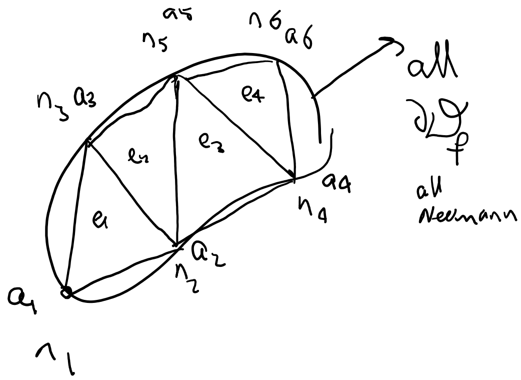
FEM2021/10/26

Tuesday, October 26, 2021 2:51 PM

At the element level we have 3 forces:

$$f^e = \underset{\text{Source}}{f^e_r} + \underset{\text{Neumann}}{f^e_q} - \underset{\text{Dirichlet}}{f^e_D}$$



$n_f = 6$

$n_p = 0$ (- - -)

$$F_r = \int_\Omega \begin{bmatrix} N_1 \\ \vdots \\ \vdots \\ N_6 \end{bmatrix} Q \, dv$$

heat conduction problem:

$r = Q$

2D heat conduction

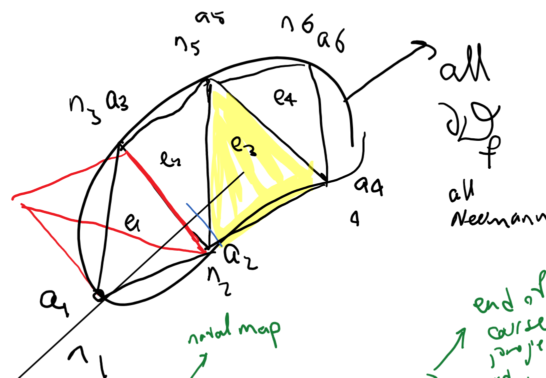
T (temperature) unknown

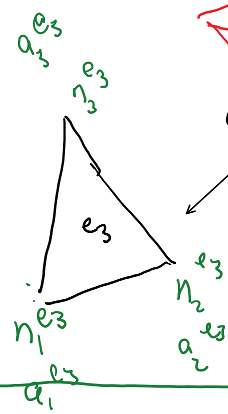
1 dof / node

$$= \underbrace{\int_{e_1} N^t Q \, dv}_{F_r^{e_1}} \; \cdots \; + \int_{e_4} N^t Q \, dv$$

$$F_r^{e_3} = \int_{e_3} \begin{bmatrix} \cancel{N_1} \to 0 \\ N_2 \\ \cancel{N_3} \\ N_4 \\ N_5 \\ \cancel{N_6} \to 0 \end{bmatrix} Q \, dv$$

$$= \int_{e_3} \begin{bmatrix} 0 \\ N_2 \\ 0 \\ N_4 \\ N_5 \\ 0 \end{bmatrix} Q \, dv$$





nodal map

$LEM_{e_3} = [2, 4, 5]$ end of course project text input

$M_{e_3} = [2, 4, 5]$

dof Map

$$= \int_{e_3} \begin{bmatrix} 0 \\ N_1^{e_3} \\ 0 \\ N_2^{e_3} \\ N_3^{e_3} \\ 0 \end{bmatrix} Q \, dv$$

$$f_r^{e_3} = \int \begin{bmatrix} N_1^{e_3} \\ N_2^{e_3} \\ N_3^{e_3} \end{bmatrix} Q \, dv = \begin{array}{c} 2 \\ 4 \\ 5 \end{array} \begin{bmatrix} \int_{e_3} N_1^{e_3} Q \, dv \\ \int N_2^{e_3} Q \, dv \\ \int N_3^{e_3} Q \, dv \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \to \int_{e_3} N_1^{e_3} Q \, dv \\ 0 \\ 0 \to \int_{e_3} N_2^{e_3} Q \, dv \\ \int_{e_3} N_3^{e_3} Q \, dv \\ 0 \end{bmatrix}$$

the local f_r^e assembled to global F_r matches contributions of element e to global F_r

Summary: Global vs Local system: equations

- From (343), (346), (350), (354), (360) we summarize element to global system relations as,

$$\mathbf{K} = \sum_{I=1}^{n_e} \mathbf{K}^{e_I} = \sum_{I=1}^{n_e} \mathbf{C}^{e_I\,\mathrm{T}} \mathbf{k}^{e_I} \mathbf{C}^{e_I} \qquad (361a)$$

$$\mathbf{F} = \mathbf{F}_n + \mathbf{F}_e, \text{ where element force contributions } \mathbf{F}_e := \sum_{I=1}^{n_e} \mathbf{C}^{e_I\,\mathrm{T}} \mathbf{f}_e^{e_I} \qquad (361b)$$

- Local stiffness matrix, $\mathbf{k}^e$, is given by (*cf.* (346b) and (340)):

$$\mathbf{k}^e = \int_e \mathbf{B}^{e\,\mathrm{T}} \mathbf{D} \mathbf{B}^e \, \mathrm{dv}, \quad \text{where} \quad \mathbf{B}^e = L_m(\mathbf{N}^e) \qquad (362)$$

- Local force vector, $\mathbf{f}_e^e$, is the assembly of all element level forces:

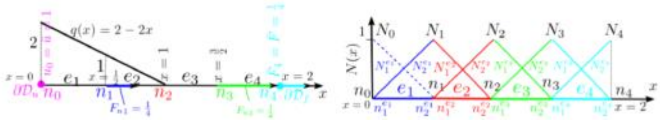
$$\mathbf{f}_e^e := \mathbf{f}_r^e + \mathbf{f}_N^e - \mathbf{f}_D^e \qquad (363a)$$

$$\mathbf{f}_r^e = \int_e \mathbf{N}^{e\,\mathrm{T}} . \mathbf{r} \, \mathrm{dv} \qquad (363b)$$

$$\mathbf{f}_N^e = \int_{\partial e_f} \mathbf{N}^{e\,\mathrm{T}} \bar{\mathbf{F}} . \, \mathrm{ds} \qquad (363c)$$

$$\mathbf{f}_D^e = \mathbf{k}^e \mathbf{a}^e \qquad (363d)$$

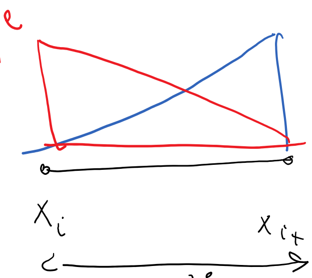
Again, going back to the same problem we were solving last time.



$$k^e = \frac{(AE)^e}{L^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

prismatic element AE constant

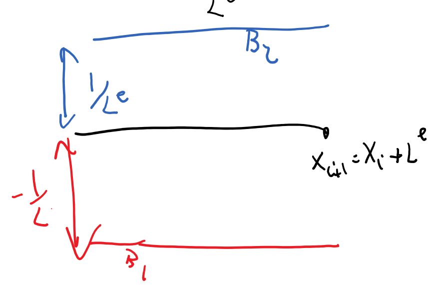
$\dfrac{x_{i+1} - x}{L_e} = N_1^e$ $\qquad N_2^e = \dfrac{x - x_i}{L_e}$



$$k^e = \int_e B^{e\,T} D B^e \, dV$$

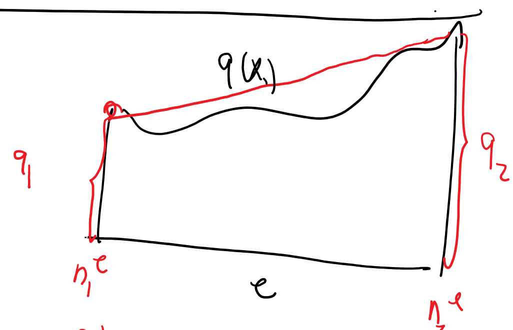
assumed constant

$$\underbrace{\int_{x_i}^{x_{i+1}} \begin{bmatrix} -1/L_e \\ 1/L_e \end{bmatrix} EA \begin{bmatrix} -\frac{1}{L_e} & \frac{1}{L_e} \end{bmatrix} dx}_{\text{constant}} \qquad B = \frac{d}{dx} N$$



$$\frac{L_e (AE)}{L_e^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \left(\frac{AE}{L}\right)^e \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$\tilde{q} \rightarrow$ linear takes values q_1 & q_2 at n_1 & n_2 element



$$f^e = \int_e \begin{bmatrix} N_1^e \\ N_2^e \end{bmatrix} q(x) \, dx \approx$$

$$= \int \begin{bmatrix} N_1^e \end{bmatrix} \tilde{q}(x) \, dx$$

$\tilde{q} = q_1 N_1 + q_2 N$ case 2

$$= \int_e \begin{bmatrix} N_1^e \\ N_2^e \end{bmatrix} q(x)\, dx$$

$$= \int_e \begin{bmatrix} N_1^e \\ N_2^e \end{bmatrix} [N_1^e \ \ N_2^e] \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} dx$$

take it out

$$\tilde{q} = q_1 N_1 + q_2 N_2 = [N_1^e \ \ N_2^e] \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

case 2 source term

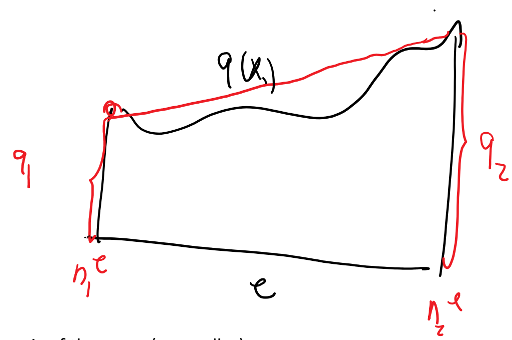
case 1 $\quad u(x) = [N_1^e(x) \ \ N_2^e(x)] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ s/1

$$= \underbrace{\left(\int_{x_i}^{x_{i+1}} \begin{bmatrix} \frac{x_{i+1}-x}{L_e} \\ \frac{x-x_i}{L_e} \end{bmatrix} \left[\frac{x_{i+1}-x}{L_e} \ \ \frac{x-x_i}{L_e} \right] dx \right)}_{r^e} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

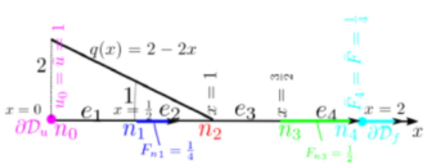
$$\underset{\substack{2\times2 \\ \text{matrix}}}{r^e} = \frac{L_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Summary

$$f_r^e = r^e \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$



Perfect approximation because if q is not linear, the error induced here is of the same (or smaller) order of FEM discretization error



From last time we got the global stiffness matrix

for all elements here $EA = 1$, $L = \frac{1}{2}$ $\rightarrow k^{e_1} = \dots = k^{e_4} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

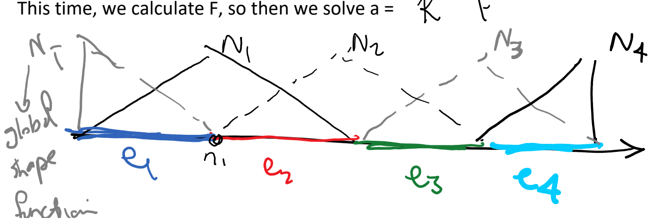
$$k^{e_1} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \quad k^{e_2} = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 2 & -2 \\ 2 & -2 & 2 \end{array} \quad k^{e_3} = \begin{array}{c|cc} & 2 & 3 \\ \hline 2 & 2 & -2 \\ 3 & -2 & 2 \end{array} \quad k^{e_4} = \begin{array}{c|cc} & 3 & 4 \\ \hline 3 & 2 & -2 \\ 4 & -2 & 2 \end{array}$$

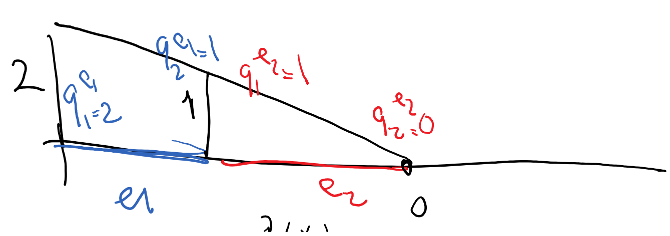
assembly

$$K = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 2+2 & -2 & & \\ 2 & -2 & 2+2 & -2 & \\ 3 & & -2 & 2+2 & -2 \\ 4 & & & -2 & 2 \end{array}$$

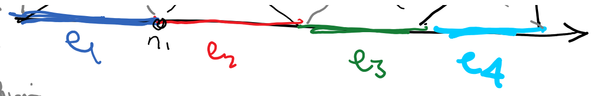
we get the same $K = \begin{bmatrix} 4 & -2 & & \\ -2 & 4 & -2 & \\ & -2 & 4 & -2 \\ & & -2 & 2 \end{bmatrix}$

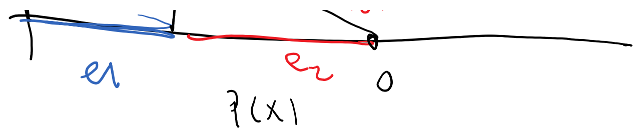
This time, we calculate F, so then we solve a = $K^{-1} F$





global shape function for tst prescribed d.o.f





$M_1 = [\bar{1}, 1]$ $M_2 : [1, 2]$ $M_3 = [2, 3]$ $M_e = [3, 4]$

forces on element 1

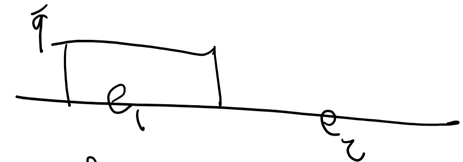
$f^{e_1} = f_r^{e_1} + \cancel{f_N^{e_1}} - f_D^{e_1}$

Not formed for 1D elements

$$f_r^{e_1} = r^{e_1} \begin{bmatrix} q_1^{e_1} \\ q_2^{e_1} \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$r^{e_1} = \frac{L_{e_1}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{1}{6 \times 2} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

side note

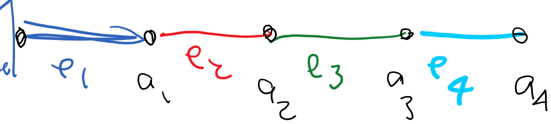


$q^{e_1} = \begin{bmatrix} \bar{q} \\ \bar{q} \end{bmatrix}$ $q^{e_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\boxed{f_r^{e_1} = \frac{1}{12} \begin{bmatrix} 5 \\ 4 \end{bmatrix}}$$

$a_{\bar{1}} = 4$

known prescribed dof



$M_{e_1} = [\bar{1}, 1]$

$$f_D^{e_1} = k^{e_1} a^{e_1} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$a^{e_1} = \begin{bmatrix} a_{\bar{1}} \\ a_1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \rightarrow$ we don't know yet

$$f^{e_1} = f_r^{e_1} + \cancel{f_N^{e_1}} - f_D^{e_1} =$$

$$\begin{bmatrix} 5/12 \\ 4/12 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -19/12 \\ 28/12 \end{bmatrix}$$

$f^{e_1} = \begin{matrix} \bar{1} \\ 1 \end{matrix} \begin{bmatrix} -19/12 \\ 28/12 \end{bmatrix}$

e_2

$f^{e_2} = f_r^{e_2} + \cancel{f_N^{e_2}} - \cancel{f_D^{e_2}}$

1D element

na prescribed dof for e_2

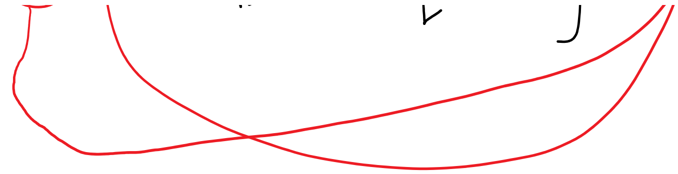
$$= f_r^{e_2} = r^{e_2} q^{e_2} = \frac{1}{12} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{matrix} 1 \\ 2 \end{matrix} \begin{bmatrix} 2/12 \\ 1/12 \end{bmatrix}$$

$$f_e = \begin{matrix} \bar{1} \\ 2 \\ 3 \\ 4 \end{matrix} \begin{bmatrix} 28/12 + 2/12 \\ 1/12 \\ \\ \\ \end{bmatrix}$$

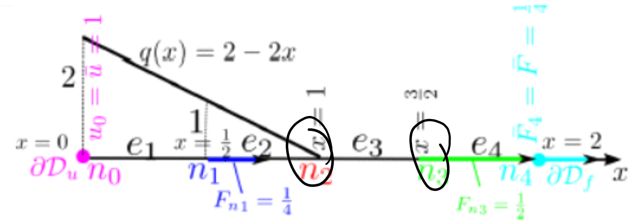
global assembly of element forces

$f^{e_3} = 0$ $f^{e_4} = 0$

$f^{e_3} = 0 \qquad f^{e_4} = 0$



$$F_e = \begin{bmatrix} 30/12 \\ 1/12 \\ 0 \\ 0 \end{bmatrix}$$



$K a = F$

$$F = F_e + F_n = \begin{bmatrix} 11/4 \\ 1/12 \\ 1/2 \\ 1/4 \end{bmatrix}$$

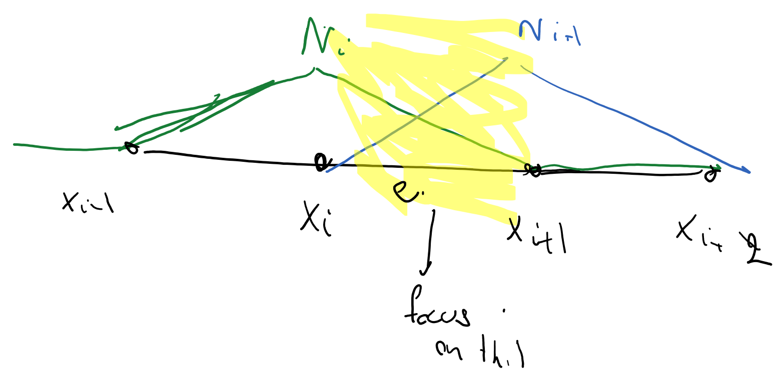
$$F_n = \begin{bmatrix} 1/4 \\ 0 \\ 1/2 \\ 1/4 \end{bmatrix}$$

We get the same solution as before ...

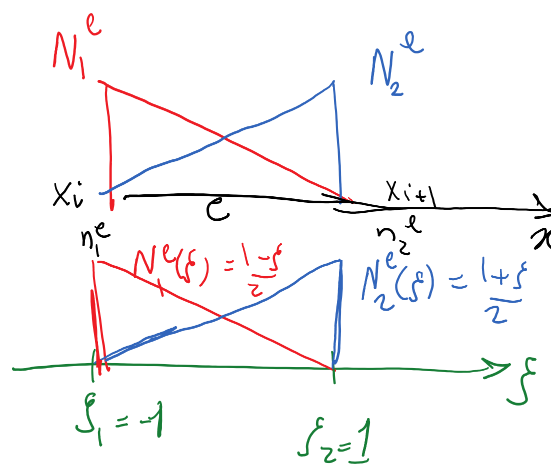
$$\mathbf{a} = \mathbf{K}^{-1}\mathbf{F} = \begin{bmatrix} \frac{43}{24} \\ \frac{53}{24} \\ \frac{31}{12} \\ \frac{65}{24} \end{bmatrix}$$ which matches our solution from global approach (321)

Formula for stiffness matrix when AE is not constant:

global



global coordinate



$N_1(\xi) = 1 \ @ \ \xi_1 = -1 \quad$ (i)

$\qquad\quad = 0 \ @ \ \xi_2 = 1 \quad$ (ii)

$N_1(\xi) = A + B\xi$

(i) $A - B = 1$ $\qquad A = \frac{1}{2} \quad B = \frac{-1}{2}$

(ii) $A + B = 0$

$$u^e = N^e a^e = \left[N_1^e \ \ N_2^e\right] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$B^e = \frac{d}{}(N^e) = \frac{d}{}\left[1-\xi \qquad 1+\xi\right]$

$$B^e = L_m(N^e) = \frac{d}{dx}\left[\frac{1-\xi}{2} \quad \frac{1+\xi}{2}\right]$$

Bar $L_m = \frac{d}{dx}$