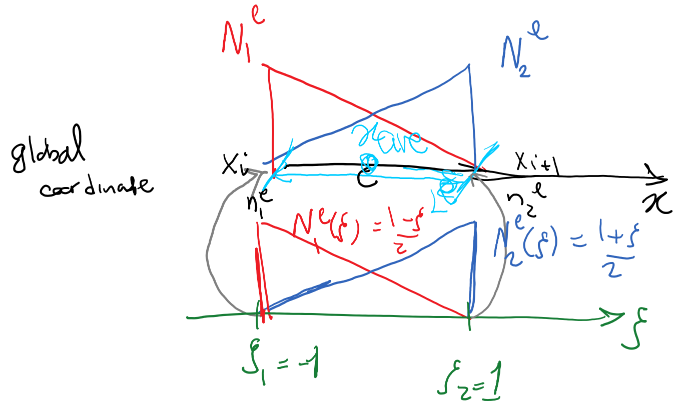


$N_1(\xi) = 1$ @ $\xi_1 = -1$ (i)

$= 0$ @ $\xi_2 = 1$ (ii)

$N_1(\xi) = A + B\xi$

(i) $A - B = 1$ $\quad A = \frac{1}{2} \quad B = \frac{-1}{2}$

(ii) $A + B = 0$

$$u^e = N^e a^e = [N_1^e \;\; N_2^e] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$B^e = L_m(N^e) = \frac{d}{dx}\left[\frac{1-\xi}{2} \quad \frac{1+\xi}{2}\right] \quad (4)$$

Bar $L_m = \frac{d}{dx}$

$x \leftrightarrow \xi$

$\xi_1 \rightarrow x_1^e = x_i$

$\xi_2 \rightarrow x_2^e = x_{i+1}$

$x = A + B\xi$

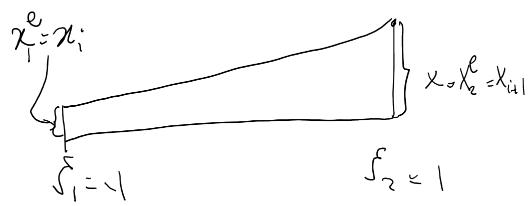
$\xi = -1 \rightarrow x_1^e = x_i \rightarrow x_i = A + B(-1)$

$\xi_2 = 1 \rightarrow x = x_2^e = x_{i+1} \rightarrow x_{i+1} = A + B$

$\rightarrow A = \frac{x_i + x_{i+1}}{2} = x_{ave}^e$

$B = \frac{x_{i+1} - x_i}{2} = L^e/2$

$$\boxed{x = x_{ave}^e + L^e/2\,\xi} \quad (2)$$



OR more easily

$$x = x_1^e N_1^e(\xi) + x_2^e N_2^e(\xi)$$

$$= x_i\left(\frac{1-\xi}{2}\right) + x_{i+1}\left(\frac{1+\xi}{2}\right) = \left(\frac{x_i + x_{i+1}}{2}\right) + \left(\frac{x_{i+1} - x_i}{2}\right)\xi$$

$$= x_{ave} + \frac{L^e}{2}\xi$$

Continue calculation of element stiffness matrix

(1) $B^e = \frac{d}{dx}[N_1^e \;\; N_2^e] = \frac{d}{dx}\left[\frac{1-\xi}{2} \quad \frac{1+\xi}{2}\right] = \underbrace{\frac{\partial}{\partial\xi}\left[\frac{1-\xi}{2} \quad \frac{1+\xi}{2}\right]}_{B_\xi} \frac{\partial\xi}{\partial x}$

(2) $x = x_{ave} + \frac{L^e}{2}\xi \longrightarrow \frac{dx}{d\xi} = \frac{L^e}{2}$

$$B_\xi^e = \frac{d}{d\xi}[N_1^e \quad N_2^e] = \left[-\frac{1}{2} \quad \frac{1}{2}\right]$$

$$B = B_\xi \frac{d\xi}{dx} = \left[-\frac{1}{2}, \frac{1}{2}\right] \frac{2}{L^e} = \frac{1}{L^e}[-1 \quad 1]$$

(2) $\boxed{\frac{dx}{d\xi} = \frac{L^e}{2}} \rightarrow \frac{d\xi}{dx} = \frac{2}{L^e}$

(3) $\boxed{B^e = \frac{1}{L^e}[-1 \quad 1]}$

$$K^e = \int_{x_i}^{x_{i+1}} B^{e^T} D B^e \, dx$$

x_i ——— x_{i+1}

$$= \int_{\xi_1=-1}^{\xi_2=1} \frac{1}{L^e}\begin{bmatrix}-1\\1\end{bmatrix} AE(\xi) \underbrace{\frac{1}{L^e}[-1 \quad 1]}\left(\frac{L^e}{2}\right) d\xi$$

$\xi_1=-1$ ——— $\xi_2=1$

$$\left|\frac{dx}{d\xi}\right| = \frac{L^e}{2} \qquad dx = \frac{dx}{d\xi}\, d\xi$$

(4) FEM stiffness matrix

$$\boxed{K^e = \frac{\int_{-1}^{1} AE(\xi)\, d\xi}{2L^e}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}}$$

for $AE(\xi)$ constant

$$K^e = \frac{AE}{L}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$$ which we obtained last time

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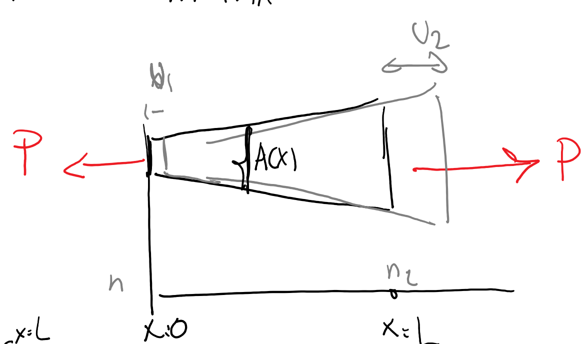
Comparision with exact stiffness matrix

P is given

$\Delta U = U_2 - U_1$

is related to P



we know

$$\varepsilon = \frac{du}{dx} \rightarrow u_2 - u_1 = \int_{x=0}^{x=L} \varepsilon\, dx$$

$$\Delta u = \int_0^L \varepsilon(x)\, dx \qquad \varepsilon(x) = \sigma(x) \qquad \sigma(x)A(x) \ldots$$

$x=0$

$$\Delta u = \int_{x=0}^{L} \varepsilon(x)\,dx \qquad \varepsilon(x) = \frac{\sigma(x)}{E(x)} \quad , \quad \sigma(x)A(x) = \text{Axial force}$$

$E(x)$: Elastic modulus $\qquad = P$ const force here

$$\Delta u = \int_{x=0}^{L} \underbrace{\left(\frac{P}{A(x)}\right)}_{\sigma(x)} \cdot \frac{dx}{E(x)}$$

5

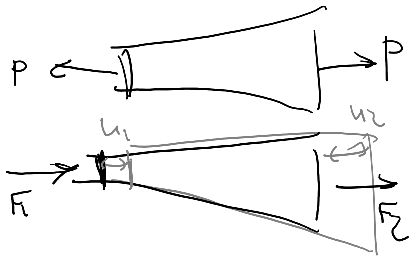
$$\Delta u = P \int_{0}^{L} \frac{dx}{AE(x)} \rightarrow \boxed{P = k_{exact}\,\Delta u \;,\; k_{exact} = \frac{1}{\int_0^L \frac{dx}{AE(x)}}}$$

$F_1 = -P$

$F_2 = P$

$\Delta u = u_2 - u_1$

@ $P = k_{exact}\,\Delta u$



$$\boxed{\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \underbrace{k_{exact} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{(K_{exact})_{2\times 2}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad k_{exact} = \frac{1}{\int_0^L \frac{dx}{AE(x)}}}$$

Compare with

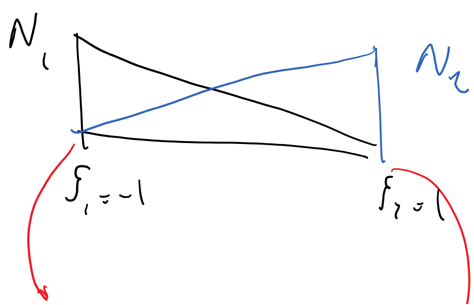
$$K_{FEM} = \int_0^L AE(x)\,dx \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Summary:

1. Finite element stiffness matrix matches the exact stiffness if the assumed shape functions of the element match those from the exact solution under the given applied loads (here linear displacement which happens for AE = constant)
2. The difference between Kexact and KFEM negligible because again the error is of the same order of discretization error …
3. Calculating stiffness from the exact approach above and assembly of the elements was how engineers first formulated FEM
4. Shape function are used for many different purposes:

a) for solution

$$u = u_1 N_1(\xi) + u_2 N_2(\xi)$$

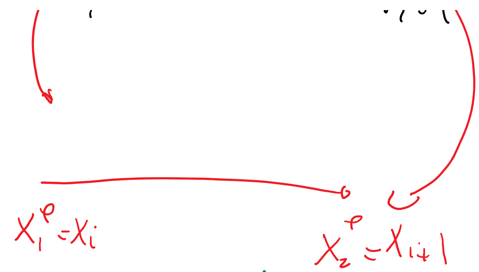


b) Geometry $\quad x = x_1^e N_1(\xi) + x_2^e N_2(\xi)$

$x(\xi)$

$x \simeq x_1 N_1(\xi) + x_2 N_2(\xi)$

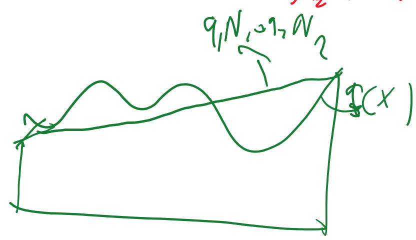
$x(\xi)$



c) Anything else, e.g. $q(x) \simeq q_1 N_1(\xi) + q_2 N_2(\xi)$

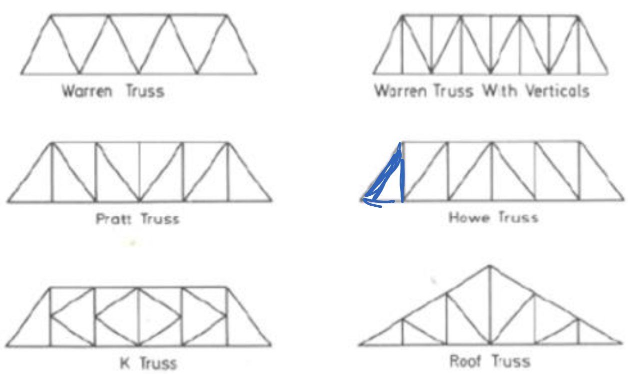
$\rightarrow f_r \simeq f_e \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$

$\frac{L^e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$



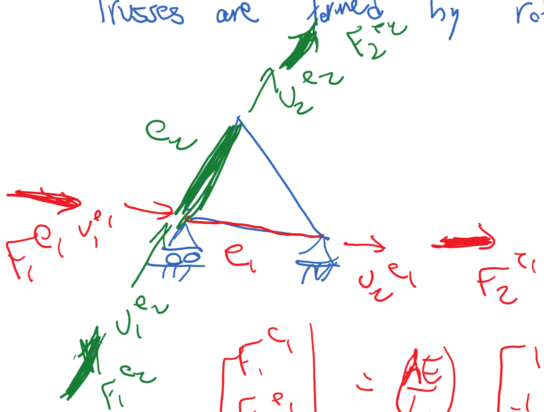
Trusses

change of coordinate system $\rightarrow$ new concept



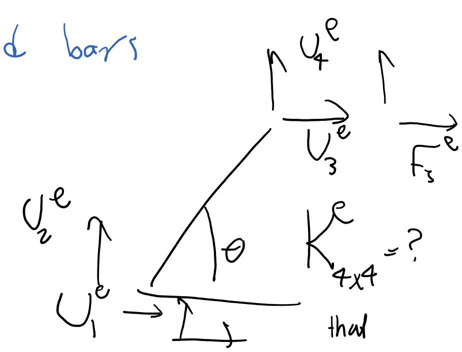
Types of simple Plane truss

Trusses are formed by rotated bars

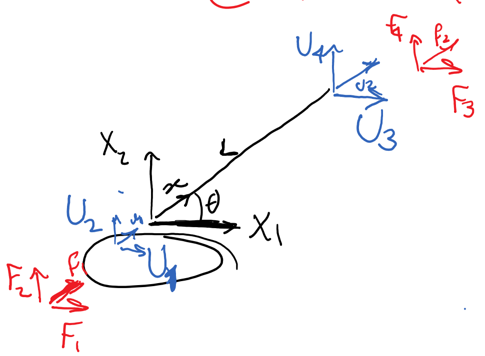


$$\begin{bmatrix} F_1^{e_1} \\ F_2^{e_1} \end{bmatrix} = \left(\frac{AE}{L}\right)_{e_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} U_1^{e_1} \\ U_2^{e_1} \end{bmatrix}$$

U_4 f_4 f_2



$K^e_{4\times4} = ?$ that relates $U^e_{4\times1} \rightarrow F^e_{4\times1}$



$$\underbrace{\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}}_{f} = K_{2\times 2} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{u} \quad (i)$$

$$K_{2\times2} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}}_{F} = K_{4\times4} \underbrace{\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}}_{U} \quad (ii) \qquad K_{2\times2} \xrightarrow{\text{derive}} K_{4\times4}$$

$c = \cos\theta$

$s = \sin\theta$

$f = ku \quad (i)$

$$\underset{2\times1}{u} = \underbrace{T_U}_{2\times4} U_{4\times1}$$

transfer matrix

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \overbrace{\begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix}}^{T_U} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}$$

$U_4 = 0$ $U_2 = ? = 0$ $U_3 = 0$

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \frac{AE}{L}\begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} \quad (iii)$$

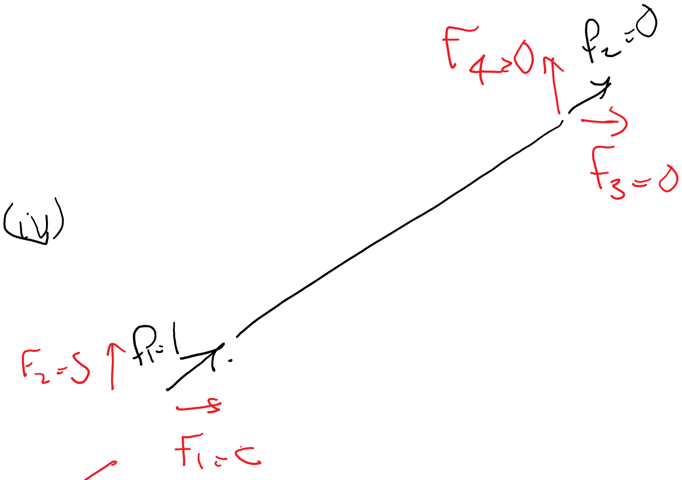
$U_2 = 0$ $u_1 = ? = \cos\theta$ $U_1 = 1$

we want $\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}$ on the LHS

$$\underbrace{\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}_{4\times1}}_{F} = T_F \underbrace{\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}}_{f} {}_{2\times1}$$

F . $D = 0$

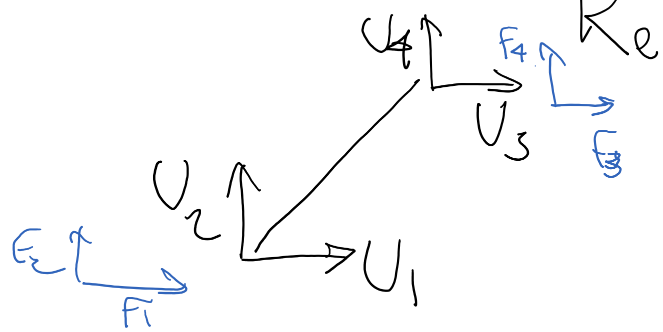
$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} c & 0 \\ s & 0 \\ 0 & c \\ 0 & s \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \quad \text{(iv)}$$



iii) & iv) $\longrightarrow$

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \frac{AE}{L} \underbrace{\begin{bmatrix} K_b & -K_b \\ -K_b & K_b \end{bmatrix}}_{K_e} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}$$

$$K_b = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}_{2\times2}$$



Coordinate is not always X_1, X_2

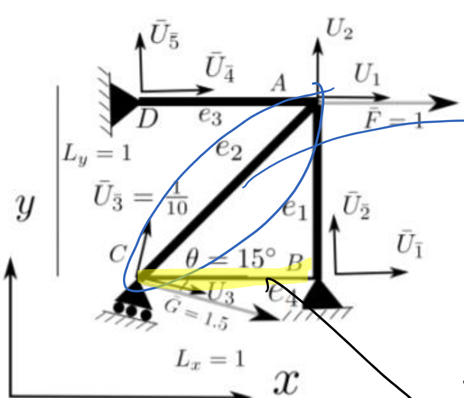
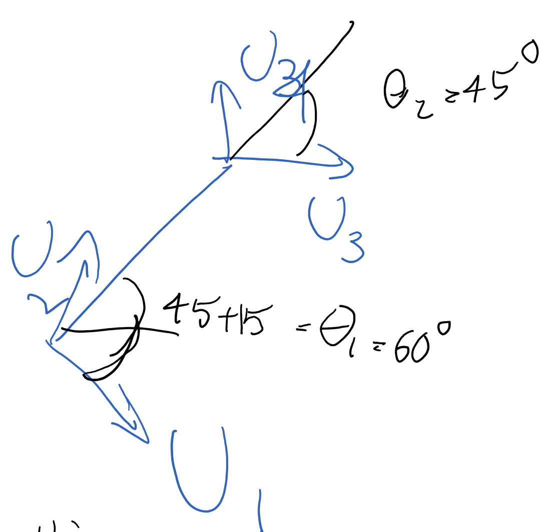
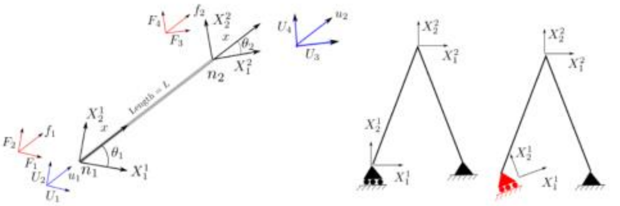


Figure 2: 3 dof truss with an angled support



Same with this

Truss element /two different coordinate systems



- In some instances we need to employ two different coordinate systems at the end points of a bar or in general coordinate system(s) that are not aligned with global coordinate system.
- For example the support highlighted in red in the right figure, do decouple displacement at the support and set the normal displacement to zero (Dirichlet BC) and tangential one free (Neumann BC) we need to employ the rotated coordinate system X_1^1, X_2^1.
- We have two different angles, θ_1 and θ_2. We define,

$$c_1 = \cos(\theta_1) \qquad s_1 = \sin(\theta_1)$$
$$c_2 = \cos(\theta_2) \qquad s_2 = \sin(\theta_2)$$

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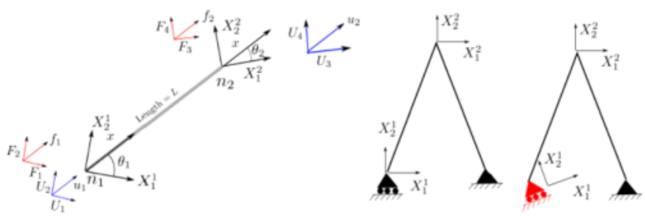
Truss element /two different coordinate systems



- As before $\mathbf{T} := \mathbf{T_{uU}} = \mathbf{T_{Ff}}$ and in this case is given by,

$$\mathbf{T} = \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ 0 & 0 & c_2 & s_2 \end{bmatrix} \tag{393}$$

- Accordingly, from $\mathbf{K} = \mathbf{T}^{\mathrm{T}}\mathbf{k}\mathbf{T}$ we obtain,

$$\mathbf{K} = \frac{AE}{L}\begin{bmatrix} c_1^2 & c_1 s_1 & -c_1 c_2 & -c_1 s_2 \\ c_1 s_1 & s_1^2 & -c_2 s_1 & -s_1 s_2 \\ -c_1 c_2 & -c_2 s_1 & c_2^2 & c_2 s_2 \\ -c_1 s_2 & -s_1 s_2 & c_2 s_2 & s_2^2 \end{bmatrix} \tag{394}$$

- Finally the axial tensile force in the bar, which is the second line of $\mathbf{k}\mathbf{T_{uU}} = \mathbf{k}\mathbf{T}$ is (compare to one global coordinate in (387)):

$$\boxed{T = AE/L\,(-c_1 U_1 - s_1 U_2 + c_2 U_3 + s_2 U_4)} \tag{395}$$

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axial force

I get this form from (iii) above

For $\theta_1 = \theta_2$ $\quad T = \frac{AE}{L}\left(c(U_3 - U_1) + s(U_4 - U_2)\right)$