FEM2021/10/12

Tuesday, October 12, 2021 2:45 PM

Appendix: Function spaces (optional)

What functions can be used for weak statement:

Galerkin Weak Statement Function spaces

1. We first reduce the highest derivative order $M = 2m$ in the strong form (and weighted residual statement) to m in the weak statement.
2. Next, we observe that the functions should only be in $H^m(\mathcal{D})$. We observed that $H^m(\mathcal{D}) \subset C^{m-1}(\mathcal{D})$. In practice, the finite element trial functions that are in $C^{m-1}(\mathcal{D})$ are also $H^m(\mathcal{D})$.

Conventional (continuus) finite element methods:

Strong Form order $M = 2m$ $\Rightarrow$
Trial functions are C^{m-1} $\longrightarrow$ instead of C^m

Bar problem

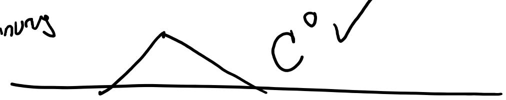
$(EAu')' + q = 0$

$M = 2 \rightarrow m = \frac{M}{2} = 1$

$\int w' EAu'$

$u, w \in C$ $\longrightarrow C^0 = m-1 = 1-1 = 0$ ✓ ✓

FEM Continuous



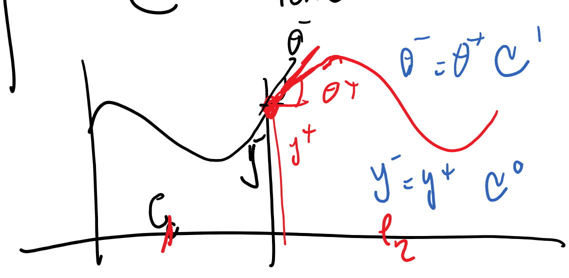
C^0 ✓

Beam problem

$(EIy'')'' + q = 0$

$M = 4 \rightarrow m = 2$

$\longrightarrow C^1$ functions



but derivative is not continuous — XC^1

we need y & y' to be continuous

1D elements

Element types:

1. 1D solid bar element.
2. Truss element.

Concepts:

1. Global (weighted residual) vs local (element level) perspectives.
2. Stiffness matrix.
3. Forces: 1.Source term; 2.Natural BC; 3.Essential BC, 4.Nodal.
4. Nodes, elements, shape function, dof.
5. Nodes with more than one dof (truss).
6. Element local coordinate system ξ (bar).
7. Rotation of element local coordinate system (truss).
8. Full stiffness $\mathbf{K}$ (free + prescribed dofs) vs (free only dofs) $\mathbf{K}_{ff}$.
9. High order differential equations (*e.g.*, C^1 beam elements).
10. Multiphysics coupling (beams: axial, bending, & torsional coupling).

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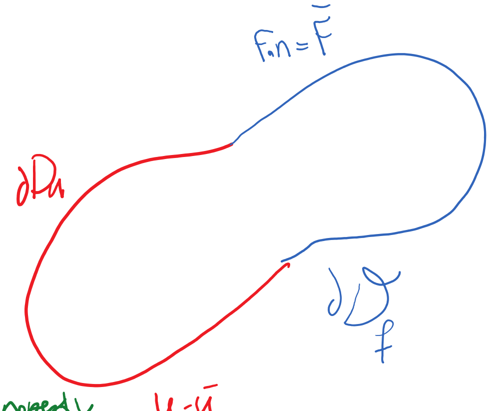
We almost always have the following form for weak statement (of self-adjoint problems):

$$\int_{\mathcal{D}} \underbrace{L_m(w)}_{\text{Differential operator for weak statement}} D\, L_m(u)\, dv = \int_{\mathcal{D}} w.q\, dv + \int_{\partial\mathcal{D}_f} w\bar{F}\, ds$$

$\mathcal{D}$ → domain

D → material (& section) property



Examples

1D bar	$w' EA u'$	$L_m = (\ )'$	$D = EA$
1D beam	$w'' EI y''$ (u'')	$L_m = (\ )''$	$D = EI$
2D, 3D elasticity	$\varepsilon(w)\, \mathcal{C}\, \varepsilon(u)$	$L_m = \varepsilon(\ ) = \frac{\nabla + \nabla^T}{2}$	$D = \mathcal{C}$ (elasticity stiffness $\tilde{\ }$)

$$\underbrace{\int_{\mathcal{D}} L_m(w)\, D\, L_m(u)\, dv}_{\text{bilinear form}} = \int_{\mathcal{D}} wq\, dv + \int_{\partial\mathcal{D}_f} w F\, ds$$

$\|$ $\|$ $\|$

$\mathcal{D}$ $\qquad\qquad\qquad\qquad$ $\mathcal{D}$ $\qquad\qquad$ $\partial\mathcal{D}_f$

bilinear form

$$\mathcal{A}(w,u) = (w,q)_\Gamma + (w,F)_N$$

$N \to$ Neumann

$$\mathcal{A}(f,g) = \int_{\mathcal{D}} L_m(f)\, D\, L_m(g)\, dV$$

$$(f,g)_\Gamma = \int_{\mathcal{D}} f g \, dV \qquad (f,g)_N = \int_{\partial\mathcal{D}_f} f g \, ds$$

Properties of $\mathcal{A}$

$$\mathcal{A}(w_1+w_2, u) = \int_{\mathcal{D}} L_m(w_1+w_2)\, D\, L_m(u)\, dV =$$

$$\overset{\text{why?}}{=} \int \left(L_m(w_1) + L_m(w_2)\right) D\, L_m(u)\, dV$$

$$= \int_{\mathcal{D}} L_m(w_1)\, D\, L_m(u)\, dV + \int_{D} L_m(w_2)\, D\, L_m(u)\, dV$$

$$= \mathcal{A}(w_1, u) + \mathcal{A}(w_2, u) \qquad \text{(i)}$$

why $\quad L_m(w_1+w_2) = L_m(w_1) + L_m(w_2)$

Examples for above $\quad$ bar $\quad L_m = (\ )'$

beam $\quad L_m = (\ )''$

2D, 3D elasticity $\quad L_m = (\nabla + \nabla^T)$

Similarly since $L_m(u_1+u_2) = L_m(u_1) + L_m(u_2) \longrightarrow$

$$\mathcal{A}(w, u_1+u_2) = \mathcal{A}(w,u_1) + \mathcal{A}(u_2) \qquad \text{(ii)}$$

So we have the linearity on w & u $\Rightarrow$ bilinear

$$\mathcal{A}(w_1+w_2, u) = \mathcal{A}(w_1, u) + \mathcal{A}(w_2, u)$$
$$\mathcal{A}(w, u_1+u_2) = \mathcal{A}(w, u_1) + \mathcal{A}(w, u_2) \qquad \text{I}$$

So, for problems with nonlinear response (nonlinear elasticity, plasticity, etc.) the linearity would only be on w and the weak statement would look different.

$$(w,q)_\Gamma = \int_\Omega wq\,dv \;\rightarrow\; (w_1+w_2, q)_\Gamma = (w_1, q)_\Gamma + (w_2, q)_\Gamma$$
$$(w,\bar F)_N = \int_{\partial\Omega_N} w\bar F\,ds \;\rightarrow\; (w_1+w_2, \bar F)_N = (w_1, \bar F) + (w_2, \bar F)_N \qquad \text{II}$$

we always have linearity w.r.t w in all terms

Let's derive the specific form of stiffness matrix and force vector for a given weak statement:

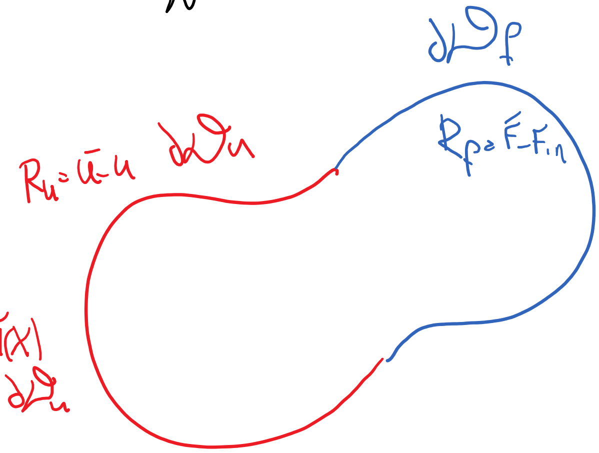
$$\forall w \in W^h \quad \mathcal{A}(w, \underbrace{u^h}_{\text{discrete solution}}) = (w,q)_\Gamma + (w,\bar F)_N$$

W^h: discrete set of weight functions

$$u^h = \phi_p + \sum_{i=1}^{n} a_i \phi_i$$

ϕ_p satisfies essential BC $\qquad \phi_p(x) = \bar u(x) \quad \forall x \in \partial\Omega_u$

ϕ_i " homog. " " $\qquad \forall i \quad \phi_i(x) = 0 \quad \forall x \in \partial\Omega_u$



$R_u = 0$ because of our choice of ϕ_p & ϕ_i's

$$\mathcal{A}(w, u^h) = (w,q)_\Gamma + (w,\bar F)_N$$

Galerkin method $\longrightarrow$ choose weight functions $\{w_1, \dots, w_n\} = \{\phi_1, \dots, \phi_n\}$

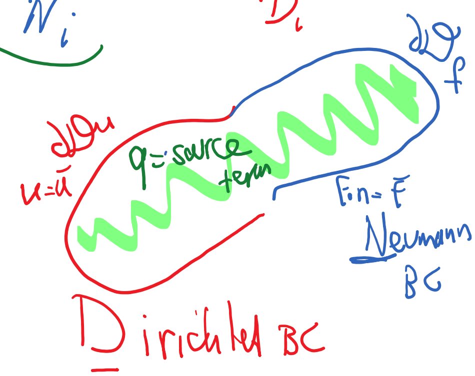
Equation $\# i \rightarrow w = \phi_i$

$$\underbrace{\mathcal{A}\left(\phi_i, \phi_p + \sum_{j=1}^{n} a_j\phi_j\right)} = (\phi_i, q)_\Gamma + (\phi_i, \bar{F})_N \qquad a_j\phi_j = \sum_{j=1}^{n} a_j \phi_j$$

$$\mathcal{A}(\phi_i, \phi_p) + \sum_{j=1}^{n} \mathcal{A}(\phi_i, \phi_j) a_j = (\phi_i, q)_\Gamma + (\phi_i, \bar{F})_N$$

$$\overbrace{\mathcal{A}(\phi_i, \phi_j) a_j}^{\sum_{j=1}^{n} \mathcal{A}(\phi_i, \phi_j) a_j} = (\phi_i, q)_\Gamma + (\phi_i, \bar{F})_N - \mathcal{A}(\phi_i, \phi_p)$$

$$\underbrace{K_{ij} a_j}_{\sum_{j=1}^{n}} = F_{\Gamma_i} + F_{N_i} - F_{D_i}$$



$$K = \int_\Omega L_m\left(\begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{bmatrix}_{n\times 1}\right) D\, L_m\Big(\overbrace{[\phi_1\ \phi_2 \cdots \phi_n]}^{1\times n}\Big)\, dV$$

$$K_{ij} = \int L_m(\phi_i)\, D\, L_m(\phi_j)\, dV \qquad K_{ji} = \int L_m(\phi_j)\, D\, L_m(\phi_i)\, dV = K_{ij}$$

self-adjoint PDE → symmetric K

$$F_\Gamma = \int_\Omega \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{bmatrix} q\, dV$$ source term force $F_{\Gamma_i} = \int_\Omega \phi_i\, q\, dV$

$$F_N = \int_{\partial\Omega_f} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{bmatrix} \bar{F}\, ds$$ Neumann BC force $F_{N_i} = (\phi_i, \bar{F})_N = \int_{\partial\Omega_f} \phi_i \bar{F}\, ds$

$$F_D = \int \left(\left(\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \right) \right) \cdots$$

$$F_D = \int_{\Omega} L_m\left(\begin{bmatrix}\phi_1\\ \phi_2 \\ \vdots \\ \phi_n\end{bmatrix}\right) D \; L_m(\phi_p)\, dV \qquad (*)$$

F_D: Dirichlet

Ω: Domain

D: material spec

Dirichlet BC

$F_{D_i} = \mathcal{A}(\phi_i, \phi_p)$

similar to

$K_{ij} = \mathcal{A}(\phi_i, \phi_j)$

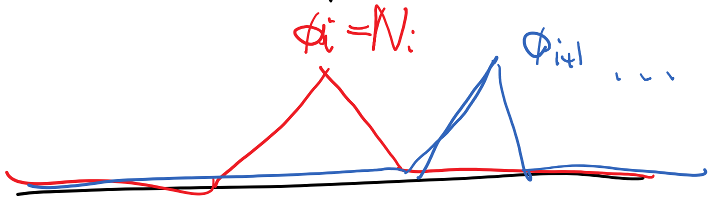
We can apply equation (*) to spectral methods (e.g. global high order polynomials, sin/cos terms, etc.) or finite element methods.

From here on, we only focus on FEM

We first take the GLOBAL approach, that is centered around the notation of basis function (called shape function in FEM) and global dofs

Global

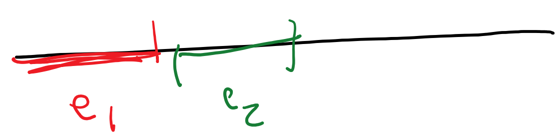
node & shape function centered



Not easy to use

Local

element-centered

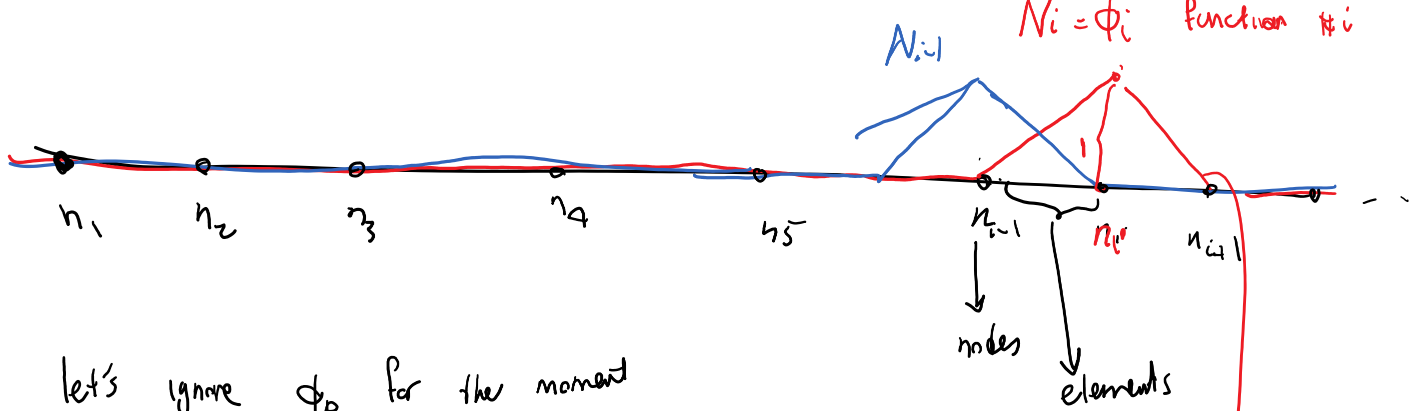


we'll transition to this later

In FEM, basis functions are called SHAPE functions and are denoted by N rather than ϕ

Shape functions take a value of 1 at one degree of freedom and zero elsewhere.

$N_i = \phi_i$ is shape (basis) function #i



let's ignore ϕ_p for the moment

nodes ↓ elements

let's ignore ϕ_p for the moment

$$u^h(x) = \cancel{\phi_p} + a_i \phi_i^{\nearrow N_i}$$

delta property

$$N_i(n_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} = \delta_{ij}$$

delta kronecker

$$\Rightarrow u^h(x) = a_i N_i(x)$$

only nonzero (eq 1) for $i=j$

$$u^h(n_j) = \sum_{i=1}^{n} a_i \underbrace{N_i(n_j)}_{\delta_{ij}}$$

$$= a_j$$

$$\boxed{a_j = u^h(n_j)}$$

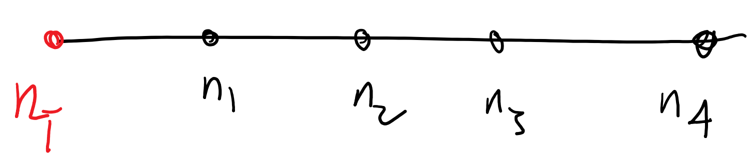
In continuous / conventional FEMs unknowns aj's are physical solutions @ dof number j. For bar problem, this is displacement at node j.

Reason for this, is the delta property mentioned above.

$$N_i(n_j) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

Definition n_f = # of free dofs (i.e. the # of unknowns)

HW3, FEM



$n_f = 4$ # unknowns

$n_p = 1$ # of prescribed dofs

Find equations for K, Fr, FN, FD (and Fn) for bar problem

A. Stiffness matrix

$$K = \int_{\mathcal{D}} L_m \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_{nf} \end{bmatrix} D \, L_m \left([N_1 \cdots \underbrace{N_{nf}}_{\#\text{ of free dofs}}] \right) dV$$

bar $D = EA$, $L_m = (\ )'$

$$K = \int \begin{bmatrix} N_1' \\ N_2' \\ \vdots \\ N_{nf}' \end{bmatrix} \underset{EA}{\underbrace{D}} \underbrace{[N_1' \cdots N_{nf}']}_{B_f} dV$$

For bar problem $\quad B_f = \frac{d}{dx} N_f$

$B_f \rightarrow$ "displacement-to-strain" matrix

why $\quad u^h = \cancel{\phi_f} + \sum_{j=1}^{nf} a_j N_j = [N_1 \cdots N_{nf}] \begin{bmatrix} a_1 \\ \vdots \\ a_{nf} \end{bmatrix}$

$$\varepsilon = (u^h)' = \left(\frac{d}{dx} [N_1 \cdots N_{nf}] \right) \begin{bmatrix} a_1 \\ \vdots \\ a_{nf} \end{bmatrix}$$

$$\boxed{\underbrace{\varepsilon(x)}_{\text{strain}} = \underbrace{[B_1(x) \cdots B_{nf}(x)]}_{\text{displacement to strain map}} \underbrace{\begin{bmatrix} a_1 \\ \vdots \\ a_{nf} \end{bmatrix}}_{\text{nodal displacements}}}$$