FEM2021/11/09

Tuesday, November 9, 2021 2:50 PM

Example

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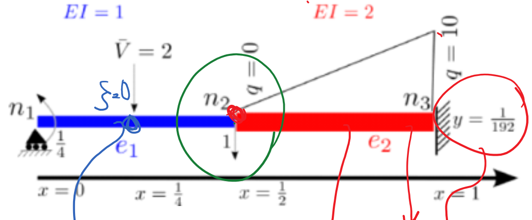
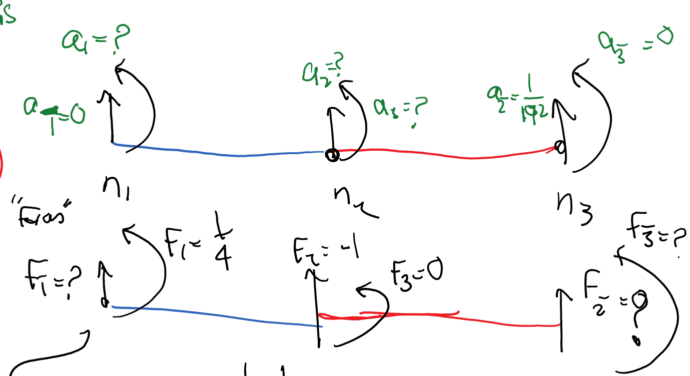
n = nnodes &ndofs/node = 6





$$F_n = \begin{bmatrix} 1/4 \\ -1 \\ 0 \end{bmatrix}$$

f_r from $\bar V$

f_r from q

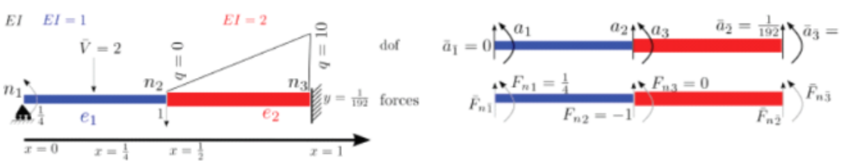
$$f_r^{e_1} \text{ is from } \bar V = \bar V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} @ (\xi=0) \Rightarrow f^{e_1} = -f_D^{e_1}$$

$$f_r^{e_2} = (r^{e_2})_{4\times 2} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \quad q_1 \to 0,\ q_2 \to =10$$

$$f_D^{e_2} = (K^{e_2})_{4\times4}\, a^{e_2}_{4\times1} = (K^{e_2})_{4\times4} \begin{bmatrix} 0 \\ 0 \\ 1/192 \\ 0 \end{bmatrix}$$

$$f^{e_2} = f_r^{e_2} - f_D^{e_2} \quad \ldots$$

Beam example: Assembly of global system



e	e_1	e_2
$\mathbf{k}^e$	$\mathbf{k}^{e_1} = \frac{1}{(\frac12)^3}\begin{bmatrix} 12 & 3 & -12 & 3 \\ 3 & 1 & -3 & 0.5 \\ -12 & -3 & 12 & -3 \\ 3 & 0.5 & -3 & 1 \end{bmatrix}$ $= \begin{array}{c\|cccc} & \bar 1 & 1 & 2 & 3 \\ \hline \bar 1 & 96 & 24 & -96 & 24 \\ 1 & 24 & 8 & -24 & 4 \\ 2 & -96 & -24 & 96 & -24 \\ 3 & 24 & 4 & -24 & 8 \end{array}$	$\mathbf{k}^{e_2} = \frac{2}{(\frac12)^3}\begin{bmatrix} 12 & 3 & -12 & 3 \\ 3 & 1 & -3 & 0.5 \\ -12 & -3 & 12 & -3 \\ 3 & 0.5 & -3 & 1 \end{bmatrix}$ $= \begin{array}{c\|cccc} & 2 & 3 & \bar 2 & \bar 3 \\ \hline 2 & 192 & 48 & -192 & 48 \\ 3 & 48 & 16 & -48 & 8 \\ \bar 2 & -192 & -48 & 192 & -48 \\ \bar 3 & 48 & 8 & -48 & 16 \end{array}$
$\mathbf{f}_r^e$	(440) (1st eqn) ($\xi = 0$) $\bar V \begin{bmatrix} N_1^{e_1}(\xi_0) \\ N_2^{e_1}(\xi_0) \\ N_3^{e_1}(\xi_0) \\ N_4^{e_1}(\xi_0) \end{bmatrix} = -2 \begin{bmatrix} \frac12 \\ \frac{\frac12}{8} \\ \frac12 \\ -\frac{\frac12}{8} \end{bmatrix} = \begin{array}{c} \bar 1 \\ 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} -1 \\ -\frac18 \\ -1 \\ \frac18 \end{bmatrix}$	equation (433); $\mathbf{r}^e \begin{bmatrix} q_1 & q_2 \end{bmatrix}^T$ $\frac12 \begin{bmatrix} \frac{7}{20} & \frac{3}{20} \\ \frac{1}{40} & \frac{1}{60} \\ \frac{3}{20} & \frac{7}{20} \\ -\frac{1}{60} & -\frac{1}{40} \end{bmatrix} \begin{bmatrix} 0 \\ 10 \end{bmatrix} = \begin{array}{c} 2 \\ 3 \\ \bar 2 \\ \bar 3 \end{array} \begin{bmatrix} \frac34 \\ \frac{1}{12} \\ \frac74 \\ -\frac18 \end{bmatrix}$
$\mathbf{f}_D^e$	$\mathbf{k}^{e_1}\mathbf{a}_1^e =$ $\begin{bmatrix} 96 & 24 & -96 & 24 \\ 24 & 8 & -24 & 4 \\ -96 & -24 & 96 & -24 \\ 24 & 4 & -24 & 8 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{array}{c} 1 \\ \bar 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\mathbf{k}^{e_2}\mathbf{a}_2^e =$ $\begin{bmatrix} 192 & 48 & -192 & 48 \\ 48 & 16 & -48 & 8 \\ -192 & -48 & 192 & -48 \\ 48 & 8 & -48 & 16 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \frac{1}{192} \\ 0 \end{bmatrix} = \begin{array}{c} 2 \\ 3 \\ \bar 2 \\ \bar 3 \end{array} \begin{bmatrix} -1 \\ -\frac14 \\ 1 \\ -\frac14 \end{bmatrix}$
$\mathbf{f}_e^e$	$\mathbf{f}_e^{e_1} = \cancel{\mathbf{f}_r^{e_1} + \mathbf{f}_N^{e_1}} - \mathbf{f}_D^{e_1} = \begin{array}{c} \bar 1 \\ 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} -1 \\ -\frac18 \\ -1 \\ \frac18 \end{bmatrix}$	$\mathbf{f}_e^{e_2} = \mathbf{f}_r^{e_2} + \mathbf{f}_N^{e_2} - \mathbf{f}_D^{e_2} = \begin{array}{c} 2 \\ 3 \\ \bar 2 \\ \bar 3 \end{array} \begin{bmatrix} \frac74 \\ \frac13 \\ [illegible] \\ [illegible] \end{bmatrix}$

$$\mathbf{K} = \begin{bmatrix} 8 & -24 & 4 \\ & 96+192 & -24+48 \\ \text{sym.} & & 8+16 \end{bmatrix} = \begin{bmatrix} 8 & -24 & 4 \\ & 288 & 24 \\ \text{sym.} & & 24 \end{bmatrix}$$

$$\mathbf{F} = \mathbf{F}_n + \mathbf{F}_e = \begin{bmatrix} \frac14 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac18 \\ -1 + \frac74 \\ \frac18 + \frac13 \end{bmatrix} = \begin{bmatrix} -\frac18 \\ -\frac34 \\ \frac{11}{24} \end{bmatrix} \Rightarrow$$

$$\mathbf{U} = \mathbf{K}^{-1}\mathbf{F} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -\frac{7}{1152} \\ -\frac{23}{6912} \\ \frac{\cdot}{3} \end{bmatrix} = \begin{bmatrix} -.00607 \\ -.00332 \\ .023434 \end{bmatrix}$$

dofs of e_1 $\quad M_{e_1} = [\bar 1, 1, 2, 3]$

Element nodal forces and support forces:

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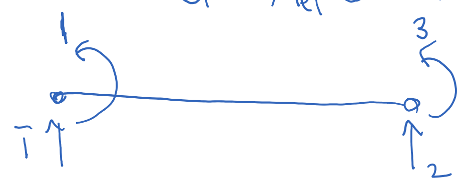
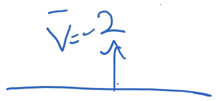
dofs of e_1 $\quad M_{e_1} = [1, 1, 2, 3]$

$a_{e_1} = [0, -.0607, -.0832, 0.0234]$

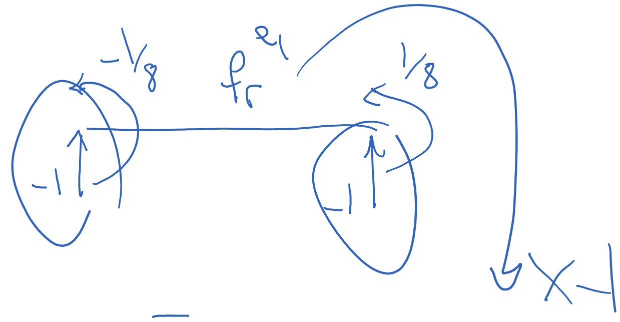


let's calculate $\quad -f_D^{e_2} = K^{e_2} a^{e_2} \quad \ldots$

what is the other contribution to nodal force

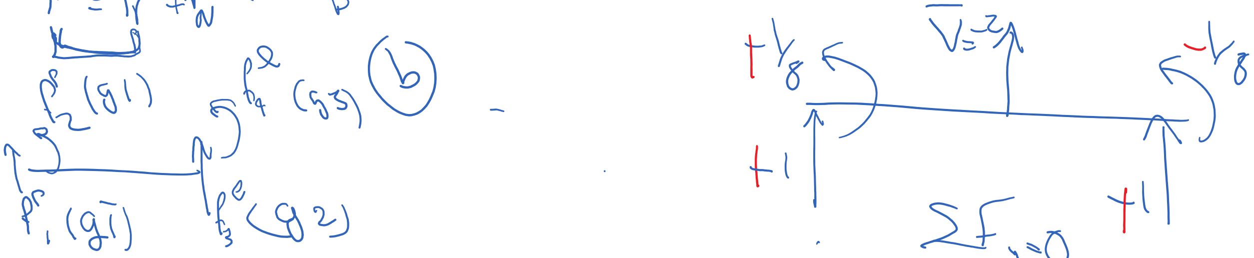


$$f_r(\bar{V}) = \begin{bmatrix} -1 \\ -1/8 \\ -1 \\ 1/8 \end{bmatrix}$$



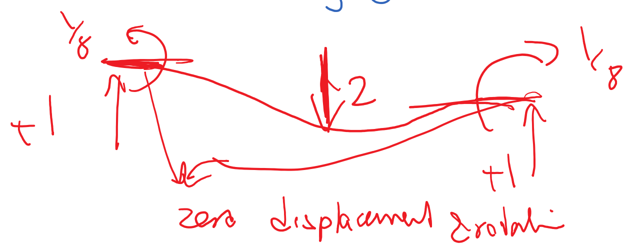
equivalent

$$f^e = f_r^e + f_N^e - f_D^e \quad (a)$$



dofMap $= [1, 1, 2, 3]$

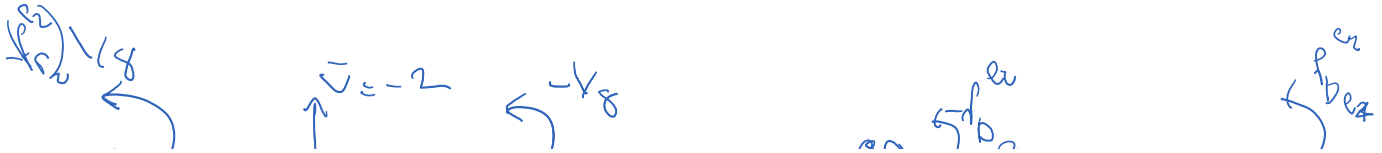
$$F^e_{\text{global}} \quad \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} + f_2^{e_1} \\ + f_3^{e_1} \\ + f_4^{e_4} \end{bmatrix}$$

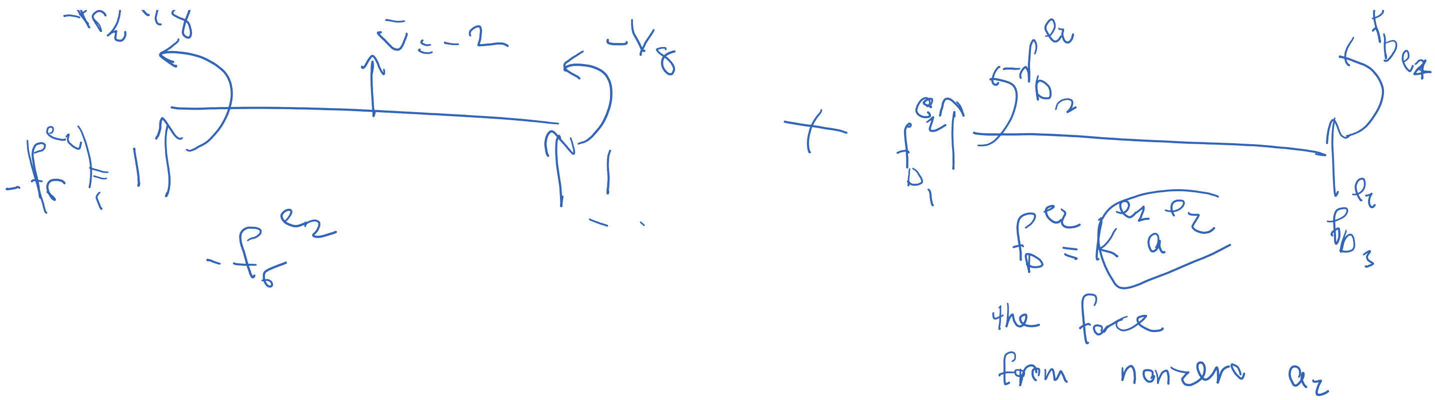


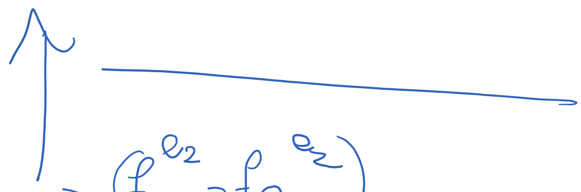
fr for an element is

1. Energetically equivalent to actual distributed force and in the act assembly we basically replace distributed force with this equivalent force.
2. If we exert the opposite of f_r to the element, it balances distributed force with all element dofs being equal to 0

after the solution to balance $\bar{V}$ we need to exert $-f_r^{e_2}$ to e_2





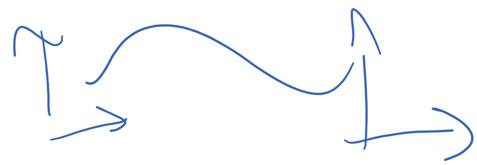


$$= \left(f_D^{e_2} - f_r^{e_2}\right)_{\text{component 1}}$$

$$= -\underbrace{\left(f_r^{e_2} - f_D^{e_2}\right)}_{\text{equivalent force}}{}_{\text{component 1}}$$

to balance

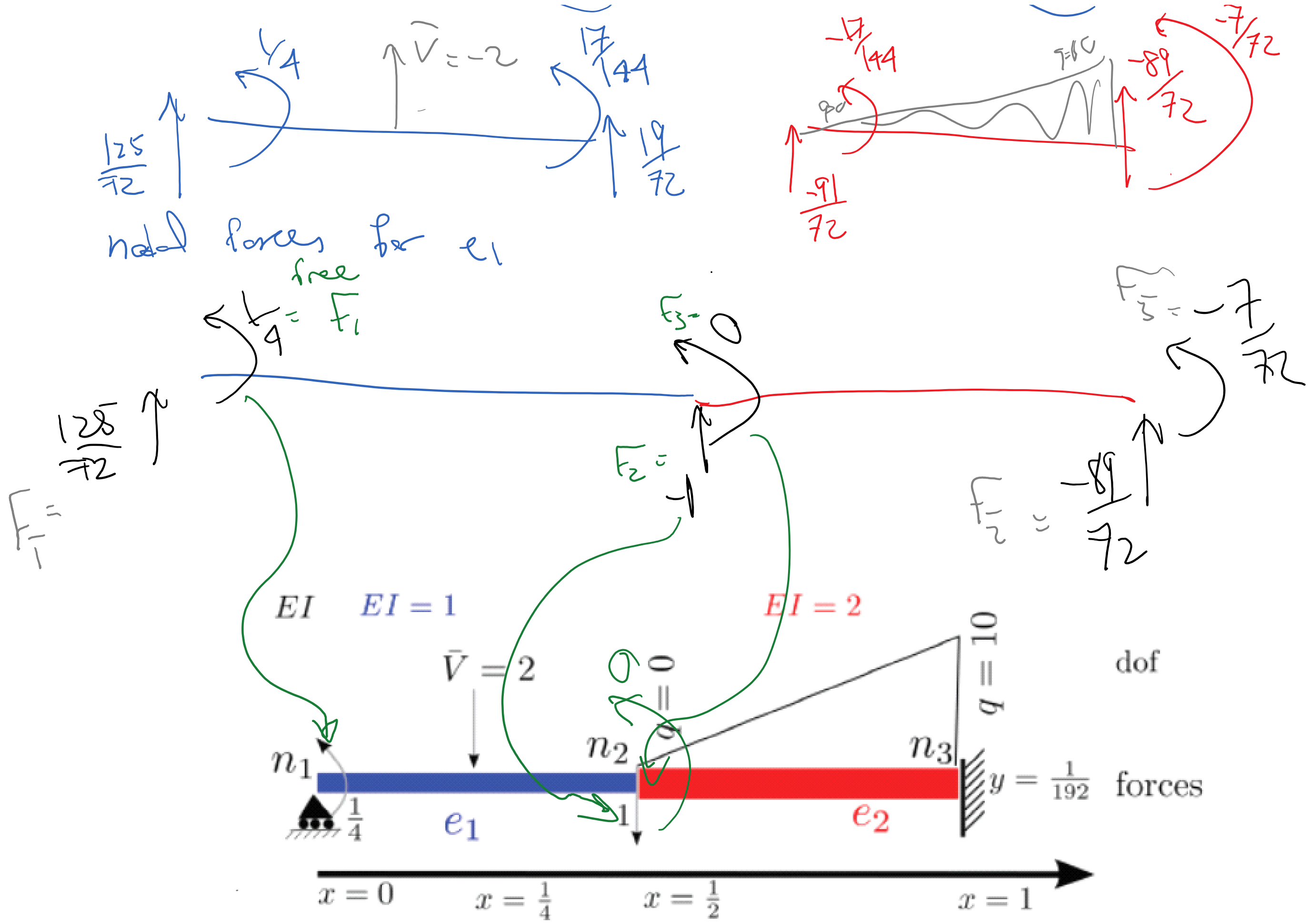
$$f^{e_2} = f_r^{e_2} + \underbrace{\cancel{f_N^{e_2}}}_{0} - f_D^{e_2}$$



To get element nodal forces just calculate

$$-f^e = -\left(f_r^e + \underbrace{f_N^e}_{0 \text{ for 1D elements}} + \cdots - f_D^e\right)$$

e	e_1	e_2
$\mathbf{u}^e$	$\begin{bmatrix} 0 \\ -\frac{7}{1152} \\ -\frac{23}{6912} \\ \frac{3}{128} \end{bmatrix}$	$\begin{bmatrix} -\frac{23}{6912} \\ \frac{3}{128} \\ \frac{1}{192} \\ 0 \end{bmatrix}$
$-f^e$	$\mathbf{k}^{e_1}\mathbf{a}_1^e - \mathbf{f}_r^{e_1} - \mathbf{f}_N^{e_1} =$ $\begin{bmatrix} 96 & 24 & -96 & 24 \\ 24 & 8 & -24 & 4 \\ -96 & -24 & 96 & -24 \\ 24 & 4 & -24 & 8 \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{7}{1152} \\ -\frac{23}{6912} \\ \frac{3}{128} \end{bmatrix} - \begin{bmatrix} -1 \\ -\frac{1}{8} \\ -1 \\ \frac{1}{8} \end{bmatrix} = \begin{array}{c} 1 \\ 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} \frac{125}{72} \\ \frac{1}{4} \\ \frac{19}{72} \\ \frac{17}{144} \end{bmatrix}$	$\mathbf{k}^{e_2}\mathbf{a}_2^e - \mathbf{f}_r^{e_2} - \mathbf{f}_N^{e_2} =$ $\begin{bmatrix} 192 & 48 & -192 & 48 \\ 48 & 16 & -48 & 8 \\ -192 & -48 & 192 & -48 \\ 48 & 8 & -48 & 16 \end{bmatrix} \begin{bmatrix} -\frac{23}{6912} \\ \frac{3}{128} \\ \frac{1}{192} \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{3}{4} \\ \frac{1}{12} \\ \frac{7}{4} \\ -\frac{1}{8} \end{bmatrix} = \begin{array}{c} 2 \\ 3 \\ 2 \\ 3 \end{array} \begin{bmatrix} -\frac{91}{72} \\ -\frac{17}{144} \\ -\frac{89}{72} \\ -\frac{7}{72} \end{bmatrix}$



HW4, problem 5

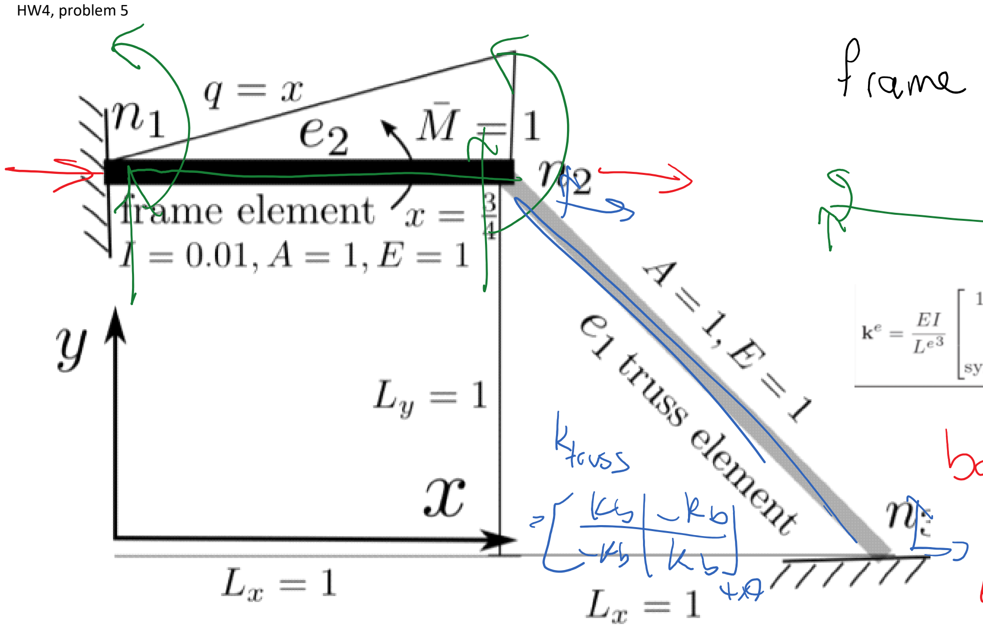


Figure 3: Frame and truss example.

frame = bar + beam

$$\mathbf{k}^e = \frac{EI}{L^{e3}} \begin{bmatrix} 12 & 6L^e & -12 & 6L^e \\ & 4L^{e2} & -6L^e & 2L^{e2} \\ & & 12 & -6L^e \\ \text{sym.} & & & 4L^{e2} \end{bmatrix} \quad \text{for constant } E \text{ and } I \tag{427}$$

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bar

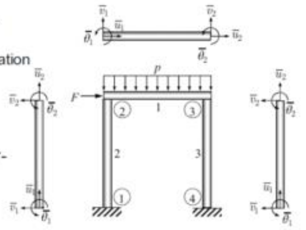
$$k = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Please don't complicate things by going to frame element stiffness matrix for this problem. Treat e2 and a bar and beam element occupying the same place.

But for your term project, you need to work with frame elements

Frames: 2D frame elements

- Beam
  - Vertical deflection and slope. No axial deformation
- Frame structure
  - Can carry axial force, transverse shear force, and bending moment (Beam + Truss)
- Assumption
  - Axial and bending effects are uncoupled
  - Reasonable when deformation is small
- 3 DOFs per node
  $\{u_i, v_i, \theta_i\}$
- Need coordinate transformation like plane truss



source: Nam-Ho Kim, Raphael T. Haftka; http://www2.mae.ufl.edu/nkim/eml5526/ sec. 5
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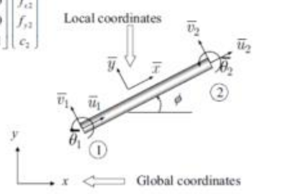
We need transfer matrix

- Element-fixed local coordinates $\bar{x} - \bar{y}$
- Local DOFs $\{\bar{u}, \bar{v}, \bar{\theta}\}$ Local forces $\{f_{\bar{x}}, f_{\bar{y}}, \bar{c}\}$
- Transformation between local and global coord.

$$\begin{Bmatrix} f_{\bar{x}1} \\ f_{\bar{y}1} \\ \bar{c}_1 \\ f_{\bar{x}2} \\ f_{\bar{y}2} \\ \bar{c}_2 \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 & 0 & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\phi & \sin\phi & 0 \\ 0 & 0 & 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} f_{x1} \\ f_{y1} \\ c_1 \\ f_{x2} \\ f_{y2} \\ c_2 \end{Bmatrix}$$

$$\boxed{\{\bar{\mathbf{f}}\} = [\mathbf{T}]\{\mathbf{f}\}}$$

$$\boxed{\{\bar{\mathbf{q}}\} = [\mathbf{T}]\{\mathbf{q}\}}$$



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2D frame: local & global coordinate stiffness matrices

- Element matrix equation (local coord.)

$$\begin{bmatrix} a_1 & 0 & 0 & -a_1 & 0 & 0 \\ 0 & 12a_2 & 6La_2 & 0 & -12a_2 & 6La_2 \\ 0 & 6La_2 & 4L^2a_2 & 0 & -6La_2 & 2L^2a_2 \\ -a_1 & 0 & 0 & a_1 & 0 & 0 \\ 0 & -12a_2 & -6La_2 & 0 & 12a_2 & -6La_2 \\ 0 & 6La_2 & 2L^2a_2 & 0 & -6La_2 & 4L^2a_2 \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{\theta}_1 \\ \bar{u}_2 \\ \bar{v}_2 \\ \bar{\theta}_2 \end{Bmatrix} = \begin{Bmatrix} \bar{f}_{x1} \\ \bar{f}_{y1} \\ \bar{c}_1 \\ \bar{f}_{x2} \\ \bar{f}_{y2} \\ \bar{c}_2 \end{Bmatrix} \qquad a_1 = \frac{EA}{L} \quad a_2 = \frac{EI}{L^3}$$

$$\boxed{[\bar{\mathbf{k}}]\{\bar{\mathbf{q}}\} = \{\bar{\mathbf{f}}\}}$$

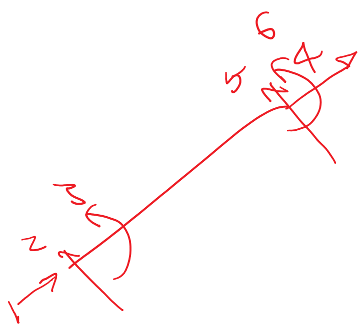
- Element matrix equation (global coord.)

$$[\bar{\mathbf{k}}][\mathbf{T}]\{\mathbf{q}\} = [\mathbf{T}]\{\mathbf{f}\} \implies [\mathbf{T}]^T[\bar{\mathbf{k}}][\mathbf{T}]\{\mathbf{q}\} = \{\mathbf{f}\} \implies [\mathbf{k}]\{\mathbf{q}\} = \{\mathbf{f}\}$$

$$\boxed{[\mathbf{k}] = [\mathbf{T}]^T[\bar{\mathbf{k}}][\mathbf{T}]}$$

Use this for term project

- Same procedure for assembly and applying BC



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