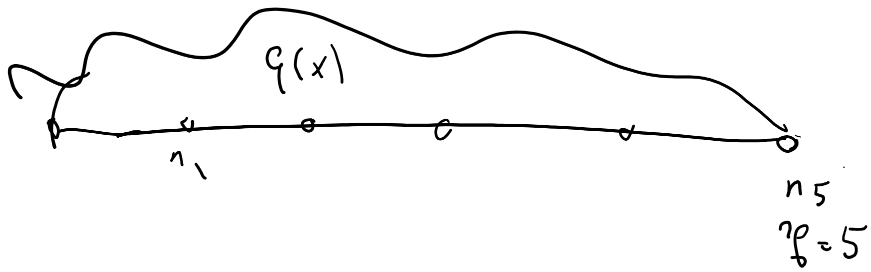
FEM2021/10/19

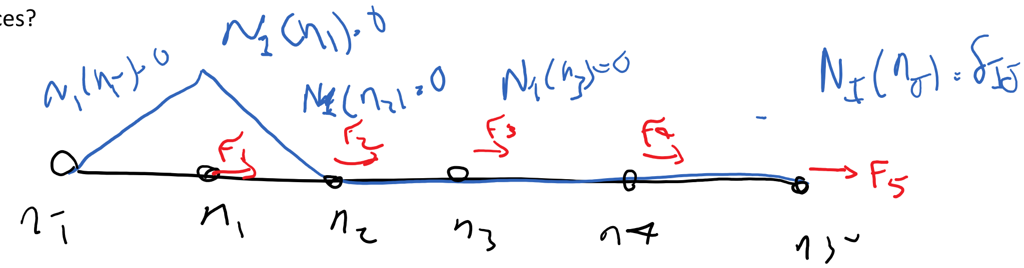
Tuesday, October 19, 2021 2:51 PM

From last time:

$$F_r = \int \begin{bmatrix} N_1 \\ \vdots \\ N_{nf} \end{bmatrix} q(x)\, dx \quad (1)$$



What about the case that we have point sources?



$$q(x) = F_1 \delta(x - x_1) + F_2 \delta(x - x_2) + \dots + F_5 \delta(x - x_5)$$

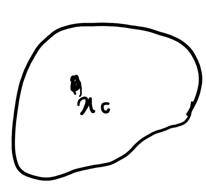
$$q(x) = \sum_{J=1}^{nf} F_J \delta(x - x_J) \qquad \text{source term corresponding to point sources}$$

from (1) $\quad (F_r)_I = \int_{\Omega} N_I(x)\, q(x)\, dx =$

$$\int_{\Omega} N_I(x) \left\{ \sum_{J=1}^{nf} F_J \delta(x - x_J) \right\} dx =$$

$$= \sum_{J=1}^{nf} \underbrace{N_I(x_J)} F_J$$

$$= \sum_{J=1}^{nf} \delta_{IJ} F_J = F_I$$



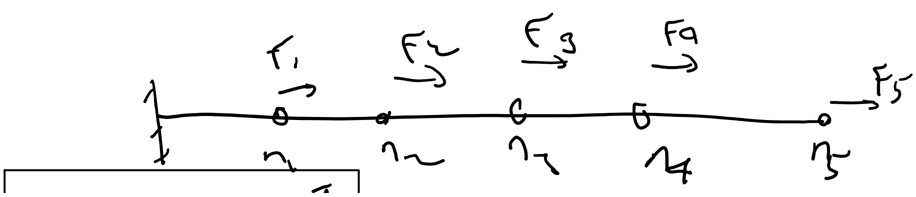
Note

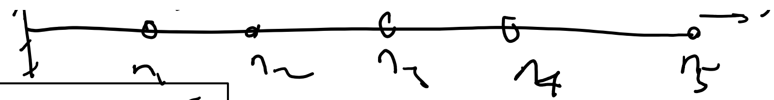
$$\int_{\Omega} \delta(x - x_0)\, g(x) = g(x_0) \quad \text{if } x_0 \in \Omega$$

$$= 0 \quad \text{if } x_0 \notin \Omega$$

$$F_{r_I} = F_I$$

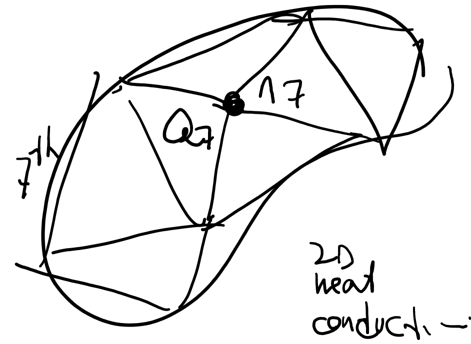
we denote this particular form of F_r as F_n (nodal forces)





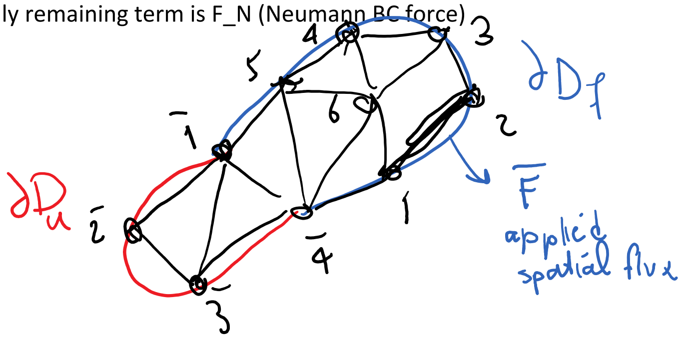
$$F_n = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{bmatrix}$$

$$F_n = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ Q_7 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \quad 7^{th}$$



2D heat conduction

The only remaining term is F_N (Neumann BC force)



$\bar{F}$ applied spatial flux

$n_f = 6$

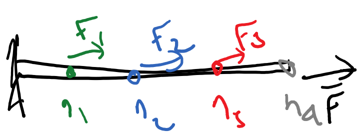
$n_p = 4$

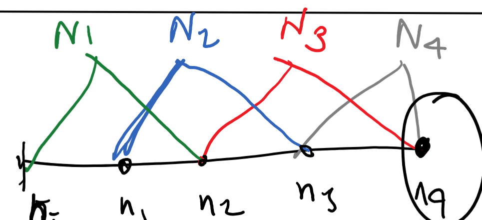
$$F_N = \int_{\partial D_f} \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_{n_f} \end{bmatrix} \bar{F}\, ds$$

here $F_N = \int_{\partial D_f} \begin{bmatrix} N_1 \\ \vdots \\ N_6 \end{bmatrix} \bar{q}\, ds$

$\bar{q}$: if 2D heat conduction

Example: 1D bar





$n_f = 4$

$n_p = 1$

$$F_n = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ \bar{F} \end{bmatrix}$$

F_n: nodal forces

$$\cancel{F_N = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \bar{F} \Big|_{x=L} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \bar{F} \end{bmatrix}}$$

$x = L$ is ∂D_f

don't form it in 1D

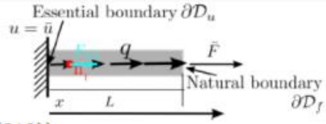
$$F = (F_r + F_N - F_D) + F_n$$

in 1D problems don't even form F_N and add its contribution to F_n only.

Summary: Force vectors

- Force vector is given by:

$$\boxed{\mathbf{F} = \mathbf{F}_r + \mathbf{F}_N + \mathbf{F}_n - \mathbf{F}_D} \quad (311)$$



- $\mathbf{F}_r$, $\mathbf{F}_N$, $\mathbf{F}_n$ and $\mathbf{F}_D$ are given by (*cf.* (301) and (310))

$$\mathbf{F}_r = \left(\mathbf{N}^{\mathrm{T}}, q\right)_r = \int_{\mathcal{D}} \mathbf{N}^{\mathrm{T}} q \,\mathrm{dv} = \int_0^L \begin{bmatrix} N_1 \\ \vdots \\ N_{n_\mathrm{f}} \end{bmatrix} q \,\mathrm{d}x \quad (312a)$$

$$\mathbf{F}_N = \left(\mathbf{N}^{\mathrm{T}}, \bar{F}\right)_N = \int_{\partial\mathcal{D}_f} \mathbf{N}^{\mathrm{T}} \bar{\mathbf{F}}.\mathbf{N} \,\mathrm{ds} = \left(\begin{bmatrix} N_1 \\ \vdots \\ N_{n_\mathrm{f}} \end{bmatrix} \bar{F} \right)_{x=L} \quad (312b)$$

$$\mathbf{F}_D = \mathcal{A}\left(\mathbf{N}^{\mathrm{T}}, \phi_p\right) = \int_{\mathcal{D}} \frac{\mathrm{d}}{\mathrm{d}x}\mathbf{N}^{\mathrm{T}} EA \frac{\mathrm{d}}{\mathrm{d}x}\phi_p \,\mathrm{dv} \quad (312c)$$

$$= \left\{ \int_{\mathcal{D}} \mathbf{B}^{\mathrm{T}} EA \bar{\mathbf{B}} \,\mathrm{dv} \right\} \bar{\mathbf{a}} = \left\{ \int_0^L \begin{bmatrix} B_1 \\ \vdots \\ B_{n_\mathrm{f}} \end{bmatrix} EA \begin{bmatrix} \bar{B}_{\bar{1}} & \cdots & \bar{B}_{\bar{n}_\mathrm{p}} \end{bmatrix} \mathrm{d}x \right\} \begin{bmatrix} \bar{a}_{\bar{1}} \\ \vdots \\ \bar{a}_{\bar{n}_\mathrm{p}} \end{bmatrix} = \mathbf{K}_{fp}\bar{\mathbf{a}}$$

$$\mathbf{F}_n = \begin{bmatrix} F_{n\,1} \\ \vdots \\ F_{n\,n_\mathrm{f}} \end{bmatrix} \quad (312d)$$

Force Essential Boundary Condition

- We have used (309) in (312c) to write,

$$\boxed{\mathbf{F}_D = \mathcal{A}\left(\mathbf{N}^{\mathrm{T}}, \phi_p\right) = \mathbf{K}_{fp}\bar{\mathbf{a}}} \quad (313)$$

- The prescribed to free stiffness matrix $\mathbf{K}_{fp}$ is an $n_\mathrm{f} \times n_\mathrm{p}$ matrix given by,

$$\mathbf{K}_{fp} = \int_{\mathcal{D}} \mathbf{B}^{\mathrm{T}} EA \bar{\mathbf{B}} \,\mathrm{dv} = \int_0^L \begin{bmatrix} B_1 \\ \vdots \\ B_{n_\mathrm{f}} \end{bmatrix} EA \begin{bmatrix} \bar{B}_{\bar{1}} & \cdots & \bar{B}_{\bar{n}_\mathrm{p}} \end{bmatrix} \mathrm{d}x \quad (314)$$

- From (306) we had,

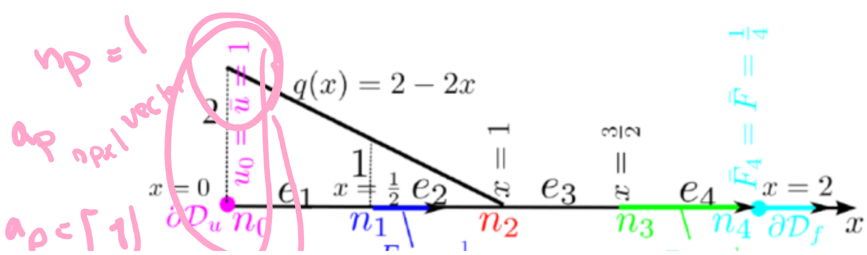
$$\mathbf{K} = \mathcal{A}\left(\phi^{\mathrm{T}}, \phi\right) = \int_{\mathcal{D}} \mathbf{B}^{\mathrm{T}} EA \mathbf{B} \,\mathrm{dv} = \int_0^L \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{n_\mathrm{f}} \end{bmatrix} EA \begin{bmatrix} B_1 & B_2 & \cdots & B_{n_\mathrm{f}} \end{bmatrix} \mathrm{d}x$$

where $\mathbf{K}$ was an $n_\mathrm{f} \times n_\mathrm{p}$ matrix.

- "Prescribed" dofs $\bar{i}$ do not go into $\mathbf{K}$ because their value $\bar{a}_{\bar{i}}$ are already known.
- This is opposite to dofs $I = 1, \ldots, n_\mathrm{f}$ which correspond to "free" dofs.

Numerical example from slide 251:

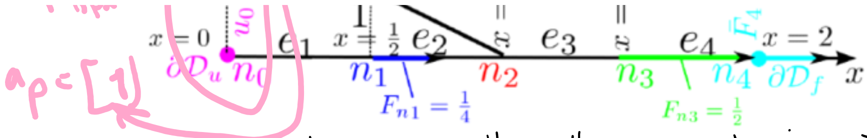
Bar Example: Overview



$$F = F_n + \cancel{F_N} + F_r - F_D$$

$K\,(K_{ff})$

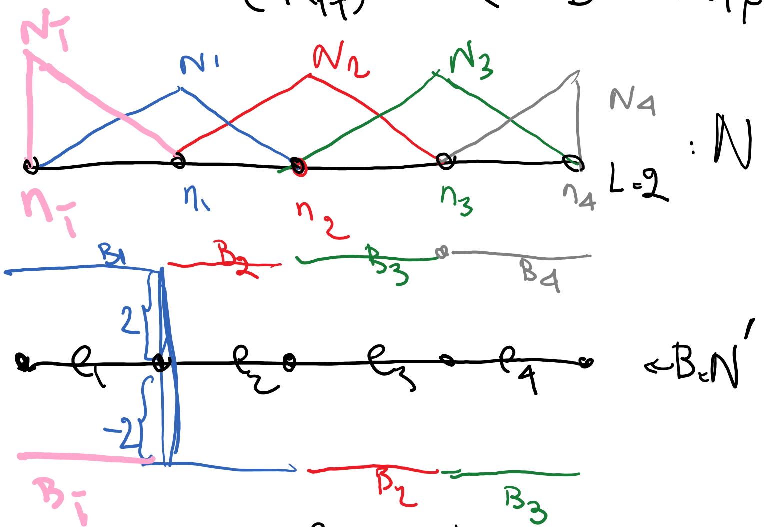
$n_f = 4$



similar to HW3 with the exception of added / different point forces

$$F_n = \begin{bmatrix} 1/4 \\ 0 \\ 1/2 \\ 1/4 \end{bmatrix}$$

Next: $K\ (K_{ff})$ & $F_D = K_{fp}\, a_p$



global view

color coding is based on node/dof #

Recall $K_{ff} = \int B_f^t\, D\, B_f\, dv$

for problem $B_f = L_m(N_f) = N_f'$

$D = EA$

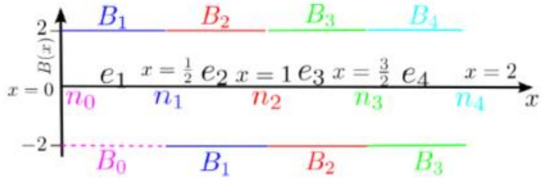
$$K_{ff} = \int_0^2 \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} \underset{= 1 \text{ for this problem}}{EA} [B_1\ B_2\ B_3\ B_4]\, dx$$

$K_{11} = \int_0^2 B_1 B_1\, dx = \int_{e_1} B_1 B_1\, dx + \int_{e_2} B_1 B_1\, dx = \int_{e_1} (2)(2)\, dx + \int_{e_2} (-2)(-2)\, dx = \frac{1}{2}(2)(2) + \frac{1}{2}(-2)(-2) = 4$

$K_{12} = \int_0^2 B_1 B_2\, dx = \int_{e_2} B_1 B_2\, dx = \int_{e_2} (-2)(2)\, dx = \frac{1}{2}(-2)(2) = -2$

the other components look the same and we get

$$K = \begin{bmatrix} 4 & -2 & 0 & 0 \\ & 4 & -2 & 0 \\ & & 4 & -2 \\ \text{sym} & & & 2 \end{bmatrix}$$



$$\mathbf{K} = \int_0^2 \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} EA \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \end{bmatrix} \mathrm{d}x = \begin{bmatrix} \int_0^2 B_1 B_1 \,\mathrm{d}x & \int_0^2 B_1 B_2 \,\mathrm{d}x & \int_0^2 B_1 B_3 \,\mathrm{d}x & \int_0^2 B_1 B_4 \,\mathrm{d}x \\ & \int_0^2 B_2 B_2 \,\mathrm{d}x & \int_0^2 B_2 B_3 \,\mathrm{d}x & \int_0^2 B_2 B_4 \,\mathrm{d}x \\ \text{sym.} & & \int_0^2 B_3 B_3 \,\mathrm{d}x & \int_0^2 B_3 B_4 \,\mathrm{d}x \\ & & & \int_0^2 B_4 B_4 \,\mathrm{d}x \end{bmatrix}$$

$$= \begin{bmatrix} \int_{e_1} B_1 B_1 \,\mathrm{d}x + \int_{e_2} B_1 B_1 \,\mathrm{d}x & \int_{e_2} B_1 B_2 \,\mathrm{d}x & 0 & 0 \\ & \int_{e_2} B_2 B_2 \,\mathrm{d}x + \int_{e_3} B_2 B_2 \,\mathrm{d}x & \int_{e_3} B_2 B_3 \,\mathrm{d}x & 0 \\ \text{sym.} & & \int_{e_3} B_3 B_3 \,\mathrm{d}x + \int_{e_4} B_3 B_3 \,\mathrm{d}x & \int_{e_4} B_3 B_4 \,\mathrm{d}x \\ & & & \int_{e_4} B_4 B_4 \,\mathrm{d}x \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}.(2).(2) + \frac{1}{2}.(-2).(-2) & \frac{1}{2}.(-2).(2) & 0 & 0 \\ & \frac{1}{2}.(2).(2) + \frac{1}{2}.(-2).(-2) & \frac{1}{2}.(-2).(2) & 0 \\ \text{sym.} & & \frac{1}{2}.(2).(2) + \frac{1}{2}.(-2).(-2) & \frac{1}{2}.(-2).(2) \\ & & & \frac{1}{2}.(2).(2) \end{bmatrix} \Rightarrow$$

$$\mathbf{K} = \begin{bmatrix} 4 & -2 & 0 & 0 \\ & 4 & -2 & 0 \\ \text{sym.} & & 4 & -2 \\ & & & 2 \end{bmatrix} \tag{316}$$

$$F_D = K_{fp} \, a_p$$

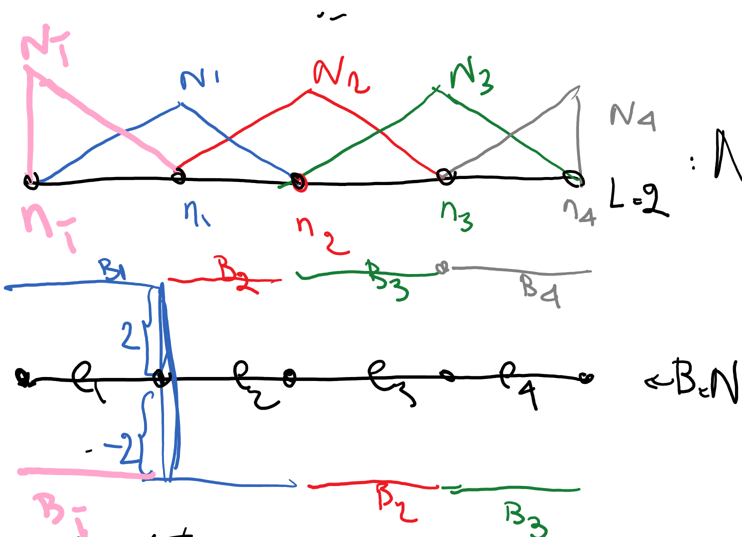
$$K_{fp} = \int_\Omega B_f \, D \, B_p \, dv$$

$$= \int_0^L [B_{\bar{1}}] \underset{1}{EA} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} dx$$

here

$$= \begin{bmatrix} \int_{e_1} B_{\bar{1}} B_1 \, dx \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \int_{e_1} \overset{B_{\bar{1}}}{(-2)} \overset{B_1}{(2)} \, dx \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow K_{fp}$$

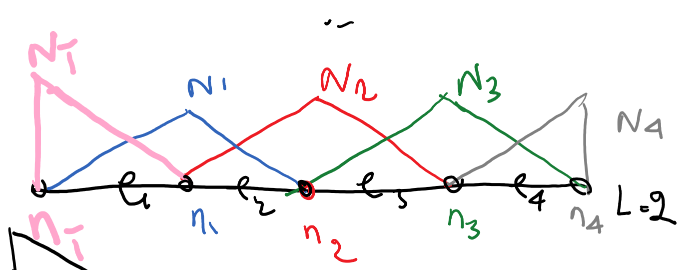
prescribed dofs ← $a_p = [1]$



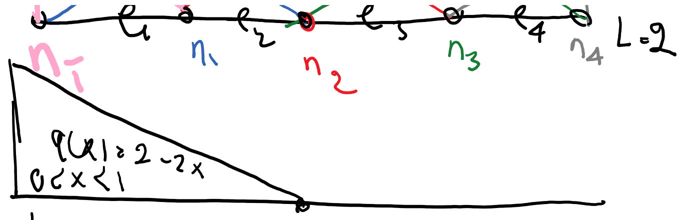
$$\rightarrow F_D = K_{fp} a_p = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix} [1] = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

$$F_r = ?$$

$$F_r = \int^2 \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} q(x) \, dx$$



$$f_r = \int_0 \begin{bmatrix} N_2 \\ N_3 \\ N_4 \end{bmatrix} q(x)\, dx$$

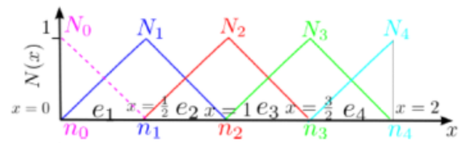


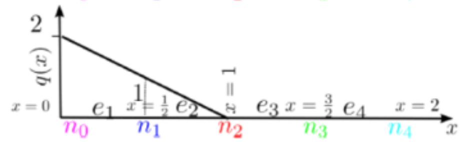
$$= \begin{bmatrix} \int_{e_1} N_1 q(x)\,dx + \int_{e_2} N_1 q\,dx \\ \int_{e_2} N_2 q(x)\,dx \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/12 \\ 0 \\ 0 \end{bmatrix}$$

Similar to HW3

More detailed calculation of this is below:

Bar Example: Step 2.1: Source term force





From (312a),

$$\mathbf{F}_r = \int_0^L \begin{bmatrix} N_1 \\ \vdots \\ N_{n_f} \end{bmatrix} q\,dx = \begin{bmatrix} \int_0^2 N_1(x)q(x)\,dx \\ \int_0^2 N_2(x)q(x)\,dx \\ \int_0^2 N_3(x)q(x)\,dx \\ \int_0^2 N_4(x)q(x)\,dx \end{bmatrix} = \begin{bmatrix} \int_{e_1} N_1(x)q(x)\,dx + \int_{e_2} N_1(x)q(x)\,dx \\ \int_{e_2} N_2(x)q(x)\,dx \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1/2}{6}\left((2).(0).(2) + (2).(1).(1) + (0).(1) + (1).(2)\right) + \frac{1/2}{6}\left((2).(1).(1) + (2).(0).(0) + (1).(0) + (0).(1)\right) \\ \frac{1/2}{6}\left((2).(0).(1) + (2).(1).(0) + (0).(0) + (1).(1)\right) \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$

$$\mathbf{F}_r = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{12} \\ 0 \\ 0 \end{bmatrix} \qquad (317)$$

255 / 456

$$K = K_{ff} = \begin{bmatrix} 4 & -2 & 0 & 0 \\ & 4 & -2 & 0 \\ \text{sym} & & 4 & -2 \\ & & & +2 \end{bmatrix}$$

$$F = \underbrace{F_r + \overset{1D}{\cancel{F_N}} - F_D}_{F_e \text{ element contributions}} + F_n = \begin{bmatrix} 1/2 \\ 1/12 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1/4 \\ 0 \\ 1/2 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 11/4 \\ 1/12 \\ 1/2 \\ 1/4 \end{bmatrix}$$

$$K = \Big[\qquad \Big[\ 43/24 \ \Big]$$

$$[1/4]$$

$$Ka = F \longrightarrow \text{Solve } a = \begin{bmatrix} 43/24 \\ 53/24 \\ 31/12 \\ 65/24 \end{bmatrix}$$

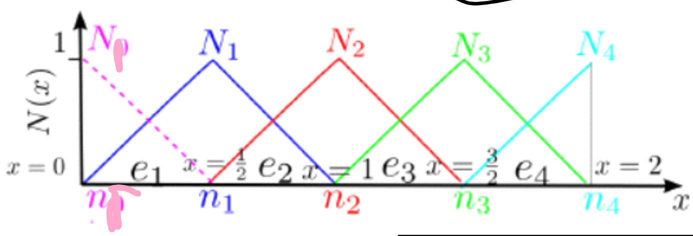
physical meaning of a's

$$u^h = N_p a_p + N_f a_f = [N_T] a_T + [N_1 \; N_2 \; N_3 \; N_4] \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

$$u^h = a_T N_T + a_p N_1 + a_2 N_2 + a_3 N_3 + a_4 N_4$$

we knew this before since $a_T = 1$

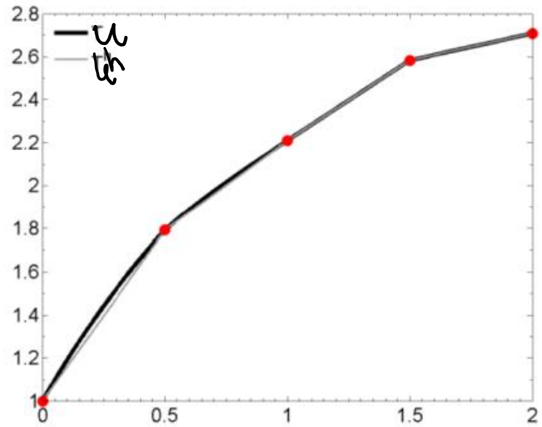
$$u^h(n_2) = \underbrace{a_T \overbrace{N_T(n_2)}^{0} + a_1 \overbrace{N_1(n_2)}^{0} + a_2 \overbrace{N_2(n_2)}^{1} + a_3 \overbrace{N_3(n_2)}^{0} + a_4 \overbrace{N_4(n_2)}^{0}}_{= a_2}$$



in fact

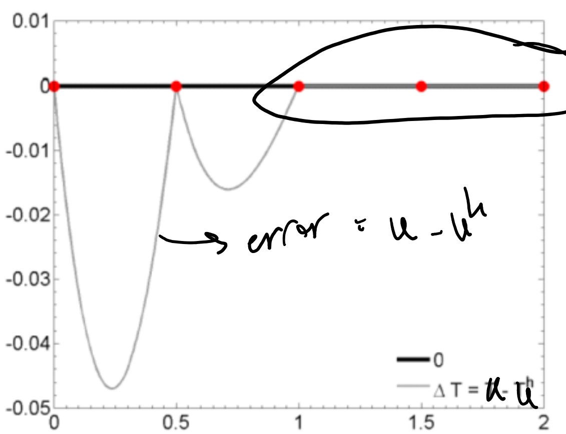
$$\boxed{\begin{array}{l} u^h(n_i) = a_i. \\ u^h(n_p) = a_T \quad \text{in course notes I call } T \text{ "0"} \end{array}}$$

Bar Example: solution values



- u^h and u match at all nodes n_0, n_1, n_2, n_3, and n_4. This holds for 1D solid elements with uniform AE and **does not hold** in general.

Bar Example: error in solution values

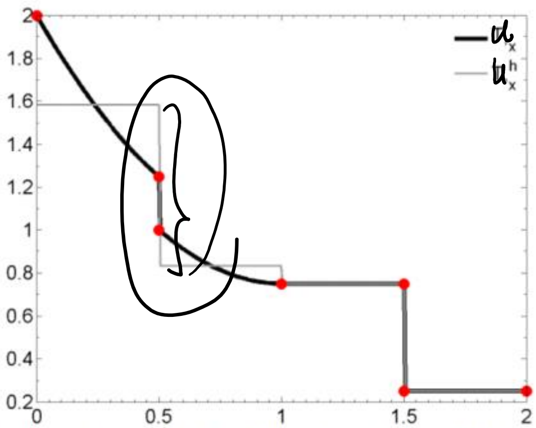


error = $u - u^h$

error = 0 @ nodes (1D elements in most cases)

error $\sqrt{\int |u^h - u|^2 dx} \propto C h^2$ for linear element

Bar Example: solution derivatives ($\propto$ axial force)



$$\varepsilon = \frac{du}{dx}$$

error in ε: $\sqrt{\int |\varepsilon^h - \varepsilon|^2 dx}$

$$\approx Ch$$

- The errors in solution derivative is larger than those in the solution itself. In general, the accuracy of FE solution decreases for solution derivatives (*e.g.*, strains, stresses, *etc.*).
- Approximate solution u^h exhibits jumps in $\frac{du^h}{dx}$ at all interior nodes. This is because the solution is piece-wise constant in $H^1([0\ 2])$.
- Even the exact solution exhibits jumps in $\frac{du}{dx}$ at n_1 and n_3 from the concentrated forces.
- The $H^1([0\ 2])$, rather than $C^1([0\ 2])$, is the right solution space for u and u^h as none of them belong to the latter space.

264 / 456

for each successive derivative we lose one order of accuracy in error convergence w.r.t. h

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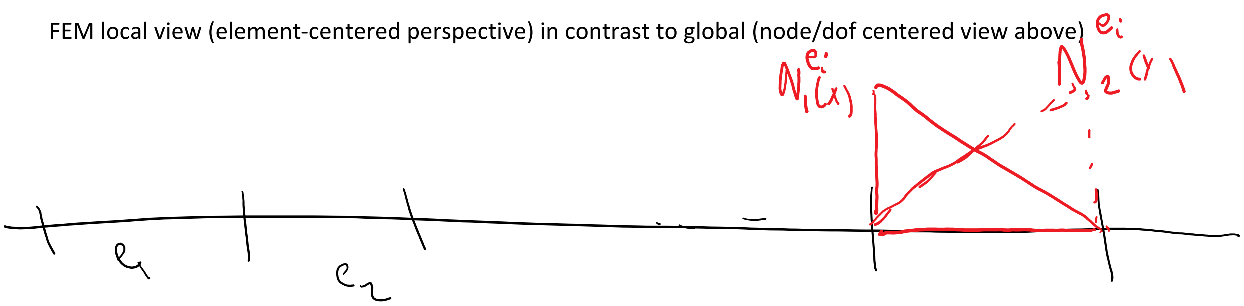
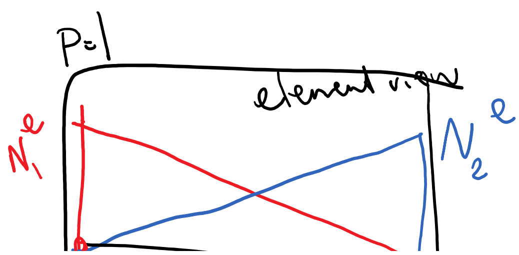
FEM local view (element-centered perspective) in contrast to global (node/dof centered view above)

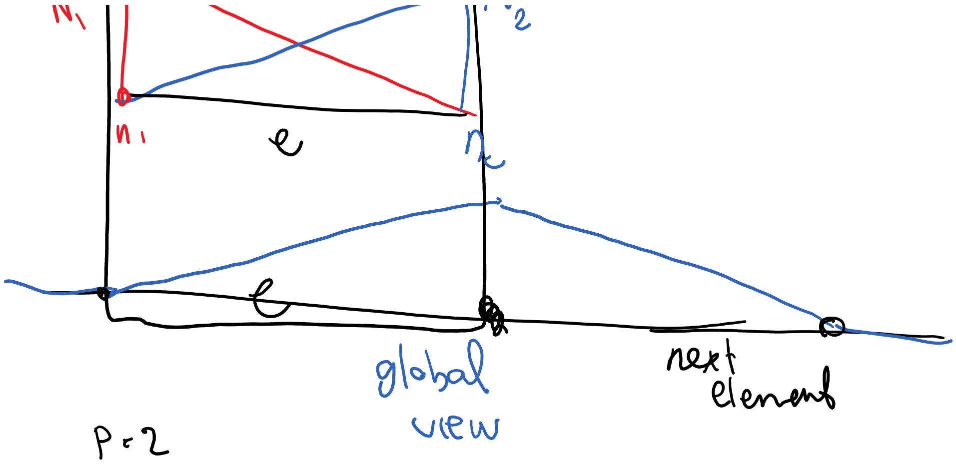


$$N_j^{e_i}(n_k^{e_i}) = \delta_{jk}$$

shape function is only nonzero @ its corresponding node (dof)

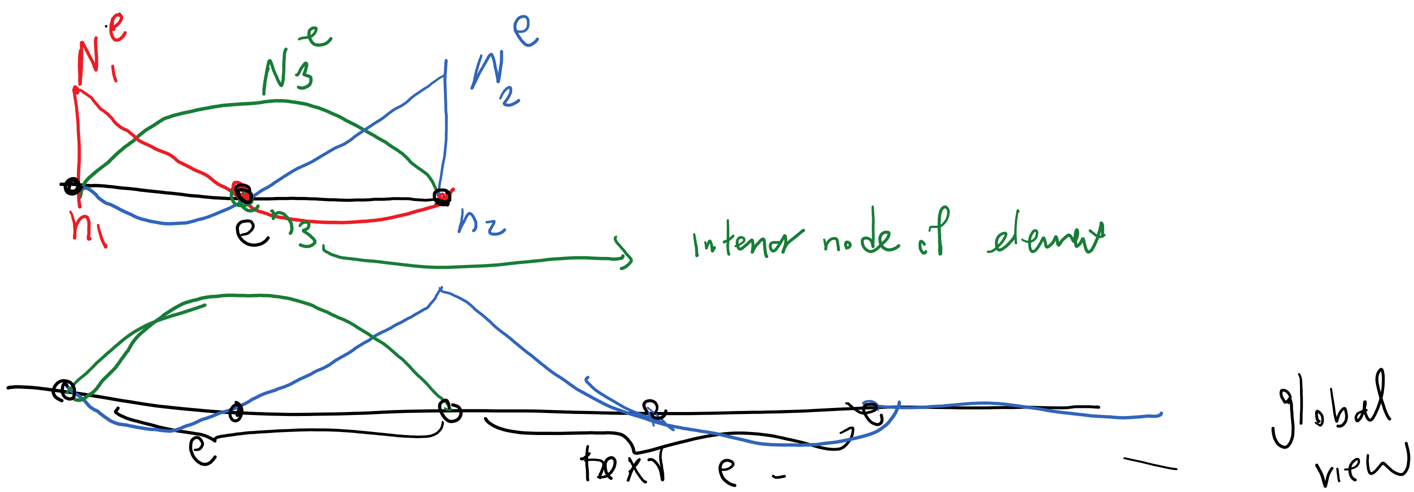
higher order elements





N_1
N_2
n_1
e
n_2
e
global view
next element

p = 2



N_1^e
N_3^e
N_2^e
n_1
e
n_3
n_2
interior node of element
e
next e
global view