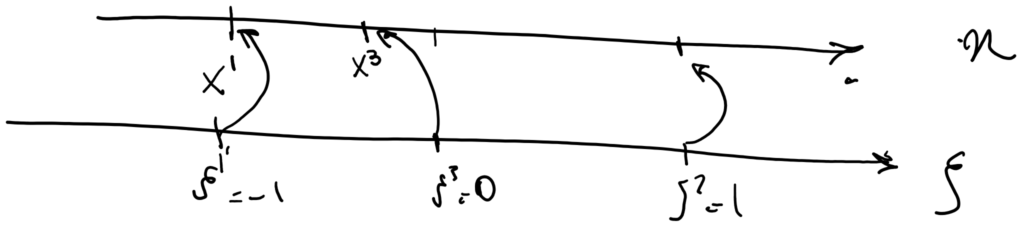
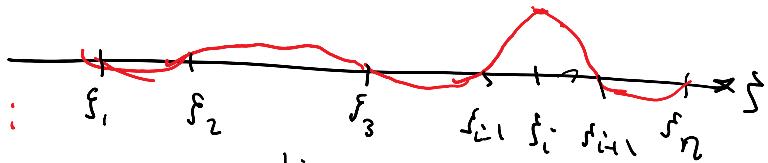
FEM2021/11/18

Thursday, November 18, 2021 2:51 PM

Basis functions for higher order 1D elements:



Lagrange Polynomials



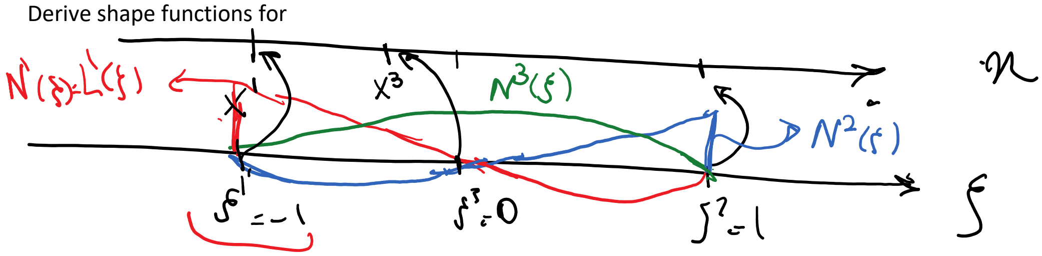
$$L_i(\xi) = \begin{cases} 1 & @\ \xi_i \\ 0 & @\ \xi_j \quad j \neq i \end{cases}$$

it's like a FE shape function.

$$L_i(\xi) = \frac{\prod_{j=1, j\neq i}^{n} (\xi - \xi_j)}{\prod_{j=1, j\neq i}^{n} (\xi_i - \xi_j)}$$

n-1'th order polynomial

Derive shape functions for



$$N^1(\xi) = L^1(\xi) = \frac{(\xi - \xi_2)(\xi - \xi_3)}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)} = \frac{(\xi - 1)(\xi - 0)}{(-1-1)(-1-0)} = \frac{\xi(\xi - 1)}{2}$$

$$N^1(x^1) = N^1(\xi = \xi_1) = N_1(-1) = \frac{(-1)(-1-1)}{2} = 1$$

$$N^1(\xi_2) = N_1(1) = \frac{1 \cdot (1-1)}{2} = 0$$

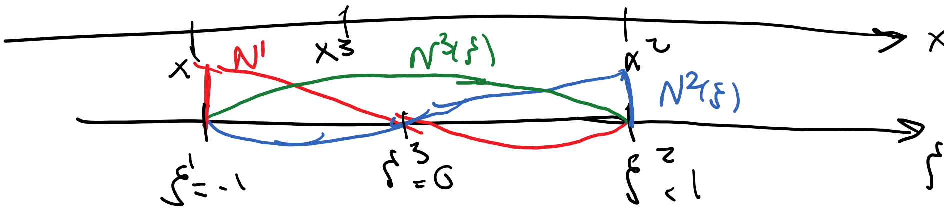
$$N^1(\xi_3) = N_1(0) = \frac{0 \cdot (0-1)}{2} = 0$$

$$N^2(\xi) = L^2(\xi) = \frac{(\xi - \xi_1)(\xi - \xi_3)}{(\xi_2 - \xi_1)(\xi_2 - \xi_3)} = \frac{(\xi - (-1))(\xi - 0)}{(1 - (-1))(1 - 0)} = \frac{\xi(\xi + 1)}{2}$$

$$N^3(\xi) = L^3(\xi) = \frac{(\xi - \xi_1)(\xi - \xi_2)}{(\xi_3 - \xi_1)(\xi_3 - \xi_2)} = \frac{(\xi - (-1))(\xi - 1)}{(0 - (-1))(0 - 1)} = 1 - \xi^2$$

Stiffness matrix for p=2, 1D bar element

Stiffness matrix for $p=2$, 1D bar element



$$N = [N^1(\xi), N^2(\xi), N^3(\xi)] = \left[\frac{\xi(\xi-1)}{2}, \frac{\xi(\xi+1)}{2}, 1-\xi^2\right] \qquad (1)$$

stiffness

Weak form: $\displaystyle \int \underbrace{\frac{dw}{dx}}_{\mathcal{L}_m} \underbrace{EA}_{D} \frac{du}{dx}\, dx = \ldots$

$\mathcal{L}_m(w) = w'$

$$B = \mathcal{L}_m(N) = \frac{dN}{dx}$$

$$\boxed{\begin{gathered} K^e = \int_e B_e^t D_e B_e \, dx \\ D_e = EA \\ B_e = \frac{d}{dx} N_e \end{gathered}} \qquad (2)$$

dropping e's

$$B = \frac{d}{dx}[N^1(\xi)\ N^2(\xi)\ N^3(\xi)] = \frac{1}{\underbrace{\left(\frac{dx}{d\xi}\right)}_{J}} \underbrace{\frac{d}{d\xi}[N^1(\xi)\ N^2(\xi)\ N^3(\xi)]}_{B_\xi}$$

$$B = \frac{1}{J} B_\xi, \qquad J = \frac{dx(\xi)}{d\xi} \text{ ? I'll discuss it later}$$

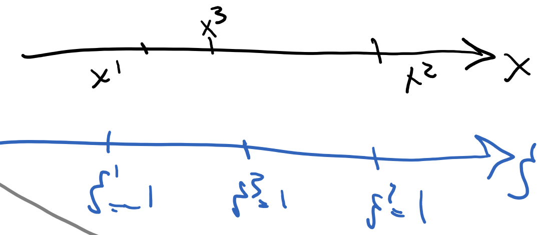
$$B_\xi = \frac{d}{d\xi}[N^1\ N^2\ N^3] = \frac{d}{d\xi}\left[\frac{\xi(\xi-1)}{2}, \frac{\xi(\xi+1)}{2}, 1-\xi^2\right] \qquad (3)$$

$$B_\xi = \left[\xi - \frac{1}{2}, \ \xi + \frac{1}{2}, \ -2\xi\right]$$

$$K = \int_e \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} D \, [B_1 \ B_2 \ B_3] \, dx$$

change of variable

$D \rightarrow AE$



$$K = \int_{\xi=-1}^{\xi=1} \underbrace{\left(\frac{1}{J} \begin{bmatrix} B^1_\xi \\ B^2_\xi \\ B^3_\xi \end{bmatrix} \right)}_{B^t} AE(\xi) \left[\frac{1}{J} \left([B^1_\xi \ \ B^2_\xi \ \ B^3_\xi] \right) \right] \underbrace{\left(\frac{dx}{d\xi} \right)}_{J} d\xi$$

$$\boxed{K = \int_{\xi=-1}^{\xi=1} \frac{EA(\xi)}{J(\xi)} \begin{bmatrix} B^1_\xi \\ B^2_\xi \\ B^3_\xi \end{bmatrix} [B^1_\xi \ \ B^2_\xi \ \ B^3_\xi] \, d\xi}$$

< ?

(4)

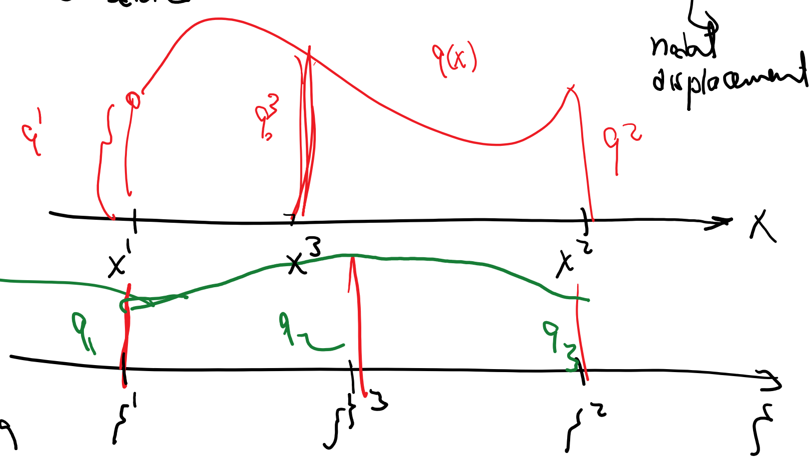
$$J(\xi) = \frac{d\, x(\xi)}{d\xi}$$

Hint

$$u^h(\xi) = [N^1(\xi) \ \ N^2(\xi) \ \ N^3(\xi)] \begin{bmatrix} a^1 \\ a^2 \\ a^3 \end{bmatrix} \quad (5a)$$

nodal displacement

1. shape functions interpolate solution

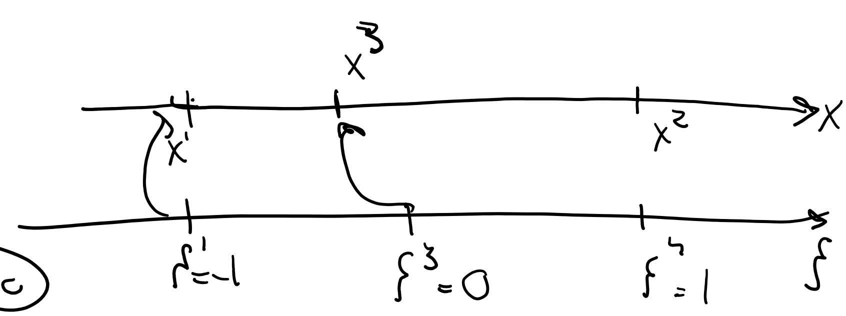


$$\tilde{q}(\xi) = [N^1(\xi) \ \ N^2(\xi) \ \ N^3(\xi)] \begin{bmatrix} q^1 \\ q^2 \\ q^3 \end{bmatrix} \quad (5b)$$

$$\tilde{q} \approx q$$

2. shape functions interpolate q & other fields

3. $x(\xi)$



$$x(\xi) = [N^1 \ \ N^2 \ \ N^3] \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} \quad (5c)$$

$x(\xi) = [N \quad N \quad N] \begin{bmatrix} x \\ x^3 \end{bmatrix}$ (5c) $\qquad \xi^1 = -1 \qquad \xi^3 = 0 \qquad \xi^2 = 1 \qquad \xi$

Isoparametric FEM formulation means that the solution (5a) is interpolated with the same set of basis functions that the geometry (x here, equation 5c) is interpolated.

(6)
$$x(\xi) = [N^1(\xi) \quad N^2(\xi) \quad N^3(\xi)] \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} = \left[\frac{\xi(\xi-1)}{2} \quad \frac{\xi(\xi+1)}{2} \quad 1-\xi^2\right] \underbrace{\begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix}}_{\text{for any element we know these values}}$$

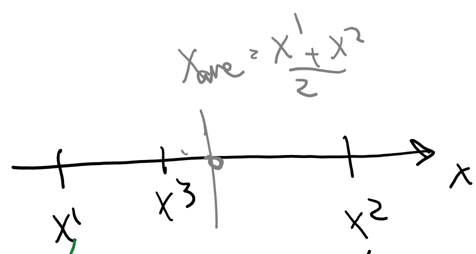
$$J(\xi) = \frac{dx(\xi)}{d\xi} = \underbrace{\frac{d}{d\xi}[N^1(\xi) \quad N^2(\xi) \quad N^3(\xi)]}_{B\xi} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

$$= \left[\xi - \frac{1}{2} \quad \xi + \frac{1}{2} \quad -2\xi\right] \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} \Rightarrow$$

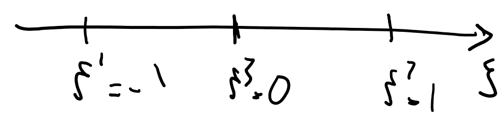
$$J(\xi) = \frac{dx(\xi)}{d\xi} = 2\left(\underbrace{\frac{x^1 + x^2}{2}} - x^3\right)\xi + \frac{1}{2}(x^2 - x^1)$$

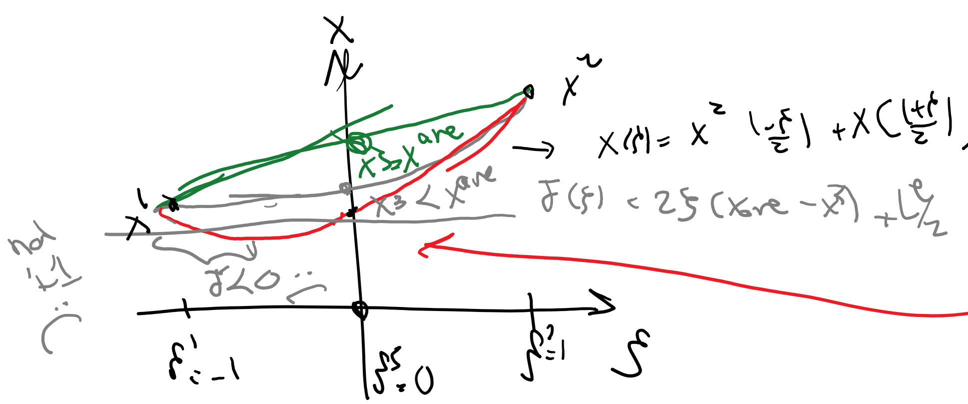
$$J(\xi) = 2\xi(X_{ave} - X^3) + \frac{L^e}{2}$$

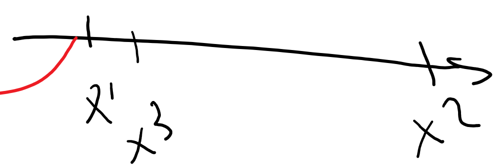
$X_{ave} = \frac{X^1 + X^2}{2}$



$L^e = X^2 - X^1$







$x(\xi) = x^2 \left(\frac{1-\xi}{2}\right) + x\left(\frac{1+\xi}{2}\right), \; J = L/2$

$J(\xi) < 2\xi(x_{ave} - x^3) + L^e/2$

this was your HW 5 assignment:

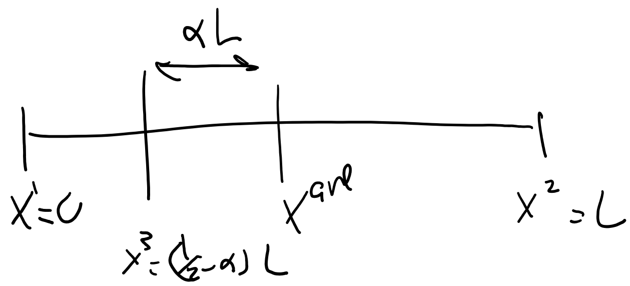
How far can x_3 go to left or right so that

J stays positive ... all

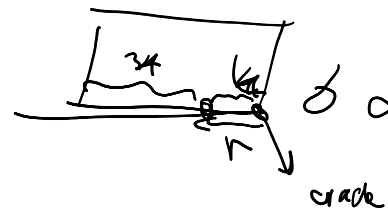
J stays positive

$$-0.25 \le \alpha \le 0.25$$

maximum distortion possible



In fracture mechanics if we put the "mid-point" at 1/4 distance
we can capture stress field singularity



$$\sigma \propto \frac{1}{\sqrt{r}}$$

Going back to element K

$$K^e_{3\times3} = \int_{\xi=-1}^{1} \frac{EA(\xi)}{\underbrace{2\xi(x_{ave}-x^3)+\frac{L_e}{2}}_{J(\xi)}} \underbrace{\begin{bmatrix}\xi-\frac{1}{2}\\ \xi+\frac{1}{2}\\ -2\xi\end{bmatrix}}_{B_\xi^t} \underbrace{\overbrace{\left[\xi-\frac{1}{2} \quad \xi+\frac{1}{2} \quad -2\xi\right]}^{\text{order } p-1}}_{B_\xi} d\xi \qquad (7)$$

element was order $p=2$

$B_\xi = \frac{d}{d\xi}N_\xi$

$EA(\xi)$ = constant : homogeneous material

Undistorted element ($x^3 = x_{ave} \to J = \frac{L_e}{2}$)

$$K^e = \int_{-1}^{1} \frac{2AE}{L^e} \begin{bmatrix}\xi-\frac{1}{2}\\ \xi+\frac{1}{2}\\ -2\xi\end{bmatrix} \left[\xi-\frac{1}{2} \quad \xi+\frac{1}{2} \quad -2\xi\right] d\xi = \frac{AE}{L}\begin{bmatrix}7/3 & -8/3 & 1/3\\ & 7/3 & -8/3\\ \text{sym} & & 16/3\end{bmatrix}$$

In general for order p: J = const, EA = constant

$$K^e = \int_{-1}^{1} \frac{2AE}{L} \underbrace{(B_e)^t}_{\text{order } p-1} \underbrace{(B_e)}_{\text{order } p-1} d\xi = \frac{2AE}{L}\int_{-1}^{1} \underbrace{B_e^t \, B_e}_{\text{order} = 2(p-1)} d\xi$$

the order of integrand for J = constant, $D = EA_z$ = is full order

for 1D bar element of order p full order $= 2(p-1)$

Quadrature: Numerical integration

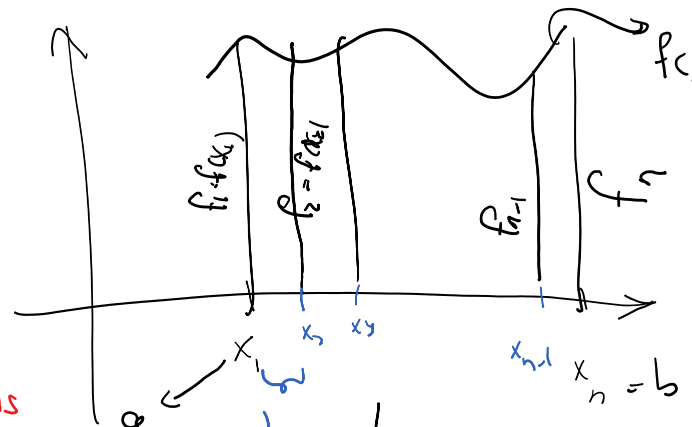
Newton-Cotes quadrature rules:

$$\int_{x_1}^{x_n} f(x)\,dx = L \sum \underbrace{w_i}_{\text{quadrature weights}} f(\underbrace{x_i}_{\text{quadrature points}})$$

$x_i = x_1 + (i-1)h$

known

unknown but we'll derive it



$h = \frac{L}{n-1}$ → # segments

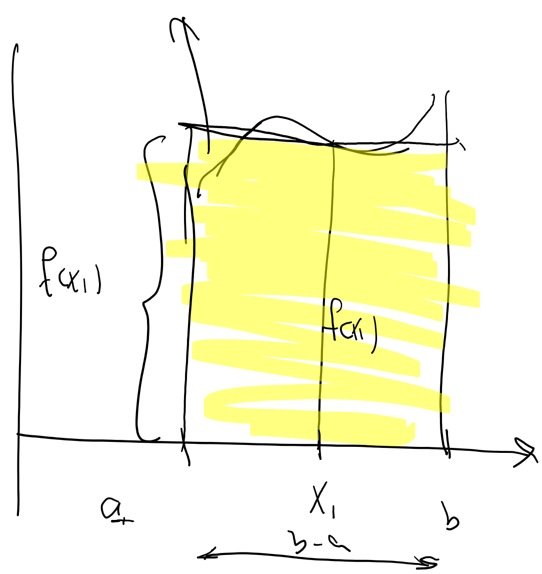
$L = x_n - x_1$

Examples of Newton-Cotes

$n = 1$

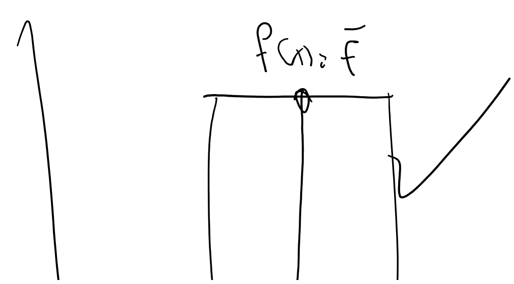
$$\int_a^b f(x) \approx (b-a)\, w_1 f(x_1), \quad w_1 = 1$$

area $= f(x_1)(b-a)$

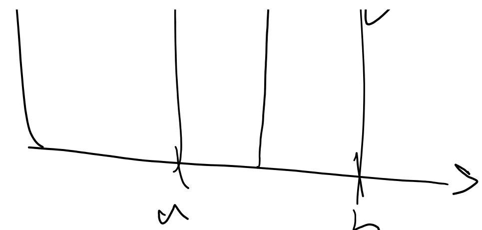


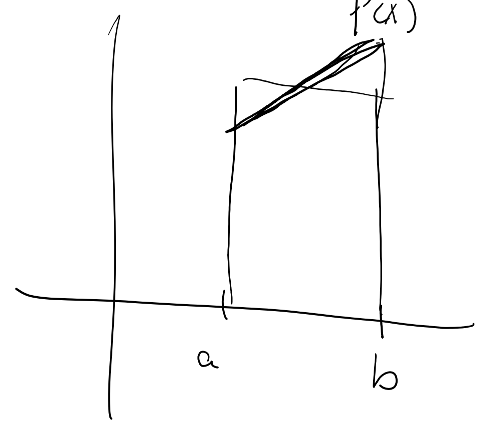
What polynomials a 1 point rule integrates exactly

const

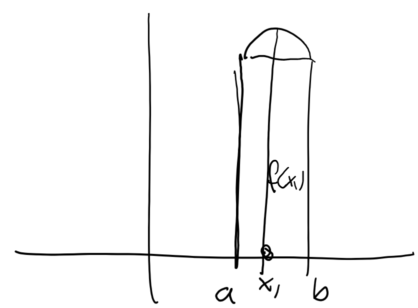


linear





zero order



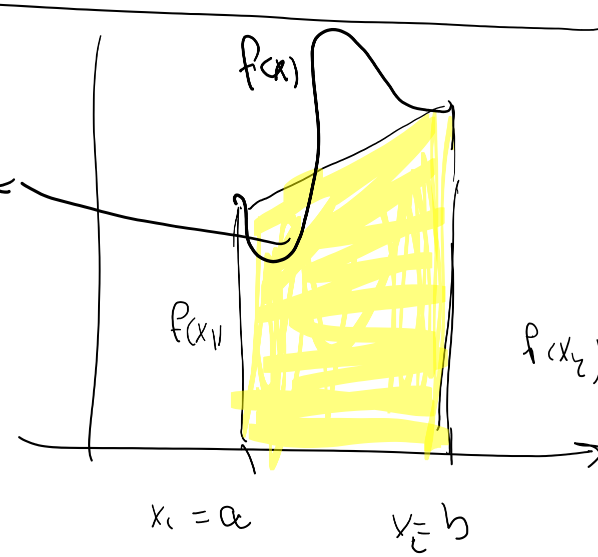
this "rectangular rule" integrates polynomials of order 0 & 1 exactly.

Trapezoidal rule

$$\int_a^b f(x)\,dx \approx \left[\frac{f(x_1)+f(x_2)}{2}\right](b-a) \quad \text{approx integral}$$

$$=(b-a)\left\{\frac{1}{2}f(x_1)+\frac{1}{2}f(x_2)\right\}$$

$$=(b-a)\left\{w_1 f(x_1)+w_2 f(x_2)\right\}$$



$w_1 = \frac{1}{2}$, $w_2 = \frac{1}{2}$ for trapezoidal rule (NC for $n=2$)