4\end{bmatrix} dx + \begin{bmatrix}1\\4\end{bmatrix}\begin{bmatrix}1 & 4\end{bmatrix} \Rightarrow K = \begin{bmatrix}1 & 4\\4 & 24\end{bmatrix}$$

$$F = -\int_0^1 \underbrace{\begin{bmatrix}0\\2\end{bmatrix}}_{} (2-2x)\, dx - \underbrace{\begin{bmatrix}-1\\-2x\end{bmatrix}}_{}\cdot 1 \Bigg|_{x=2} = \begin{bmatrix}1\\2\end{bmatrix}$$

$q(x) = \ldots \quad x<1$

$q(x) = 0 \quad x>1$

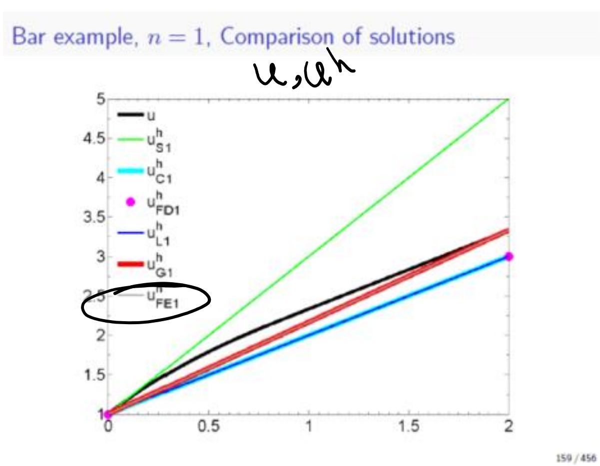
$w = L^t_\psi(\phi) = (\phi')^t$

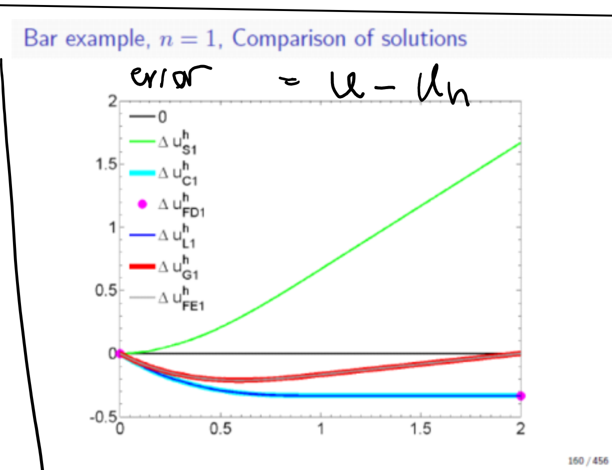
$w_f = -L_\phi(\phi) = -\phi'$

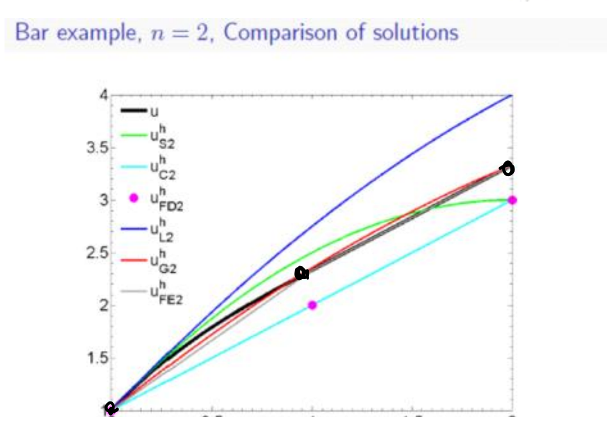
$$Ka = F \rightarrow a = \begin{bmatrix}2\\-\frac{1}{4}\end{bmatrix} \rightarrow u^h_{L2} = 1 + 2x - \frac{1}{4}x^2$$

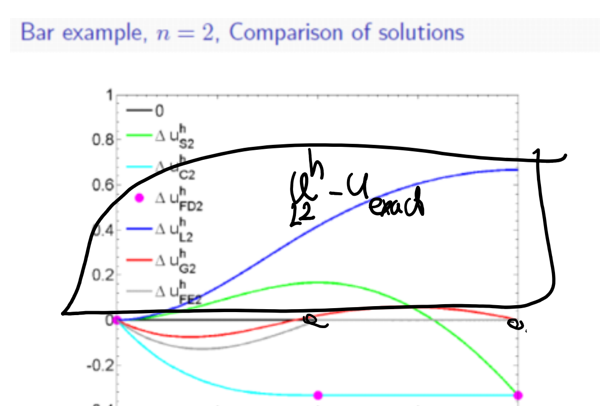
we're going to get the same solution we got

with brute-force (i.e. calculating $R^2(a_1, a_2)$, $\nabla R^2 = 0, \ldots$) last time

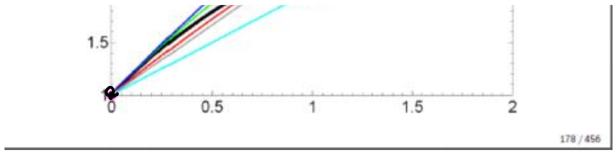


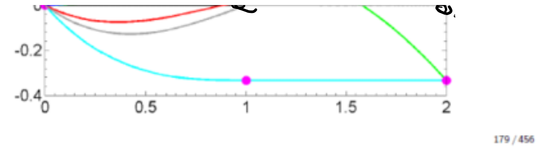




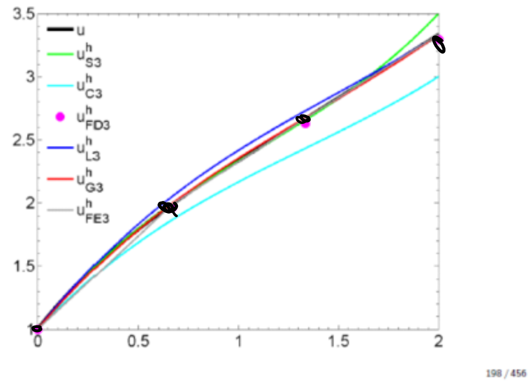


error of LS is large





Bar example, $n = 3$, Comparison of solutions



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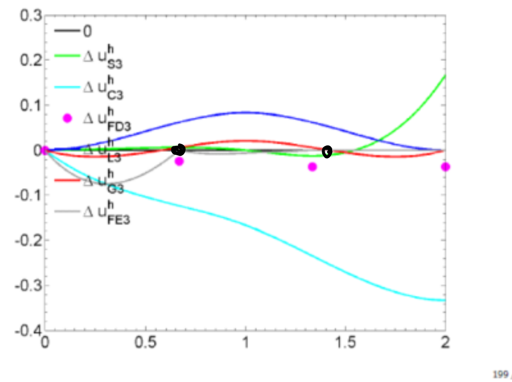
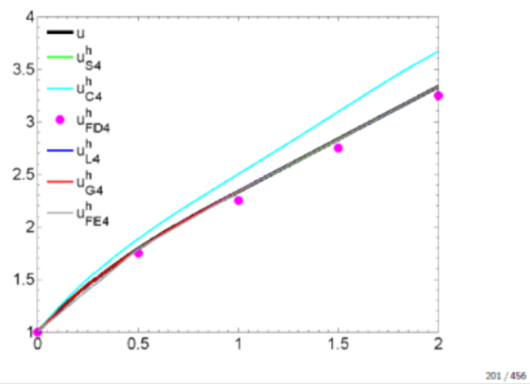
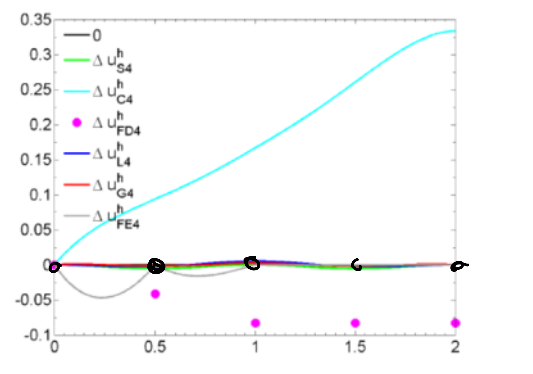


Bar example, $n = 4$, Comparison of solutions



Bar example, $n = 4$, Comparison of solutions



Exact sln

$$u(x) = \begin{cases} \frac{x^3}{3} - x^2 + 2x + 1 & 0 \le x \le 1 \\ x + \frac{4}{3} & 1 < x \le 2 \end{cases} \qquad (179)$$

142 / 456

1D FEM in most cases matches the exact solution at the nodes

LS:

within full space

$u^h = \phi_p + a_1\phi_1 + a_2\phi_2$, a_1, a_2 arbitrary are minimized

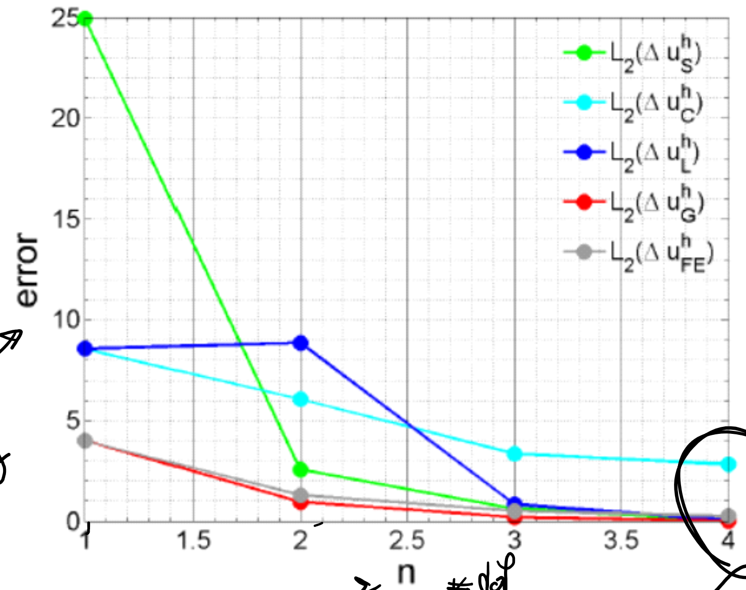
$R^2(a_1, a_2)$

why is it's solution error not the smallest?

We minimize R2 (error in PDE + natural BC) but this does not result in smallest error in solution (uh - u)

Talking about errors:

Bar example, Error Convergence



log

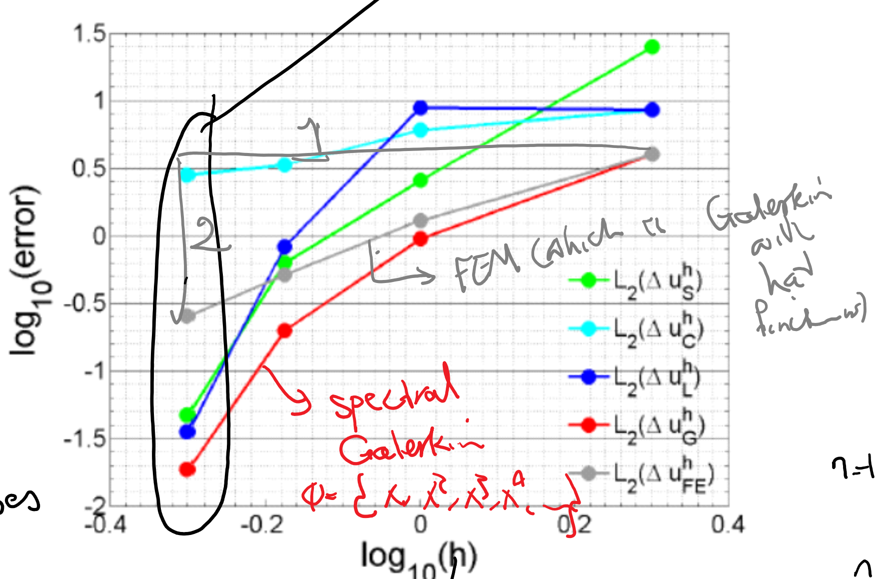
\# dof

log

not clear

203 / 456

Bar example, Error Convergence



error decreases

$1/h$

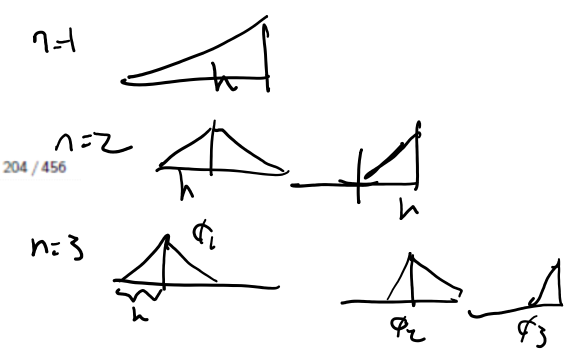
204 / 456

more dofs

h = element size

L/n

has meaning for FEM



All except FEM use spectral basis function $1 \to x, x^2 \to x, x^2, x^3, \ldots$

this results in higher than linear convergence rate.

this results in higher than linear convergence rate.

— FEM piecewise linear

$\boxed{p=1}$ order of element

$$\log(\text{error}) = A + 2 \log(h)$$

slope

$e^{(\ )}$

$$\text{error} = \underbrace{e^{A}}_{C} h^{2}$$

$$\boxed{\text{error} = C h^{2}}$$

linear FEM

for order p FEM

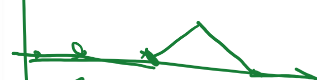
$$\boxed{\text{error} = C h^{ap+b}}$$

a, b depend on error type & PDE

Stiffness matrices

$f_i'' \approx \frac{f_{i+1} + f_{i-1} - 2f_i}{h^2}$

sparse matrices



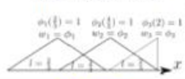
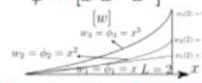
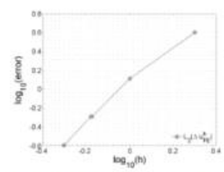
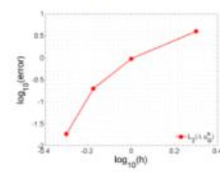
	S	C	FD
$\mathbf{K}$	$\begin{bmatrix} 0 & 1 & \frac{3}{4} & \frac{1}{2} \\ 0 & 1 & \frac{9}{4} & \frac{7}{2} \\ 0 & 1 & \frac{15}{4} & \frac{19}{2} \\ -1 & -3 & -\frac{27}{4} & -\frac{27}{2} \end{bmatrix}$	$\begin{bmatrix} 0 & 2 & 3 & 3 \\ 0 & 2 & 6 & 12 \\ 0 & 2 & 9 & 27 \\ -1 & -4 & -12 & -32 \end{bmatrix}$	$\begin{bmatrix} -8 & 4 & 0 & 0 \\ 4 & -8 & 4 & 0 \\ 0 & 4 & -8 & 4 \\ 0 & 0 & 2 & -2 \end{bmatrix}$
$\mathbf{F}^T$	$\begin{bmatrix} -\frac{3}{4} & -\frac{1}{4} & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -5 & 0 & 0 & -1 \end{bmatrix}$
$\mathbf{a}^T$	$\begin{bmatrix} 2 & -\frac{13}{12} & \frac{1}{2} & -\frac{1}{12} \end{bmatrix}$	$\begin{bmatrix} \frac{7}{3} & -\frac{3}{2} & \frac{5}{6} & -\frac{1}{6} \end{bmatrix}$	$u_i = \begin{bmatrix} \frac{7}{4} & \frac{9}{4} & \frac{11}{4} & \frac{13}{4} \end{bmatrix}$
	LS	G (spectral Galerkin)	FE hat-function - Galerkin
$\mathbf{K}$	$\begin{bmatrix} 1 & 4 & 12 & 32 \\ 4 & 24 & 72 & 192 \\ 12 & 72 & 240 & 672 \\ 32 & 192 & 672 & \frac{9728}{5} \end{bmatrix}$	$\begin{bmatrix} 2 & 4 & 8 & 16 \\ 4 & \frac{32}{3} & 24 & \frac{256}{5} \\ 8 & 24 & \frac{288}{5} & 128 \\ 16 & \frac{256}{5} & 128 & \frac{2048}{7} \end{bmatrix}$	$\begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix}$
$\mathbf{F}^T$	$\begin{bmatrix} 1 & 2 & 10 & 30 \end{bmatrix}$	$\begin{bmatrix} \frac{7}{3} & \frac{25}{6} & \frac{81}{10} & \frac{241}{15} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} & \frac{1}{12} & 0 & 1 \end{bmatrix}$
$\mathbf{a}^T$	$\begin{bmatrix} 2 & -\frac{17}{16} & \frac{23}{48} & -\frac{5}{64} \end{bmatrix}$	$\begin{bmatrix} \frac{97}{48} & -\frac{9}{8} & \frac{17}{32} & -\frac{35}{384} \end{bmatrix}$	$\begin{bmatrix} \frac{19}{24} & \frac{4}{3} & \frac{11}{6} & \frac{7}{3} \end{bmatrix}$

→ if the problem is self adjoint

- System Matrix $\mathbf{K}$ is nonsymmetric for Subdomain, Collocation and Finite Difference methods.
- System Matrix $\mathbf{K}$ is always symmetric for Least Square method.
- For this self adjoint problem $\mathbf{K}$ is symmetric for Galerkin methods ($\mathbf{w} = [x^i]$ and FE hat functions).
- Finite Element trial functions are local leading to sparse structure of $\mathbf{K}$ matrix.
- Spectral trial functions are continuus and span the entire domain. The matrix $\mathbf{K}$ is dense.
- Spectral methods have better convergence properties than FE methods, while their use is most often is limited to simple geometries.

Always symmetric

Observations: FE versus spectral methods

Feature	Finite Element	Spectral Methods
Trial Functions Example	Local / Finite Regularity hat functions 	Globally Smooth $\phi = [x\ x^2\ x^3]$ 
Matrix $\mathbf{K}$ Example	Sparse $\begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix}$	Full (diagonal for orthogonal ϕ) $\begin{bmatrix} 2 & 4 & 8 & 16 \\ 4 & \frac{32}{3} & 24 & \frac{256}{5} \\ 8 & 24 & \frac{288}{5} & 128 \\ 16 & \frac{256}{5} & 128 & \frac{2048}{7} \end{bmatrix}$
order of accuracy of u^h (p)	fixed ($e.g.$, $p = 1$)	$\nearrow$ vs. n ($e.g.$, $p = n$)
Convergence Example	Linear: $e = Ch^\alpha$ $\alpha = 2$ 	higher than linear exponential 
Geometry	Very general geometries	simple (e.g., rectangular) in practice to get diagonal $\mathbf{K}$

Diagonal matrix for spectral methods

- The global nature of trial functions ϕ in spectral method results in full $\mathbf{K}$ matrices that are expensive to solve.
- To circumvent this problem we employ trial functions that make $\mathbf{K}$ diagonal.
- In weak statement $K_{ij} := \mathcal{A}(\phi_i, \phi_j) = \int_{\mathcal{D}} L_m^w(\phi_i) L_m(\phi_j)\,\mathrm{dv}$.
- If the problem is self-adjoint $\mathcal{A}(.,.)$ is an inner product and we can construct an orthogonal trial function basis ϕ_i for example using Gram Schmidt method.
- Given the particular form of $\mathcal{A}$ (from L_m^w and L_m) and domain of integration $\mathcal{D}$ ([0 1], [−1 1], semi-infinite, infinite, *etc.*) we employ various trigonometric and orthogonal polynomial spaces. Some examples are:
  - $\phi_k(x) = e^{ikx}$ Fourier spectral method.
  - $\phi_k(x) = T_k(x)$ Chebyshev spectral method.
  - $\phi_k(x) = L_k(x)$ or $P_k(x)$ Legendre spectral method.
  - $\phi_k(x) = \mathcal{L}_k(x)$ Laguerre spectral method.
  - $\phi_k(x) = H_k(x)$ Hermite spectral method.

  where $T_k(x)$, $L_k(x)(P_k(x))$, $\mathcal{L}_k(x)$, and $H_k(x)$ are the Chebyshev, Legendre, Laguerre, and Hermite polynomials of degree k, respectively.
- The orthogonal property of these functions is for simple geometries. That is why spectral methods are more popular for simple geometries where we can take advantage of their exponential convergence property while keeping computational costs low by using orthogonal trial functions.

For simple geometries we can get exponential convergence with spectral methods with diagonal or close to diagonal matrix equations, but for complex geometries FEM is very versatile.

The error of FEM is in fact:

solution regularity

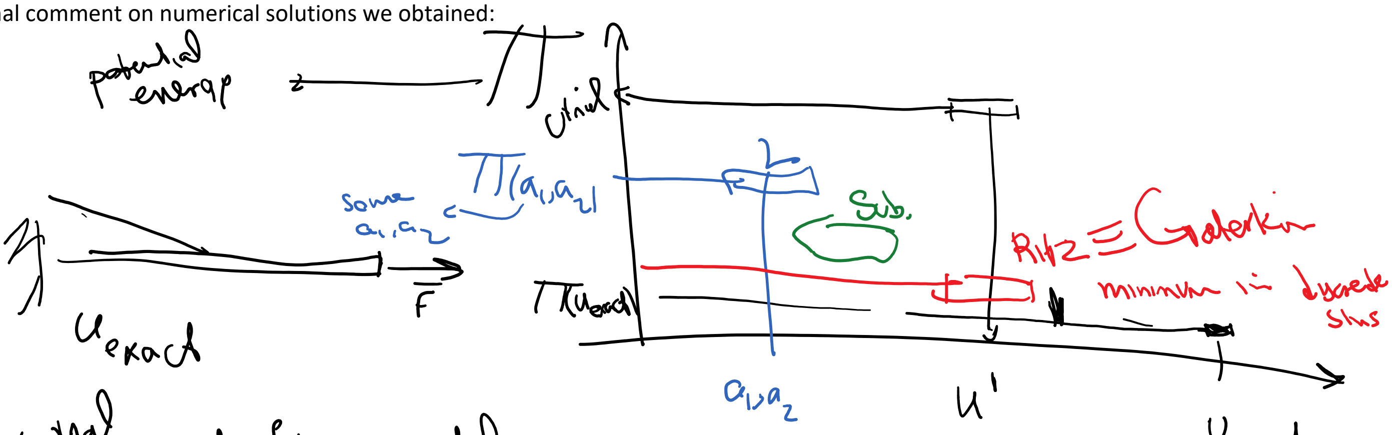
FEM

$$\text{error} = C h^{\alpha \operatorname{Min}(p, s) + b}$$

Basically, if the solution is not smooth enough (hydraulic jumps, shocks, …), it's NOT worth using higher order FEMs, Discontinuous

Galerkin methods, …, high order spectral methods, etc. because the error does not decrease that fast.

Final comment on numerical solutions we obtained:



$u' = u^{trial}$ satisfies essential BC

$u^h = \phi_0 + a_1 \phi_1 + a_2 \phi_2$ e.g. $\phi = \{x, x^2\}$

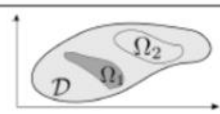
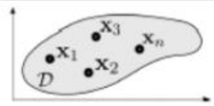
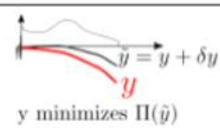
$\phi_0 = 1$

discrete trial function for n=2

Galerkin ≡ Ritz has the least energy compared to other approaches

In fact, one can show that for Galerkin method "energy of error" is minimum

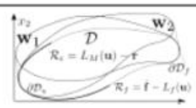
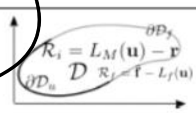
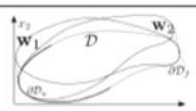
$$\Pi(\overbrace{u^h - u^{exact}}^{error})$$

Approach	Equation	Figure	Discretization
Balance Law (20)	$\forall\Omega \subset \mathcal{D}: \int_{\partial\Omega}(\mathbf{f}.\mathbf{n})\mathrm{ds} - \int_{\Omega}\mathbf{r}\,\mathrm{dv} = \mathbf{0}$		Change $\forall\Omega$ to $\{\Omega_1, \Omega_2, \dots, \Omega_n\}$
Strong Form (23)	$\forall\mathbf{x} \in \mathcal{D}: \nabla.\mathbf{f} - \mathbf{r} = \mathbf{0}$		Change $\forall\mathbf{x}$ to $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$
Energy Method (80)	$\forall\tilde{y} \in \mathcal{V}: \ \Pi(y) \le \Pi(\tilde{y})$		$\forall\{\tilde{a}_1, \dots, \tilde{a}_n\}$: $\Pi(a_1, \dots, a_n) \le \Pi(\tilde{a}_1, \dots, \tilde{a}_n) \Rightarrow \frac{\partial\Pi}{\partial a_1} = \dots = \frac{\partial\Pi}{\partial a_n} = 0$

Subdomain

Collocation

Galerkin = Ritz

Approach	Equation	Figure	Discretization
Weighted Residual Method (45)	$\forall\mathbf{w} \in \mathcal{W}$: $\int_{\mathcal{D}}\mathbf{w}.\mathcal{R}_i\ \mathrm{dv} + \int_{\partial\mathcal{D}_f}\mathbf{w}^f.\mathcal{R}_f\ \mathrm{ds} = \mathbf{0}$		Change $\forall\mathbf{w}$ to $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$
Least Square (51)	$R^2 = \int_{\mathcal{D}}\mathcal{R}_i^2\ \mathrm{dv} + \int_{\partial\mathcal{D}_f}\mathcal{R}_f^2\ \mathrm{ds} = \mathbf{0}$		Change $R^2 = 0$ to $\forall\{\tilde{a}_1, \dots, \tilde{a}_n\}$: $R^2(a_1, \dots, a_n) \le R^2(\tilde{a}_1, \dots, \tilde{a}_n) \Rightarrow \frac{\partial R^2}{\partial a_1} = \dots = \frac{\partial R^2}{\partial a_n} = 0$
Weak Form (74)	$\forall\mathbf{w} \in \mathcal{W}$ $\int_{\mathcal{D}} L_m^w(\mathbf{w}) L_m(\mathbf{u})\ \mathrm{dv} = \int_{\mathcal{D}}\mathbf{w}.\mathbf{r}\mathrm{dv} + \int_{\partial\mathcal{D}_f}\mathbf{w}.\bar{\mathbf{f}}\mathrm{ds}$		Change $\forall\mathbf{w}$ to $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$

$w = L_m^+(\phi)$

$w_f = -L_f^+(\phi)$

Galerkin

the only option for FEM