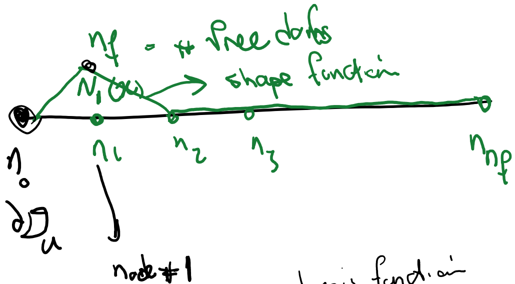
FEM2021/10/14

Thursday, October 14, 2021 2:52 PM

Bar problem



$$K = \int_0^L \underbrace{\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ \vdots \\ N_{nf} \end{bmatrix}'}_{B^t} \underbrace{EA}_{D} \underbrace{[N_1 \cdots N_{nf}]'}_{B} dx$$

D section - material property

basis function

$u^h = \varphi_p + \sum \phi_i a_i$ general

$u^h = \varphi_p + \sum N_i a_i$ FEM

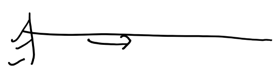
↳ shape function

$$K = \int_0^L B^t D B \, dx$$

As I'll discuss this is a general formula for stiffness

Bar problem

Weak statement (LHS)



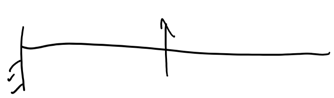
$$\int_0^L \underbrace{w'}_{L_m(w)} \underbrace{EA}_{D} \underbrace{u'}_{L_m(u)} dx = \cdots$$

$L_m() = ()'$

$N = [N_1 \; N_2 \cdots N_{nf}] \rightarrow B = L_m(N) = N'$

$= [N_1' \; N_2' \cdots N_{nf}']$

Beam problem



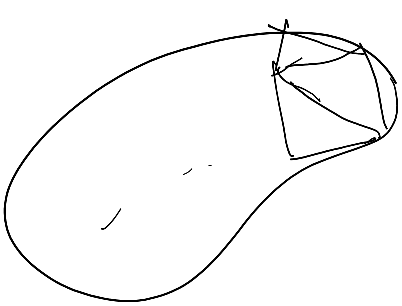
$$\int_0^L \underbrace{w''}_{L_m(w)} \underbrace{EI}_{D} \underbrace{y''}_{L_m(y)} dx = \cdots$$

$L_m() = ()''$

$N = [N_1, \ldots, N_{nf}] \rightarrow B = L_m(N) = N''$

$= [N_1'' \; N_2'' \cdots N_{nf}'']$

2D/3D heat conduction



$$\int_\Omega \underbrace{\nabla w}_{L_m(w)} \; \underbrace{k}_{D} \; \underbrace{\nabla T}_{L_m(T)} \, dV$$

conductivity $D = k_{2\times 2}$

pos. def

sym.

matrix

$L_m() = \nabla$

$N = [N_1, \ldots, N_{nf}] \rightarrow$

$$B = \nabla N = \begin{bmatrix} N_{1,1} & N_{2,1} & \cdots & N_{nf,1} \\ N_{1,2} & N_{2,2} & \cdots & N_{nf,2} \end{bmatrix}$$

$2 \times nf$

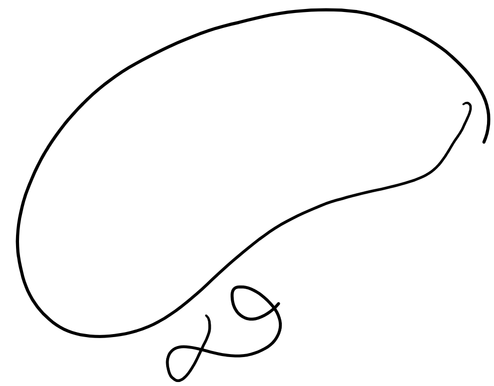
$$N_{i,j} = \frac{\partial N_i(x)}{\partial x_j}$$

$$\underset{n_f \times n_f}{K} = \int (B^t) D \underset{2\times n_f}{B} \, dV$$

$$\underset{n_f \times n_f}{K} = \int \underset{n_f \times 2}{(B)^t} \underset{2\times 2}{D} \underset{2 \times n_f}{B} \, dv$$

In general

$$K_{ff} = \int_{\partial\Omega} B_f^t \, D \, B_f \, dv \qquad L_m$$



	B	D
1D bar	$B = \frac{d}{dx} N$	EA
1D beam	$B = \frac{d^2}{dx^2} N$	EI
2D heat conduction	$B = \nabla N$	k
general	$B = L_m N$	. .

f refers to "free" (unknown) dof and many times, it simply is dropped

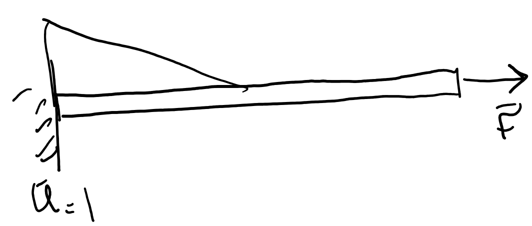
---

B. Essential Boundary Condition

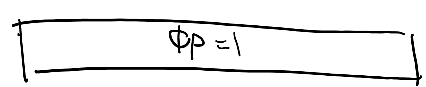
prev. section

$$u^h = \Phi_p + \sum a_i \phi_i$$

we chose $\Phi_p = 1$



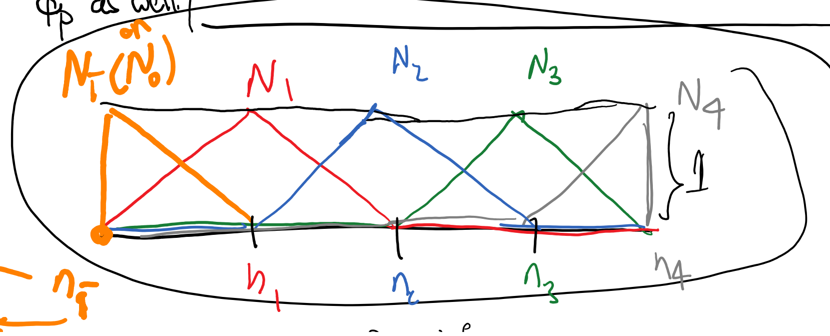
to fully transition to FEM we use shape functions to form Φ_p as well.



$$u^h = \Phi_p + (N_1 \, N_2 \, N_3 \, N_4) \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

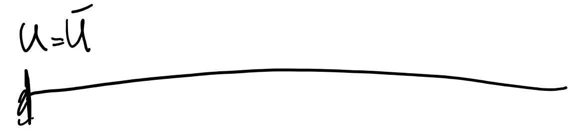
prescribed dof (node)

(-1)



n_0 $\quad n_1 \dots n_4$ are free dofs $\rightarrow n_f = 4$

$$\phi_p = \bar{u}\, N_0(x)$$

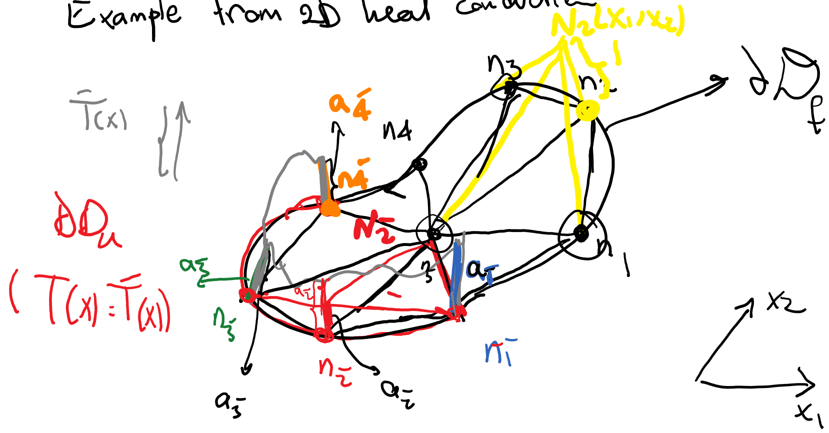


$$\phi_p(x=0) = \bar{u}\, \overbrace{N_0(x=0)}^{1} = \bar{u}$$

:)

In this example there is just 1 prescribed dof ($n_p = 1$)

Example from 2D heat conduction



① number free dofs (where we don't know T)

$n_f = 5$

② number prescribed dofs

$n_{\bar1}, n_{\bar2}, n_{\bar3}, n_{\bar4}$

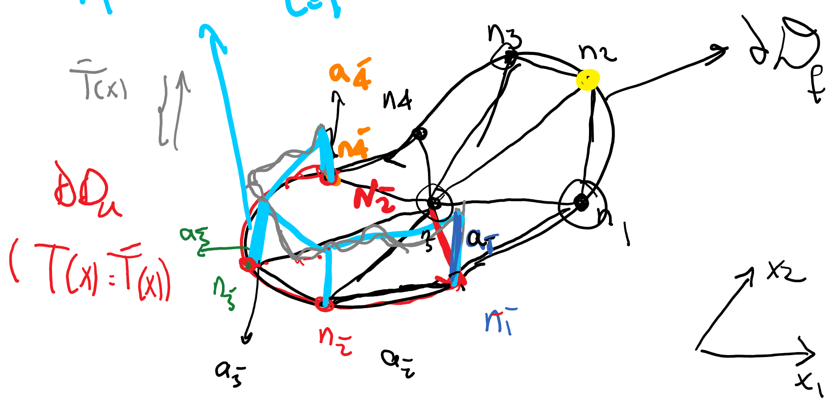
(can interpret $\bar3$ as -3)

$f(\vec{x}) = a_{\bar1} N_{\bar1}(\vec{x}) + a_{\bar2} N_{\bar2}(\vec{x}) + a_{\bar3} N_{\bar3}(x) + a_{\bar4} N_{\bar4}(x)$ what does this function do?

$f(n_{\bar1}) = a_1 \overbrace{N_{\bar1}(n_{\bar1})}^{1} + a_2 \overbrace{\cancel{N_{\bar2}(n_{\bar1})}}^{0} + a_{\bar3} \overbrace{\cancel{N_{\bar3}(n_{\bar1})}}^{0} + a_{\bar4} \overbrace{\cancel{N_{\bar4}(n_{\bar1})}}^{0} = a_{\bar1}$

$f(n_{\bar2}) = a_2 \underbrace{\quad n_{\bar2} \quad}_{0} \; a_{\bar2} \underbrace{N_{\bar2}(n_{\bar2})}_{1} + \underbrace{\quad}_{0} \; \underbrace{n_{\bar2} \quad n_{\bar2}}_{0} = a_{\bar2}$

$$\phi_p(\vec{x}) = \sum_{i=1}^{n_p} a_{\bar{i}}\, N_{\bar{i}}(\vec{x}).$$



①

②

$n_f = 5$

$n_p = 4$

as we can see $a_{\bar{1}} N_{\bar{1}} + a_{\bar{2}} N_{\bar{2}} + \dots + a_{\bar{4}} N_{\bar{4}}$

MATCHES ESSENTIAL BC @ all prescribed nodes :)

but it may not match it in between :(

Let's say we have the following errors
Satisfying essential BC: 0.002 (This is an acceptable error)
Discretization error (nf = 4 as opposed to infinity): 0.005
.... 0.003
...

In general, the error induced by potentially not satisfying essential BC is of the same order of other relevant errors and we are fine with it.

Summary:

General

$$T^h = \phi_p(\vec{x}) + \sum_{i=1}^{n_f} a_i \phi_i$$

$$\sum_{i=1}^{n_p} a_{\bar{i}} N_{\bar{i}}(\vec{x}) + \sum_{i=1}^{n_f} a_i N_i(\vec{x})$$

FEM

$$T^h = N_p a_p + N_f a_f$$

$$N_p = [N_{\bar{1}} \ \cdots \ N_{\bar{n}_p}] \quad a_p = \begin{bmatrix} a_{\bar{1}} \\ \vdots \\ a_{\bar{n}_p} \end{bmatrix}$$

$$N_f = [N_1 \ \cdots \ N_{n_f}] \quad a_f = \begin{bmatrix} a_1 \\ \vdots \\ a_f \end{bmatrix}$$

heat conduction

weak statement

$$\int_\Omega \nabla w \, k \, \nabla T \, dV$$

$L_m(w)$ $\quad L_m(T)$

$L_m = \nabla$

L_m

$$\nabla T^h = \nabla (N_p a_p + N_f a_f)$$

$$= (\nabla N_p) a_p + (\nabla N_f) a_f$$

$$\boxed{\nabla T^h = B_p a_p + B_f a_f}$$

In general

$$T^h = \sum^{n_p} \dots \sum^{n_f} \dots$$

In general

$$u^h = \underbrace{\sum_{i=1}^{n_p} a_{\bar{i}} N_{\bar{i}}(\bar{x})}_{\phi_p} + \sum_{i=1}^{n_f} a_i N_i$$

$$F_D = \mathcal{A}(N^t, \phi_p) = \int_{\mathcal{D}} L_m\left(\begin{bmatrix} N_1 \\ \vdots \\ N_{n_f} \end{bmatrix}\right) D \, L_m(\phi_p) \, dV$$

$$\phi_p = [N_{\bar{1}} \cdots N_{n_p}] \begin{bmatrix} a_{\bar{i}} \\ | \\ a_{\bar{n}_p} \end{bmatrix}$$

Linear differential operator

$$F_D = \int_{\mathcal{D}} \underbrace{L_m\left(\begin{bmatrix} N_1 \\ \vdots \\ N_{n_f} \end{bmatrix}\right)}_{B_f = (L_m(N_f))^t} D \underbrace{L_m([N_{\bar{1}} \cdots N_{\bar{n}_p}])}_{\underbrace{L_m(N_p)}_{B_p}} \begin{bmatrix} a_{\bar{1}} \\ a_{\bar{2}} \\ | \\ a_{\bar{n}_p} \end{bmatrix} dV$$

$$L_m(\ ) = B$$

$$F_D = \underbrace{\left(\int_{\mathcal{D}} B_f^t \, D \, B_p \, dV\right)}_{K_{fp}} a_p$$

Recall $\quad K_{ff} = \int_{\mathcal{D}} B_f^t \, D \, B_f \, dV$

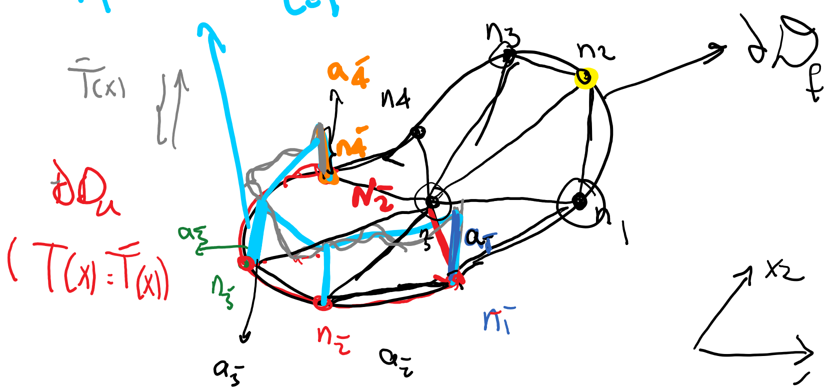
Summary of K and FD

$$K_{ff} = \int_\Omega \underbrace{B_f^t}_{n_f\times c} \underbrace{D}_{c\times c} \underbrace{B_f}_{c\times n_f} dV$$

$\underbrace{K_{ff}}_{n_f\times n_f}$

$$\underbrace{F_\Omega}_{n_f\times 1} = \underbrace{K_{fp}}_{n_f\times n_p} \underbrace{a_p}_{n_p\times 1} \qquad K_{fp} = \int_\Omega \underbrace{B_f^t}_{n_f\times c} \underbrace{D}_{c\times c} \underbrace{B_p}_{c\times n_p} dV$$

$$\Phi_p(\vec{x}) = \sum_{i=1}^{n_p} a_{\bar{i}} N_{\bar{i}}(\vec{x}).$$



①

②

$n_f = 5$

$n_p = 4$

$(K_{ff})_{5\times5}$

$K_{fp\,5\times4},\ a_{p\,4\times1} \to F_{D\,5\times1} = K_{fp}a_p$

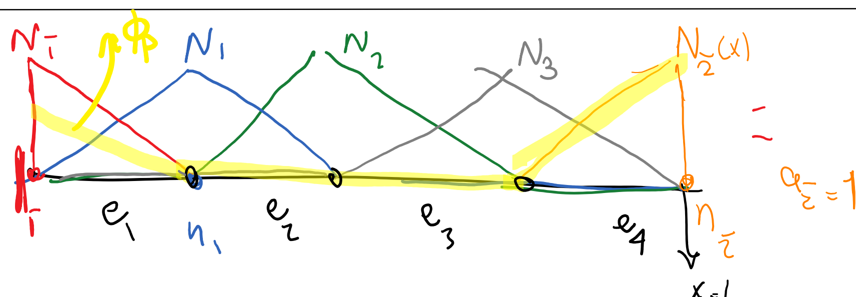
2D heat conduction

c = 2

1D bar example

$n_f = 3$

$a_{\bar 1} = .5$



$a_{\bar 2} = 1$

$$\Phi_p = N_{\bar 1} a_{\bar 1} + N_{\bar 2} a_{\bar 2} = .5 N_{\bar 1} + 1 N_{\bar 2}$$

$$(K_{ff})_{3\times3} = \int_0^L \underbrace{\begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}'}_{B_f^t = L_m(N_f)^t} EA \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix}' dx$$

$$(K_{fp})_{3\times2} = \int \underbrace{\begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}'}_{B_f^t} EA \underbrace{\begin{bmatrix} N_{\bar 1} & N_{\bar 2} \end{bmatrix}'}_{B_p} dx \qquad a_{\bar p} = \begin{bmatrix} .5 \\ 1 \end{bmatrix}_{2\times1}$$

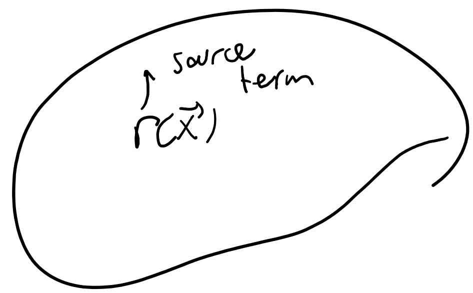
$$(F_D)_{3\times1} = (K_{fp})_{3\times2} (a_p)_{2\times1}$$

15f Dp 2x1 3x1 3x2 rx1

$K \qquad F_D \qquad F_r \qquad F_N \qquad\qquad F = F_r + F_N - F_D$

$$F_r = \int \begin{bmatrix} N_1 \\ \vdots \\ N_{nf} \end{bmatrix} q(x)\, dx$$

ie 2D, 3D



in general $$F_r = \int_{\Omega} \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_{nf}(\vec{x}) \end{bmatrix} r(\vec{x})\, dV$$

---

next time derive nodal forces (F_n)