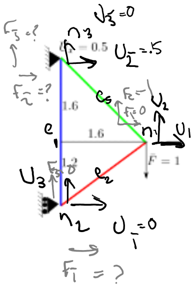
FEM2021/11/02

Tuesday, November 2, 2021 2:50 PM



n_1	1.6	0
n_2	0	-1.2

nodes of element

e1	2	3	LEM_{e_1}
e2	2	1	LEM_{e_2}
e3	3	1	

dofs

3 nodes
× 2 dof/node

$n = 6$ (total # dofs)

$n_p = 3$

$n_f = n - n_p = 3$

f dofs

$$U_f = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = ?$$

$$F_f = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}}_{F_n}$$

this is our nodal force vector

p prescribed dofs

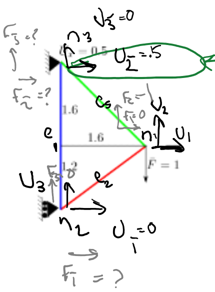
$$U_p = \begin{bmatrix} \bar{U_1} \\ \bar{U_2} \\ \bar{U_3} \end{bmatrix} = \begin{bmatrix} 0 \\ .5 \\ 0 \end{bmatrix}$$

$$F_p = \begin{bmatrix} \bar{F_1} \\ \bar{F_2} \\ \bar{F_3} \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$$ reaction forces

$F = F_e + F_n$

& $F_e = F_r + \cancel{F_N} - F_D$

truss elements are 1D / if we ignore weight $F_r = 0$



lets focus on e_3

$LEM_{e_3} = [3, 1]$

$n_1^{e_3}$ $U_2^{e_3}$ $U_1^{e_3}$ $\theta_{e_3} = -45°$ $\rightarrow c = \cos\theta = +\frac{\sqrt{2}}{2}$, $s = \sin\theta = -\frac{\sqrt{2}}{2}$

$L_{e_3} = \sqrt{2} \times 1.6$ $n_2^{e_3}$ $U_4^{e_3}$ $U_3^{e_3}$

$$a_{e_3} = \begin{bmatrix} .5 \\ 0 \\ ? \\ ? \end{bmatrix}_{4\times1} \qquad a_1^{e_3} = a_{M_{e_3}(1)}$$

$M_{e_3} = [\bar{2}, \bar{3}, 1, 2]_{1\times4}$

map of local to global dofs

$$K_{e_3} = \frac{(AE)_{e_3}}{(L)_{e_3}} \left[\begin{array}{cc|c} c^2 & cs & - \\ cs & s^2 & \\ \hline - & & + \end{array}\right] = \begin{array}{c|cccc} & \bar{2} & \bar{3} & 1 & 2 \\ \hline \bar{2} & .221 & -.221 & & \\ \bar{3} & -.221 & .221 & & \\ 1 & & & .221 & -.221 \\ 2 & & & -.221 & .221 \end{array}$$

$n_f = 3 \rightarrow K (= K_{ff})$ is 3x3

$$K = \begin{bmatrix} .221 & -.221 & \\ -.221 & .221 & \\ & & \end{bmatrix}$$

this was the contribution of element 3 to global K

contribution to global F_e

$f^{e_3} = \cancel{f_r^{e_3}} + \cancel{f_N^{e_3}} - f_D^{e_3} = -f_D^{e_3}$ / $f_D^{e_3} = k^{e_3} a^{e_3} =$

$$\left[\begin{array}{cc|c} .221 & -.221 & - \\ -.221 & .221 & \\ \hline - & & + \end{array}\right] \begin{bmatrix} .5 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

in calculating f_D

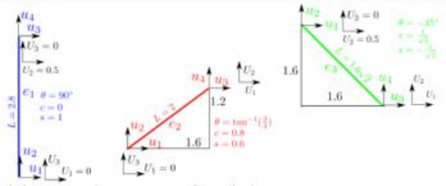
$$f_D^{e_3} = \begin{bmatrix} 0.1105 \\ -0.1105 \\ -.1105 \\ 0.1105 \end{bmatrix} \rightarrow f^{e_3} = -f_D^{e_3} = \begin{array}{c|c} \bar{2} & \cancel{-.1105} \\ \bar{3} & \cancel{.1105} \\ 1 & .1105 \\ 2 & -.1105 \end{array}$$

$n_f = 3 \rightarrow$

$$(F_e)_{3\times1} = \begin{bmatrix} .1105 \\ -.1105 \\ \\ \end{bmatrix}$$

We just need to repeat this for elements 1 and 2 to get:

Truss Example



- Table below summarizes parameters for each element:

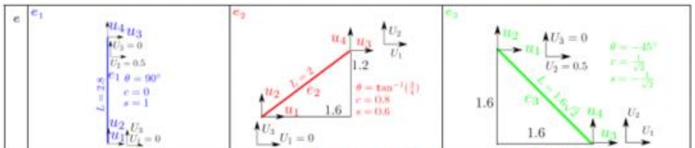
e	L^e	θ	c	s	$\mathbf{M}_t^e$
e_1	2.8	90°	0	1	$[1\ 3\ 2\ 3]$
e_2	2	$\tan^{-1}(\frac{3}{4})$	0.8	0.6	$[1\ 3\ 1\ 2]$
e_3	$1.6\sqrt{2}$	$-45°$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$[2\ 3\ 1\ 2]$

- Local stiffness matrices are given by (390):

$$\mathbf{k}^e = \frac{AE}{L}\left[\begin{array}{c|c} \mathbf{k}_b & -\mathbf{k}_b \\ \hline -\mathbf{k}_b & \mathbf{k}_b \end{array}\right],\ \mathbf{k}_b = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix} : \mathbf{k}^e = \frac{AE}{L}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ cs & s^2 & -cs & -s^2 \end{bmatrix}$$

- As mentioned for trusses generally $\mathbf{f}_r^e = 0$ (no body force), similar to bars we lump natural BC into nodal forces, and finally $\mathbf{f}_D^e = \mathbf{k}^e\mathbf{a}^e$.

Truss example: Assembly of global system

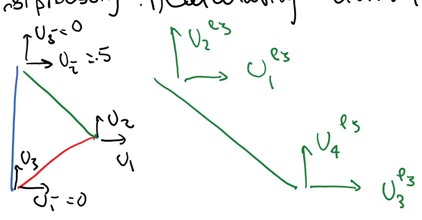


	e_1	e_2	e_3
$\mathbf{k}^e$	$\mathbf{k}^{e_1} = \frac{(1)(1)}{2.8}\begin{bmatrix} 0&0&0&0 \\ 0&1&0&-1 \\ 0&0&0&0 \\ 0&-1&0&1 \end{bmatrix}$ $= \begin{array}{c\|cccc} & 1 & 3 & 2 & 3 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0.3571 & 0 & -0.3571 \\ 2 & 0 & 0 & 0 & 0 \\ 3 & 0 & -0.3571 & 0 & 0.3571 \end{array}$	$\mathbf{k}^{e_2} = \frac{(1)(1)}{2}\begin{bmatrix} 0.64&0.48&-0.64&-0.48 \\ 0.48&0.36&-0.48&-0.36 \\ -0.64&-0.48&0.64&0.48 \\ -0.48&-0.36&0.48&0.36 \end{bmatrix}$ $= \begin{array}{c\|cccc} & 1 & 3 & 1 & 2 \\ \hline 1 & 0.32 & 0.24 & -0.32 & -0.24 \\ 3 & 0.24 & 0.18 & -0.24 & -0.18 \\ 1 & -0.32 & -0.24 & 0.32 & 0.24 \\ 2 & -0.24 & -0.18 & 0.24 & 0.18 \end{array}$	$\mathbf{k}^{e_3} = \frac{(1)(1)}{1.6\sqrt{2}}\begin{bmatrix} 0.5&-0.5&-0.5&0.5 \\ -0.5&0.5&0.5&-0.5 \\ -0.5&0.5&0.5&-0.5 \\ 0.5&-0.5&-0.5&0.5 \end{bmatrix}$ $= \begin{array}{c\|cccc} & 2 & 3 & 1 & 2 \\ \hline 2 & 0.221 & -0.221 & -0.221 & 0.221 \\ 3 & -0.221 & 0.221 & 0.221 & -0.221 \\ 1 & -0.221 & 0.221 & 0.221 & -0.221 \\ 2 & 0.221 & -0.221 & -0.221 & 0.221 \end{array}$
$\mathbf{f}_D^e$	$\mathbf{k}^{e_1}\mathbf{a}_1^e = \begin{bmatrix} 0&0&0&0 \\ 0&0.3571&0&-0.3571 \\ 0&0&0&0 \\ 0&-0.3571&0&0.3571 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0.5 \\ 0 \end{bmatrix} = \begin{array}{c} 1 \\ 3 \\ 2 \\ 3 \end{array}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\mathbf{k}^{e_2}\mathbf{a}_2^e = \begin{bmatrix} 0.32&0.24&-0.32&-0.24 \\ 0.24&0.18&-0.24&-0.18 \\ -0.32&-0.24&0.32&0.24 \\ -0.24&-0.18&0.24&0.18 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{array}{c} 1 \\ 3 \\ 1 \\ 2 \end{array}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\mathbf{k}^{e_3}\mathbf{a}_3^e = \begin{bmatrix} 0.221&-0.221&-0.221&0.221 \\ -0.221&0.221&0.221&-0.221 \\ -0.221&0.221&0.221&-0.221 \\ 0.221&-0.221&-0.221&0.221 \end{bmatrix}\begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{array}{c} 2 \\ 3 \\ 1 \\ 2 \end{array}\begin{bmatrix} 0.1105 \\ -0.1105 \\ -0.1105 \\ 0.1105 \end{bmatrix}$
$\mathbf{f}_e^e$	$\mathbf{f}_e^{e_1} = \mathbf{f}_r^{e_1} + \mathbf{f}_N^{e_1} - \mathbf{f}_D^{e_1} = \begin{array}{c} 1 \\ 3 \\ 2 \\ 3 \end{array}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\mathbf{f}_e^{e_2} = \mathbf{f}_r^{e_2} + \mathbf{f}_N^{e_2} - \mathbf{f}_D^{e_2} = \begin{array}{c} 1 \\ 3 \\ 1 \\ 2 \end{array}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\mathbf{f}_e^{e_3} = \mathbf{f}_r^{e_3} + \mathbf{f}_N^{e_3} - \mathbf{f}_D^{e_3} = \begin{array}{c} 2 \\ 3 \\ 1 \\ 2 \end{array}\begin{bmatrix} -0.1105 \\ 0.1105 \\ 0.1105 \\ -0.1105 \end{bmatrix}$

$$\mathbf{K} = \begin{bmatrix} 0.32+0.221 & 0.24-0.221 & -0.24 \\ 0.24-0.221 & 0.18+0.221 & -0.18 \\ -0.24 & -0.18 & 0.3571+0.18 \end{bmatrix} = \begin{bmatrix} 0.5410 & 0.019 & -0.24 \\ 0.019 & 0.401 & -0.18 \\ -0.24 & -0.18 & 0.5371 \end{bmatrix}$$

$$\mathbf{F} = \underbrace{\mathbf{F}_N}_{\text{Nodal forces}} + \underbrace{\mathbf{F}_e}_{f_e} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.1105 \\ -0.1105 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.1105 \\ -1.1105 \\ 0 \end{bmatrix} \Rightarrow \boxed{\mathbf{U} = \mathbf{K}^{-1}\mathbf{F} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} -0.2123 \\ -3.2980 \\ -1.200 \end{bmatrix}}$$

Post processing :1) Calculating element forces.



$M_{e_3} = [\bar{2}, \bar{3}, 1, 2]$

$a_{e_3} = [.5, 0, ?, ?]$

$a_{e_3} = \underbrace{[.5, 0, -.2123, -3.29]}_{U_{e_3}}$ after sol'n

$$\boxed{T = \frac{AE}{L}\{c(U_3 - U_1) + s(U_4 - U_2)\}} \quad (387)$$

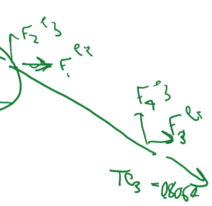
$$T_{e_3} = \left(\frac{AE}{L}\right)^{e_3} \cdot \left\{ c\left(U_3^{e_3} - U_1^{e_3}\right) + s\left(U_4^{e_3} - U_2^{e_3}\right)\right\} = +.8064$$

$\frac{1}{1.6\sqrt{2}}$; $c = \frac{\sqrt{2}}{2}$; $U_3^{e_3} = -.2123$; $U_1^{e_3} = .5$; $s = -\frac{\sqrt{2}}{2}$; $U_4^{e_3} = -3.29$; $U_2^{e_3} = 0$

for e_1 & e_2
for element see
eqn (395) slide 319

$$\boxed{T = AE/L\,(-c_1U_1 - s_1U_2 + c_2U_3 + s_2U_4)} \quad (395)$$

$T_{e_3} = 0.8064$

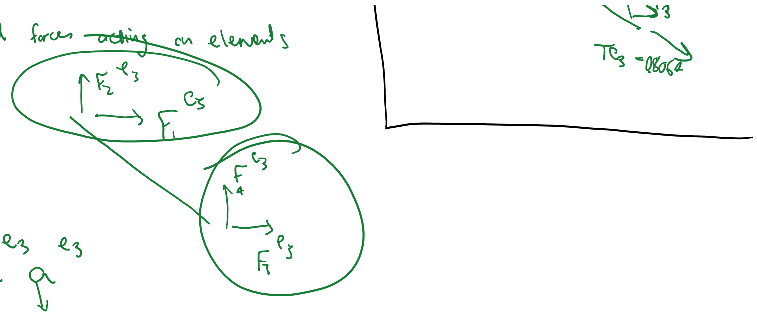


Reaction forces

first get nodal forces ~~acting~~ on elements

$F_2^{e_3}$

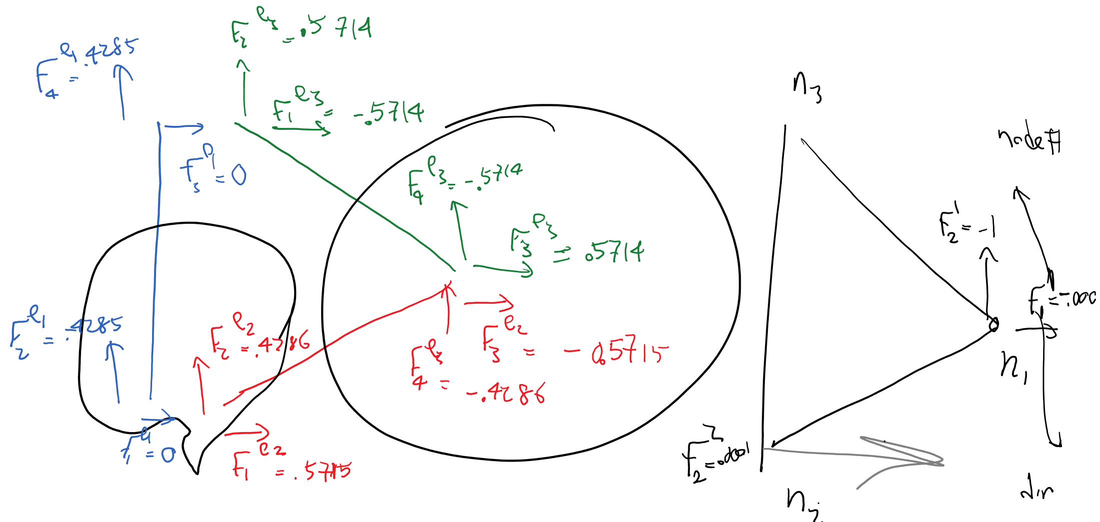
first get nodal forces ~~acting~~ on elements

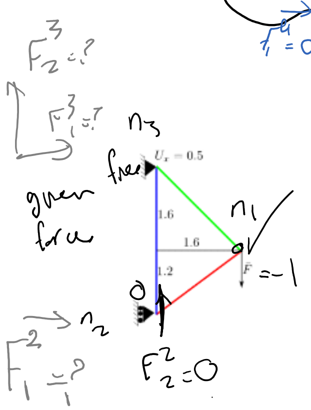


$F^{e_3} = K^{e_3} a^{e_3}$

$$= \begin{bmatrix} -.221 & - & - \\ & & \\ & & \end{bmatrix} \begin{bmatrix} .5 \\ 0 \\ -.2123 \\ -3.2980 \end{bmatrix} = \begin{bmatrix} -.5714 \\ .5714 \\ .5714 \\ -.5714 \end{bmatrix}$$

we do the same for e_1 & e_2 to get





$F_2^1 = -1$

$F_1^1 = 0$

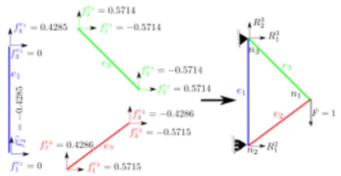
Reaction forces

$F_1^2 = F_1^{e_1} + F_1^{e_2} = .5715$

$F_1^3 =$

$F_2^3 =$

Truss Example: Reaction Forces



- First, we compute reaction forces by adding up forces from individual elements that contribute to reaction forces:

$$R_1^2 = f_1^{e1} + f_1^{e2} = 0 + 0.5715 = 0.5715 \quad (397a)$$
$$R_1^3 = f_3^{e1} + f_1^{e3} = 0 + -0.5714 = -0.5714 \quad (397b)$$
$$R_2^3 = f_4^{e1} + f_2^{e3} = 0.4285 + 0.5714 = 0.9999 \quad (397c)$$

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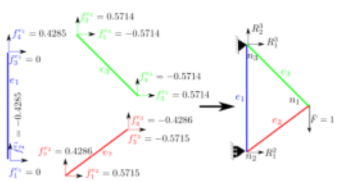
Sanity check (not really needed)

Truss Example: verification of forces at free dofs



- Also, if we want to double-check our calculations on **free dofs**. This step is not needed and it may be done as a verification for hand calculations:

$$F_1^1 = f_3^{e2} + f_3^{e3} = -0.5715 + 0.5714 = -0.0001 \quad (398a)$$
$$F_2^1 = f_4^{e2} + f_4^{e3} = -0.4286 + -0.5714 = -1 = \bar{F} \quad (398b)$$
$$R_2^2 = f_2^{e1} + f_2^{e2} = -0.4285 + 0.4286 = 0.0001 \quad (398c)$$

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Truss Example: Observations

- The fact that our free dof forces in (398) are the values enforced is a necessary (but not sufficient) check on the correctness of our FEM solutions.
- We observe that exact (direct solutions in (399)) match our FEM solution in the last table and (397).
- The reason FEM and exact solutions match is that FEM shape functions (linear displacement within each bar) can capture the exact solution. In general, whenever, FEM approximate solution space can capture exact solution, FEM recovers the exact solution.
- Small error between FEM and direct method or about 0.0001 errors in some reaction and sum of forces in FEM method are finite numerical precision error which are different from discretization errors caused by approximating an infinite solution space by FEM shape functions. The former is caused by working with finite number of digits in our calculations.

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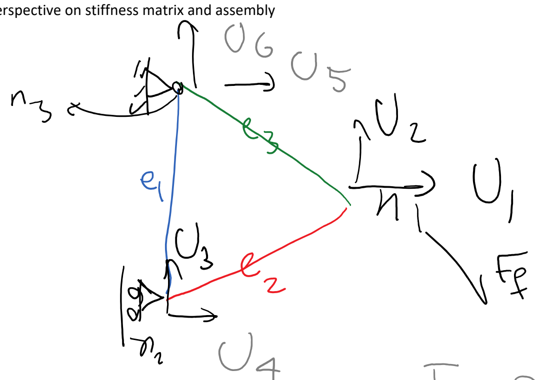
"Undergraduate" perspective on stiffness matrix and assembly



number free dofs (U_1 to U_3)

prescribed dofs continue (U_4, U_5, U_6)

$K_{6\times6}$

Unknown

$F_p = ?$

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} \quad U_f = ?$$

U_4

$F_p = ?$ reaction forces

$$\begin{bmatrix} F_4 \\ F_5 \\ F_6 \end{bmatrix} \qquad \begin{bmatrix} U_3 \\ U_4 \\ U_5 \\ U_6 \end{bmatrix} = U_p$$

Unknown

$$\begin{bmatrix} F_f \\ F_p = ? \end{bmatrix} = \begin{bmatrix} (K_{ff})_{n_f \times n_f} & (K_{fp})_{n_f \times n_p} \\ K_{pf} & (K_{pp})_{n_p \times n_p} \end{bmatrix}_{n \times n} \begin{bmatrix} U_f = ? \\ U_p \end{bmatrix}$$

at global level F_D

(i) $F_f = K_{ff} U_f + K_{fp} U_p \longrightarrow \underbrace{K_{ff} U_f = F_f - K_{fp} U_p}$

(ii) $F_p = K_{pf} U_f + K_{pp} U_p$

similar to

$K U = F$

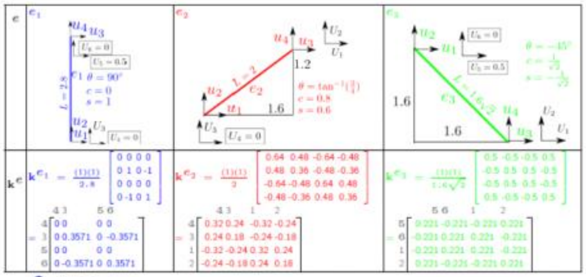
if we only assemble free dofs

numbering free dofs $\rightarrow$ we only assemble K_{ff}

- F_D automatically assembled
- much cheaper

For the actual numerical example on this, please refer to

Truss example: Assembly of global system $(f+p)$



- Global stiffness matrix:

$$\mathbf{K} = \begin{bmatrix} 0.32+0.221 & 0.24-0.221 & -0.24 & -0.32 & -0.221 & 0.221 \\ 0.24-0.221 & 0.18+0.221 & -0.18 & -0.24 & 0.221 & -0.221 \\ -0.24 & -0.18 & 0.3571+0.18 & 0+0.24 & 0 & -0.3571 \\ -0.32 & -0.24 & 0+0.24 & 0+0.32 & 0 & 0 \\ -0.221 & 0.221 & 0 & 0 & 0+0.221 & 0-0.221 \\ 0.221 & -0.221 & -0.3571 & 0 & 0-0.221 & 0.3571+0.221 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5410 & 0.019 & -0.24 & -0.32 & -0.221 & 0.221 \\ 0.019 & 0.401 & -0.18 & -0.24 & 0.221 & -0.221 \\ -0.24 & -0.18 & 0.5371 & 0.24 & 0 & -0.3571 \\ -0.32 & -0.24 & 0.24 & 0.32 & 0 & 0 \\ -0.221 & 0.221 & 0 & 0 & 0.221 & -0.221 \\ 0.221 & -0.221 & -0.3571 & 0 & -0.221 & 0.5781 \end{bmatrix}$$

- Global force vector. The element vectors that would assemble to $\mathbf{F}_e$ are $f_e^e - f_r^e + f_N^e$ and we do not include $-f_D^e$ in it as it would be directly taken care of subsequently. Since all $-f_r^e$ and f_N^e are zero, $\mathbf{F}_e$ is identically zero.

$$\mathbf{F} = \mathbf{F}_N + \mathbf{F}_e = \begin{bmatrix} 0 \\ -1 \\ 0 \\ R_1^2 \\ R_1^3 \\ R_2^3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ R_1^2 \\ R_1^3 \\ R_2^3 \end{bmatrix}$$

Truss example: Solution of global system $(f+p)$

- The global system is,

$\mathbf{KU} = \mathbf{F}$ where

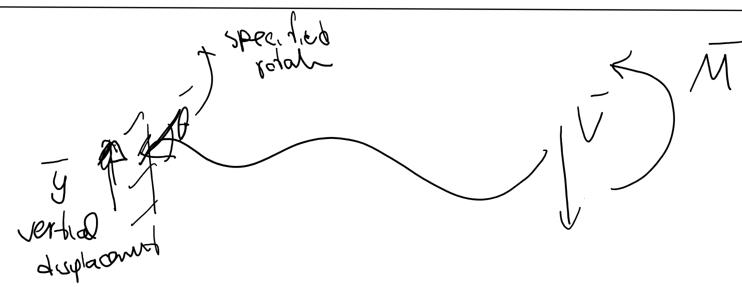
$$\mathbf{K} = \left[\begin{array}{ccc|ccc} 0.5410 & 0.019 & -0.24 & -0.32 & -0.221 & 0.221 \\ 0.019 & 0.401 & -0.18 & -0.24 & 0.221 & -0.221 \\ -0.24 & -0.18 & 0.5371 & 0.24 & 0 & -0.3571 \\ \hline -0.32 & -0.24 & 0.24 & 0.32 & 0 & 0 \\ -0.221 & 0.221 & 0 & 0 & 0.221 & -0.221 \\ 0.221 & -0.221 & -0.3571 & 0 & -0.221 & 0.5781 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{K}_{ff} & \mathbf{K}_{fp} \\ \hline \mathbf{K}_{pf} & \mathbf{K}_{pp} \end{array}\right]$$

$$\mathbf{F} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_6 \\ F_6 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \\ R_1^2 \\ R_1^3 \\ R_2^3 \end{bmatrix} = \left[\frac{\mathbf{F}_f}{\mathbf{F}_p}\right]$$

- The unknown quantities $\mathbf{F}_p$ and $\mathbf{U}_f$ are highlighted.

Please read from slide 329 to 335

Beam element



ODE

$$\underbrace{\frac{d^2}{dx^2}\left(EIy''\right)}\ - q = 0$$

$M = 2m = 4 \longrightarrow m = 2$

we need C^{m-1} continuity for shape (basis) functions

can reduce it by 1

can reduce it by 1

Weak statement

$$\int_0^L \underbrace{w''}_{O} \underbrace{EI}_{D} y'' \, dx = \int_0^L w q \, dx - y(L)\bar{V} + y'(L)\bar{M}$$

$L_m(w) = w''$ $L_m = (\ )''$

$$B = L_m(N) = N''$$