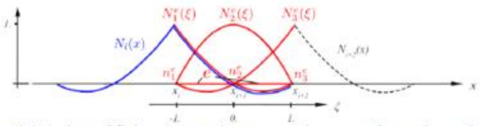
FEM2021/10/21

Thursday, October 21, 2021 2:50 PM

## Element shape functions to global shape functions



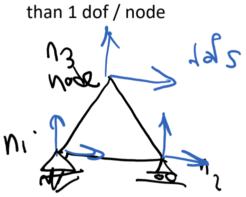

- While the global view of finite element has some advantages in mathematical analysis, we often form the shape functions at the local level and if needed form global shape functions.
- It was this local perspective that first was employed in engineering finite element analysis.
- For example, in the figure the 1D bar element has three nodes with one being internal node and has interpolation order $p = 2$.
- We observe that,

$$N_i^e(n_j^e) = \delta_{ij} \quad \Rightarrow N_I(n_J) = \delta_{IJ}$$

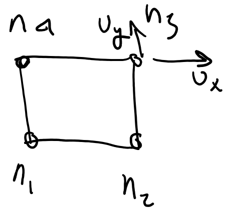
  which was the condition we first stipulated for finite elements in global view.
- As an example, we observe that the global shape function $N_i(x)$ is formed from local element shape functions.
- Notice that while local element order is $p = 2$ the global shape functions are still $C^0$ (piece-wise quadratic in this case).
- Elements can have internal nodes. This generally occurs for higher than linear elements ($p > 1$).



Nodes and dofs are distinct. In many problems there are more than 1 dof / node




2 dofs/node



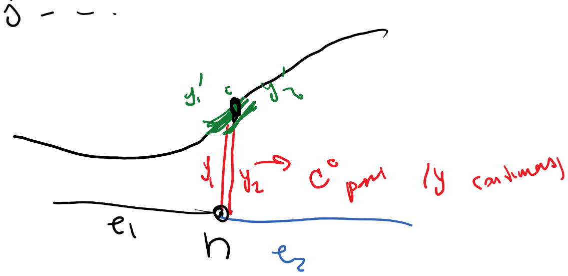

2 dofs / node → displacement

2D elasticity

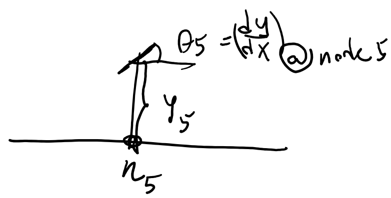
2D / 3D problems

beam : PDE $(EIy'')'' + q = 0$ $M = 4 \to m = 2$ $C^{m-1} = \underbrace{C^1}_{\text{continuity needed}}$

Recall $\int w'' EI y'' \ldots$

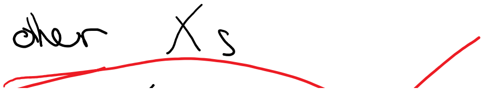



2 dofs per node




$m > 1$

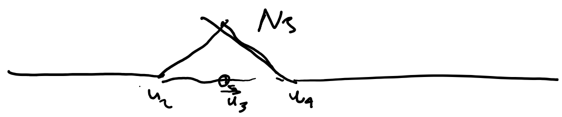
is shape functi 1 @ one X and zero @ other Xs



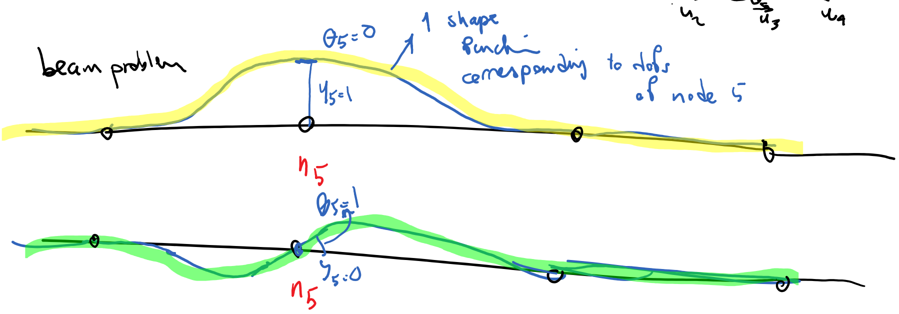
other $X_s$

① $X_i$ : dof

② $X_i$ : node

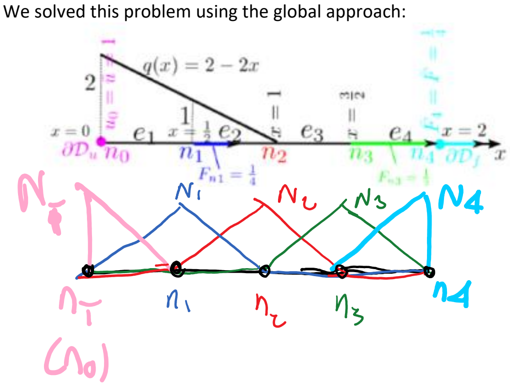

beam problem




all dofs are zero except 1st dof of node 5

all dofs are zero except 2nd dof of node 5

We solved this problem using the global approach:



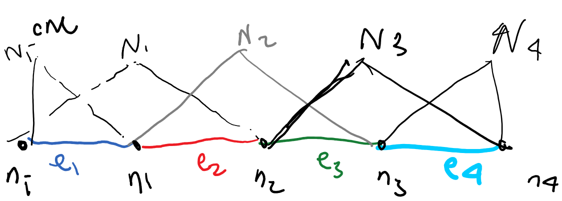

Global or node/dof centered

You are not going to use this anymore because it's cumbersome

$$K = \int_0^2 B^t D B \, dx \rightarrow K_{12} = \int_0^2 B_1 \, \widetilde{EA} \, B_2 \, dx = \int_{e_2} B_1 B_2 \, dx$$

$$K_{11} = \int_{e_1} B_1 B_1 \, dx + \int_{e_2} B_1 B_1 \, dx$$

in the "local" or "element-centered" approach we take care of all things pertained to an element and then move to the next one

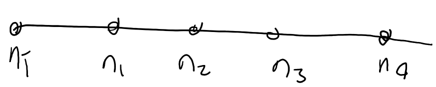



$$K = \int_{\mathcal{D}=[0,2]} B^t D B \, dx = K_{e_1} + K_{e_2} + K_{e_3} + K_{e_4}$$

$$K_{e_i} = \int_{e_i} B^t D B \, dx$$

$B_1 = 2$

$$K_{e_1} = \int_{e_1} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} [B_1 \; B_2 \; B_3 \; B_4] \, dx = \int_{e_1, x=0}^{x=\frac{1}{2}} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} [2 \; 0 \; 0 \; 0] = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

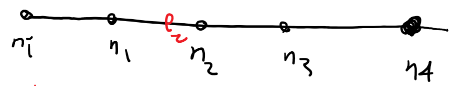



$B_2 = 2$

$$= \int_{e_1} \begin{bmatrix}2\\0\\0\\0\end{bmatrix}[2\ \ 0\ \ 0\ \ 0] = \begin{bmatrix}2&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

$B_2 = 2$

$$K_{e_2} = \int_{e_2} \begin{bmatrix}B_1\\B_2\\B_3\\B_4\end{bmatrix}[B_1\ \ B_2\ \ B_3\ \ B_4]\,dx =$$




$B_1 = -2$

$$\int_{e_2} \begin{bmatrix}-2\\2\\0\\0\end{bmatrix}[-2\ \ 2\ \ 0\ \ 0]\,dx = \begin{bmatrix}2&-2&0&0\\-2&2&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

$B_3 = 2$

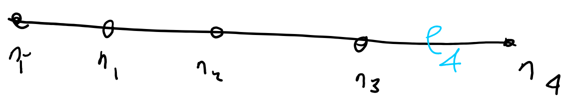
$$K_{e_3} = \int_{e_3} B^T D B = \int_{e_3} \begin{bmatrix}0\\-2\\2\\0\end{bmatrix}[0\ \ -2\ \ 2\ \ 0]$$


$B_2 = -2$

$$= \begin{bmatrix}0&0&0&0\\0&2&-2&0\\0&-2&2&0\\0&0&0&0\end{bmatrix}$$

$B_4 = 2$

$$K_{e_4} = \int_{e_4} B^T D B = \int_{e_3} \begin{bmatrix}0\\0\\-2\\2\end{bmatrix}[0\ \ 0\ \ -2\ \ 2]$$




$B_3 = -2$

$$= \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&2&-2\\0&0&-2&2\end{bmatrix}$$

$$K = K_{e_1} + K_{e_2} + K_{e_3} + K_{e_4} = \begin{bmatrix}2+2&-2&&\\-2&2+2&-2&\\&-2&2+2&-2\\&&-2&2\end{bmatrix}$$

$$K = \begin{bmatrix}4&-2&&\\-2&4&-2&\\&-2&4&-2\\&&-2&2\end{bmatrix}$$

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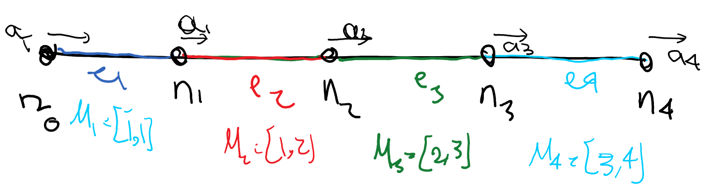
local element map

To fully utilize element-centered approach we define $\underline{\underline{2}}$ maps

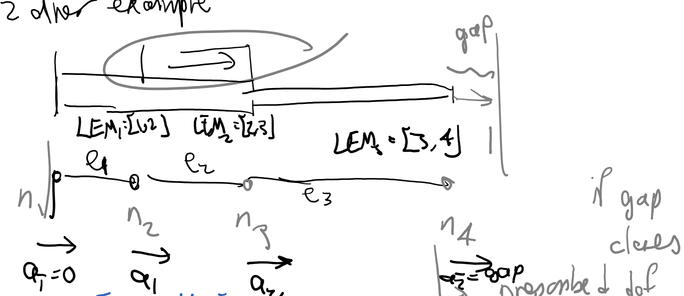
$LEM_1 = [0,1]$ $LEM_2 = [1,2]$ $LEM_3 = [2,3]$ $LEM_4 = [3,4]$




there are
- 4 free dofs ($n_f = 4$)
- 1 prescribed dof ($n_p = 1$)

M, or dof is is the map of element dofs

2 other example




$M_1=[\bar{1},1]$  $M_2=[1,2]$  $M_3=[2,\bar{2}]$

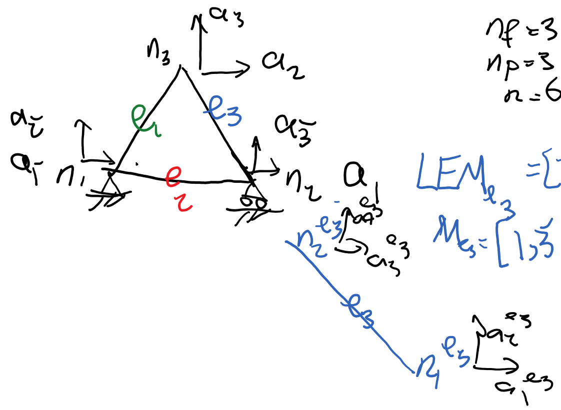
$n_f=2$
$n_p=2$

LEM = element nodal map

M = element dof map

we'll use this now

$n_f=3$
$n_p=3$
$n=6$



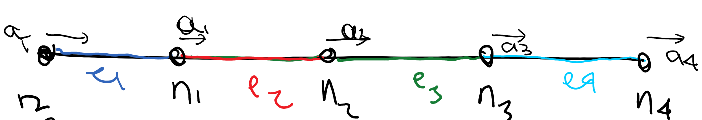

$LEM_{e_3}=[2,3]$

$M_{e_3}=[1,\bar{3},2,\bar{3}]$

element map

o-u-y utilize element-centered app

$LEM_1=[0,1]$  $LEM_2=[1,2]$  $LEM_3=[2,3]$  $LEM_4=[3,4]$




$M_1=[\bar{1},1]$  $M_2=[1,2]$  $M_3=[2,3]$  $M_4=[3,4]$

we'll show $K^e=\frac{(EA)^e}{L^e}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}$

for all elements here $EA=1$, $L=\frac{1}{2}$ $\rightarrow$ $K^{e_1}=\dots=K^{e_4}=\begin{bmatrix}2&-2\\-2&2\end{bmatrix}$

$K^{e_1}=\begin{array}{c|cc} & \bar{1} & 1\\ \hline \bar{1} & 2 & -2\\ 1 & -2 & 2\end{array}$  $K^{e_2}=\begin{array}{c|cc} & 1 & 2\\ \hline 1 & 2 & -2\\ 2 & -2 & 2\end{array}$  $K^{e_3}=\begin{array}{c|cc} & 2 & 3\\ \hline 2 & 2 & -2\\ 3 & -2 & 2\end{array}$  $K^{e_4}=\begin{array}{c|cc} & 3 & 4\\ \hline 3 & 2 & -2\\ 4 & -2 & 2\end{array}$

assembly

$K=\begin{array}{c|cccc} & 1 & 2 & 3 & 4\\ \hline 1 & 2+2 & -2 & & \\ 2 & -2 & 2+2 & -2 & \\ 3 & & -2 & 2+2 & -2\\ 4 & & & -2 & 2\end{array}$

we get the same K $\begin{bmatrix}4 & -2 & \dots\end{bmatrix}$

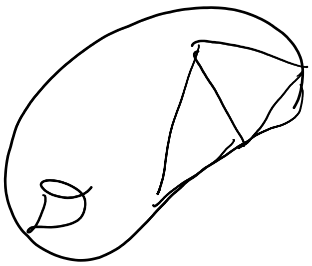
we get the same $K = \begin{bmatrix} 4 & -2 & & \\ -2 & 4 & -2 & \\ & -2 & 4 & -2 \\ & & -2 & 2 \end{bmatrix}$

we'll discuss this next

$$f_D^e = k^e a^e$$

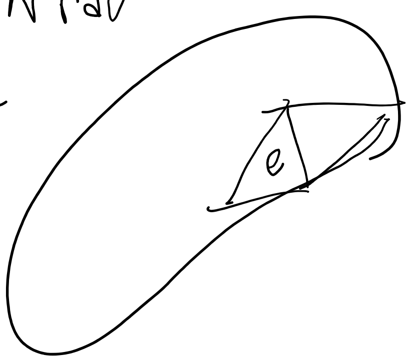
## Global Approach

$$K = \int_\Omega B^t D B \, dV$$



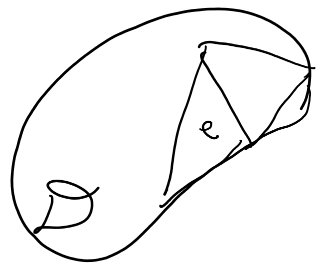

$$F = \underbrace{(F_r + F_N - F_D)}_{\text{Element contributions } F_e} + \underbrace{F_n}_{\text{nodal forces}}$$

$$F_r = \int_\Omega N^T r \, dV$$




## Local (element) approach

$$k^e = \int_e B^{eT} D B^e$$

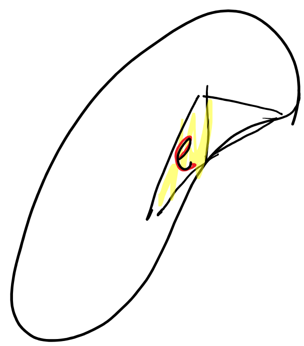



$$F = F^e + F^n$$

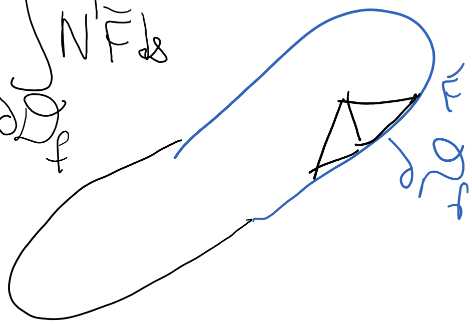
$$f^e = f_r^e + f_N^e - f_D^e$$

"assembled" to $F^e$

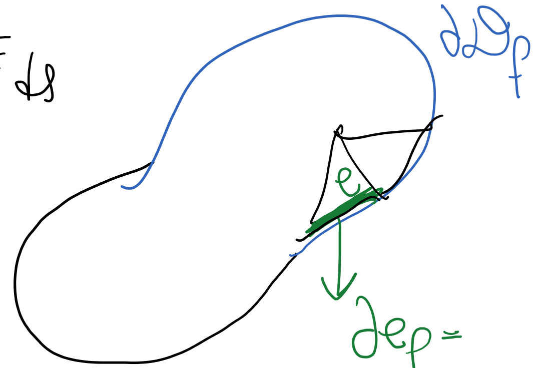
$$f_r^e = \int_e N^{eT} r \, dV$$



$$F_N = \int_{\partial\Omega_f} N^T \bar{F}\, ds$$




$$f_N^e = \int_{\partial e_f} N^{e^T} \bar{F}\, ds$$




$$\partial e_f = \partial e \cap \partial\Omega_f$$

$$F_D = K_{fp}\, a_p$$

$$K_{fp} = \int_{\Omega} B_f^T D B_p \, dV$$

$$f_D = k^e a^e$$

very simple :)