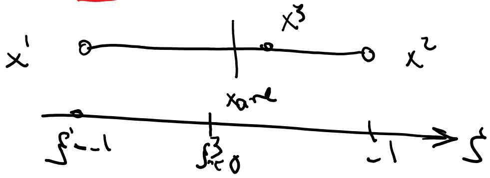
We want to numerically integrate k^e for 1D, bar element $p=2$

$L^e = x^2 - x^1$



$$J = 2\xi(x_{ave} - x^3) + \frac{L^e}{2}$$

$$k^e = \int_{-1}^{1} \overbrace{\frac{AE(\xi)}{\underbrace{2\xi(x_{ave}-x^3)+L^e/2}_{J(\xi)}} \begin{bmatrix} \xi - \frac{1}{2} \\ \xi + \frac{1}{2} \\ -2\xi \end{bmatrix} \underbrace{\left[\xi - \tfrac{1}{2} \quad \xi + \tfrac{1}{2} \quad -2\xi\right]}_{B_\xi}}^{I(\xi)} d\xi$$

Bar Element $\boxed{p=2}$ $N = \left[\frac{\xi(\xi-1)}{2} \quad \frac{\xi(\xi+1)}{2} \quad 1-\xi^2\right]$

in general 1D bar
order p

$L_m = (\ )'$ $B_\xi = \frac{\partial}{\partial \xi} N = \left[\xi - \frac{1}{2} \quad \xi + \frac{1}{2} \quad -2\xi\right]$ $p-1$

k^e

$$\int \frac{AE(\xi)}{J(\xi)} \underbrace{B^T(\xi)\, B(\xi)}_{2(p-1)}$$

ignoring this part → full integration order

Element order	Integrand order (o)	# Quad Pts (n)
p	$2(p-1)$	Gauss

$$n = \text{ceil}\left(\frac{o+1}{2}\right)$$
$$= \text{ceil}\left(\frac{2(p-1)+1}{2}\right)$$
$$= \text{ceil}\left(p - \tfrac{1}{2}\right) = p$$

Gauss

NC

o even

$$n = o + 1 = 2(p-2)+1$$
$$= 2p - 1$$

Summary

1D for order p needs

p	Gauss	Pts
$2p-1$	NC	Pts

For the example above $p=2$ so we need

2 Gauss Pts

NCs $2\times2-1=3$ pts

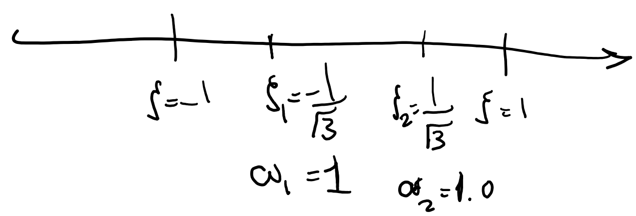
Quadrature using Gauss pts:

2 Gauss Pts

TABLE 5.6 *Sampling points and weights in Gauss-Legendre numerical integration (interval −1 to +1)*

number

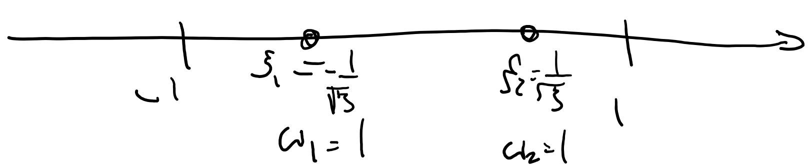
n	r_i			α_i		
1	0.	(15 zeros)		2.	(15 zeros)	
2	±0.57735	02691	89626	1.00000	00000	00000
3	±0.77459	66692	41483	0.55555	55555	55556
	0.00000	00000	00000	0.88888	88888	88889
4	±0.86113	63115	94053	0.34785	48451	37454
	±0.33998	10435	84856	0.65214	51548	62546
5	±0.90617	98459	38664	0.23692	68850	56189
	±0.53846	93101	05683	0.47862	86704	99366
	0.00000	00000	00000	0.56888	88888	88889
6	±0.93246	95142	03152	0.17132	44923	79170
	±0.66120	93864	66265	0.36076	15730	48139
	±0.23861	91860	83197	0.46791	39345	72691



Gauss scheme

$$K^e=\int_{-1}^{1} I(\xi)\,d\xi = \sum \omega_i\, I(\xi_i)$$

$$=\omega_1 I(\xi_1)+\omega_2 I(\xi_2) = 1.0\, I\left(-\frac{1}{\sqrt{3}}\right)+1.0\, I\left(\frac{1}{\sqrt{3}}\right)$$



NC $n=3$

TABLE 5.5 *Newton-Cotes numbers and error estimates*

n # pts interval pts	Number of intervals	C_0^n	C_1^n	C_2^n	C_3^n	C_4^n	C_5^n	C_6^n	Upper bound on error R_n as a function of the derivative of F
2	1	$\frac{1}{2}$	$\frac{1}{2}$						$10^{-1}(b-a)^3F^{II}(r)$
3	2	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$					$10^{-3}(b-a)^5F^{IV}(r)$
4	3	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$				$10^{-3}(b-a)^5F^{IV}(r)$
5	4	$\frac{7}{90}$	$\frac{32}{90}$	$\frac{12}{90}$	$\frac{32}{90}$	$\frac{7}{90}$			$10^{-6}(b-a)^7F^{VI}(r)$
6	5	$\frac{19}{288}$	$\frac{75}{288}$	$\frac{50}{288}$	$\frac{50}{288}$	$\frac{75}{288}$	$\frac{19}{288}$		$10^{-6}(b-a)^7F^{VI}(r)$
7	6	$\frac{41}{840}$	$\frac{216}{840}$	$\frac{27}{840}$	$\frac{272}{840}$	$\frac{27}{840}$	$\frac{216}{840}$	$\frac{41}{840}$	$10^{-9}(b-a)^9F^{VIII}(r)$

Simpson's rule

NC $\quad k^e = \int_{-1}^{1} I(\xi)\,d\xi = \underbrace{(1-(-1))}_{\text{NC always have } b-a \text{ upfront}}\left(\omega_1 f(\xi_1) + \omega_2 f(\xi_2) + \omega_3 f(\xi_3)\right)$

$= 2\left(\frac{1}{6}f(-1) + \frac{4}{6}f(0) + \frac{1}{6}f(1)\right)$

$\xi_1 = -1 \qquad \xi_2 = 0 \qquad \xi_3 = 1$

$\omega_1 = \frac{1}{6} \qquad \omega_2 = \frac{4}{6} \qquad \omega_3 = \frac{1}{6}$

$$\boxed{k = \frac{1}{3}f(-1) + \frac{4}{3}f(0) + \frac{1}{3}f(1)}$$

$\frac{1}{3} \qquad \frac{4}{3} \qquad \frac{1}{3}$

$\xi_1 = 0 \qquad \xi_2 = 0 \qquad \xi_3 = 1$

weighted pts

NC's

$$k^e = \int_{-1}^{1} \overbrace{\frac{AE(\xi)}{\underbrace{2\xi(x_{ave} - x^3) + L^e/2}_{J(\xi)}} \begin{bmatrix} \xi - \frac{1}{2} \\ \xi + \frac{1}{2} \\ -2\xi \end{bmatrix} \underbrace{\left[\xi - \tfrac{1}{2} \quad \xi + \tfrac{1}{2} \quad -2\xi\right]}_{B_\xi}}^{I(\xi)} d\xi$$

When is the full integration scheme exact (k^e is exactly calculated)?

1. AE should be constant (Homogeneous material)

2. J constant (undistorted element)

3. (Not seen here for nonlinear problems like plasticity, nonlinear elasticity, non linear effects are small

then K is exact (we obtained this before

$$\frac{AE}{L}\begin{bmatrix} 7/3 & -8/3 & \cdot \; - \\ - & & - \\ - & & - \end{bmatrix}$$

Q2: How many more points are needed to integrate K exactly when $J \neq cte$ ($X_3 \neq X_{ave}$)

With no # of quad pts we integrate this exactly

Q3 How many pts should we use when when $J \neq cte$ or $AE \neq cte$

Almost always full integration order suffices

$|\text{quadrature error}| \leq |\text{discretization error}|$

in terms of order of magnitude

Q4 Can we even use fewer quad pts to be more efficient?

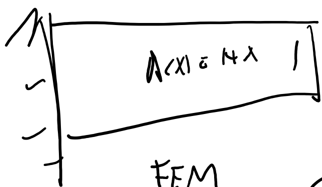
Cases for this:

a) Reducing computational costs

b)

Stiffness-based FEM results in over stiff solutions

$F = Ku$



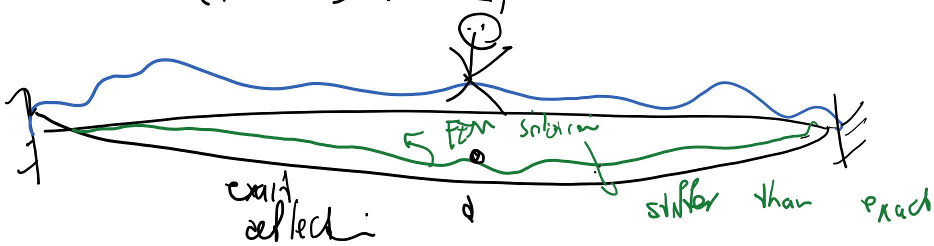
FEM $k = 1.5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Exact $1.4\ldots \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$1/\ln 2$

k^{FEM} stiffer $\text{exact}\ k$

$(1.5 > 1.4\ldots)$

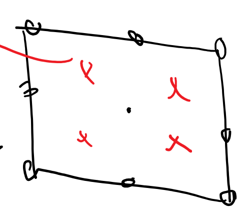


Reduced order integration can make the structure less stiff:)

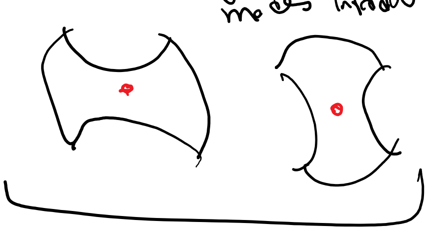
Cases against reduced order integration:

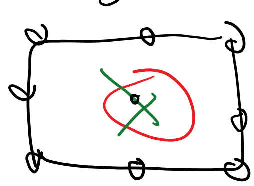
It can result in nonphysical solution by introducing nonphysical ZERO ENERGY MODES

full integration



hour glass modes introduced go with 1 pt



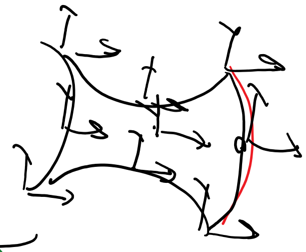


These are the shapes that have zero strain right at the reduced scheme Gauss points (here just 1 point) but realistically have nonzero strains elsewhere. But with reduced order integration we miss these points.

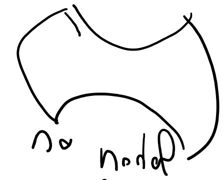
elsewhere. But with reduced order integration we miss these points.

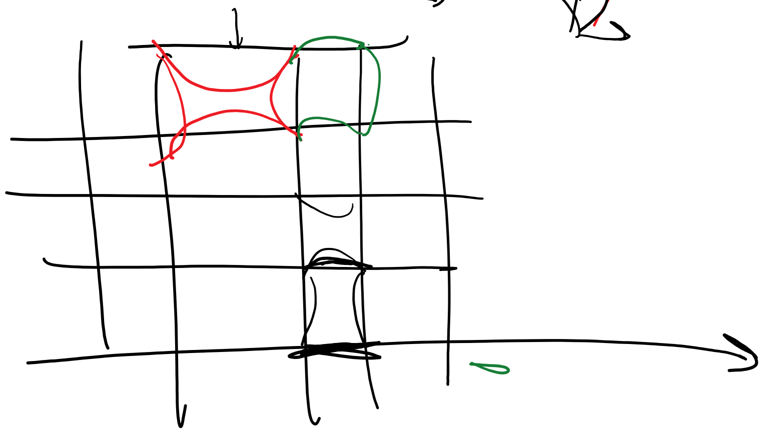
18 dofs

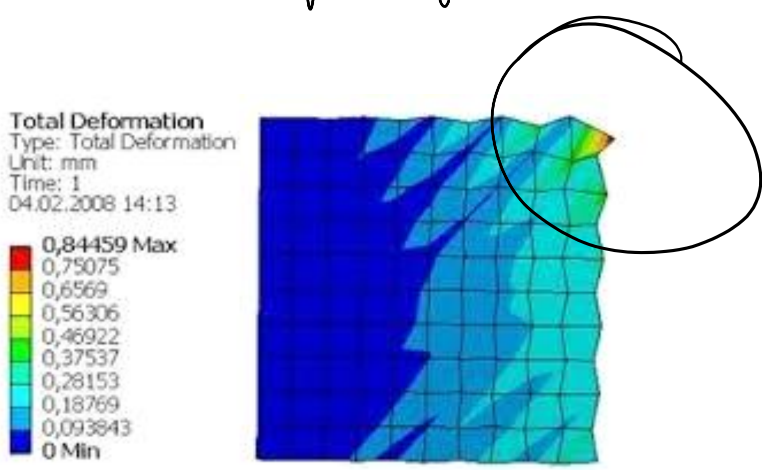
$u_1 \cdots u_{18} \neq 0$

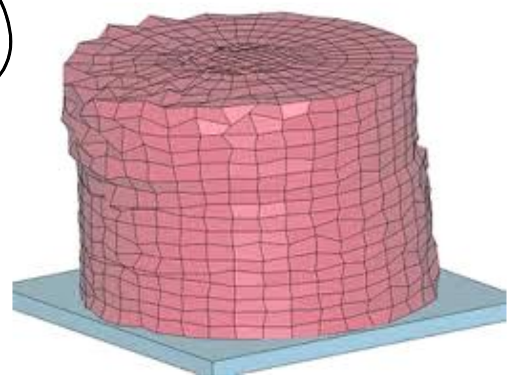


Forces $= 0$









How to check if the under-integration is safe, that is, it does not introduce nonphysical zero modes

$p = 1$ element

Are there any nonzero u's $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \neq 0$

$\ni \quad F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = 0$

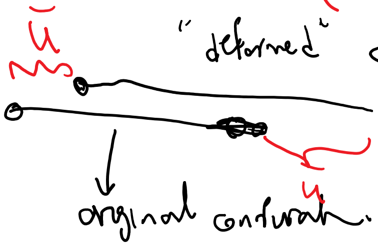
$k^e = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ $\qquad F = k^e u$

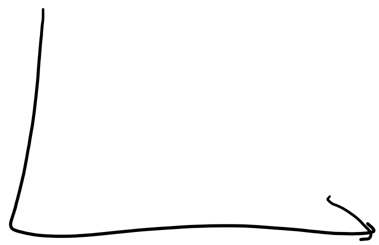
$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{AE}{L}\begin{bmatrix} u_1 - u_2 \\ u_2 - u_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{AE}{L}\begin{pmatrix} u_1 - u_2 \\ u_2 - u_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \boxed{u_1 = u_2}$$

If $u_1 = u_2$ $\left(u = \tilde{u}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \rightarrow F = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$



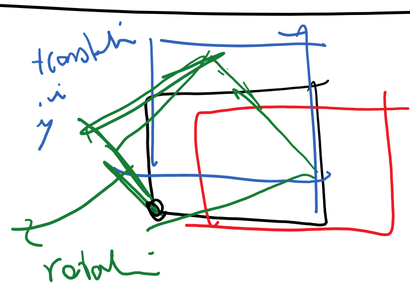


$$\varepsilon = \frac{\Delta u}{L} = \frac{u_2 - u_1}{L}$$

$\varepsilon = 0 \rightarrow \sigma = 0 \rightarrow F = 0$

Rigid motion

2D



Translation in x

3 rigid modes in 2D

6 " " " 3D

more generally Look for solutions for which $\boxed{L_m(u) = 0}$

weak condition $L_m = \nabla$

only 1 zero mode

Going back to p=1 element

\# zero modes = \# zero eigenvalues = 1

n = size of k $\qquad k = \frac{AE}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$ $\quad (n=2)$

$$\text{rank}(K) = n - \#\text{ zero modes}$$

$K_{n\times n}$

Matlab has a rank functi

K known $\quad n$ ✓ $\quad \underbrace{\text{rank}(k)}_{\text{matlab}} \rightarrow \#$ zero modes $= n - \text{rank}(K)$

$p=2$

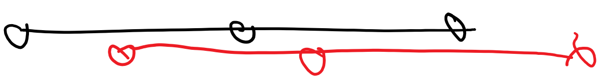
element

$K_{3\times3}$ $\quad \boxed{n=3}$

what if rank(k) = 1

zero modes = 3 − 1 = 2

should be 1



$u = \bar{u}\begin{bmatrix}1\\1\\1\end{bmatrix}$

and only this one

anything beyond this is nonphysical zero mode!

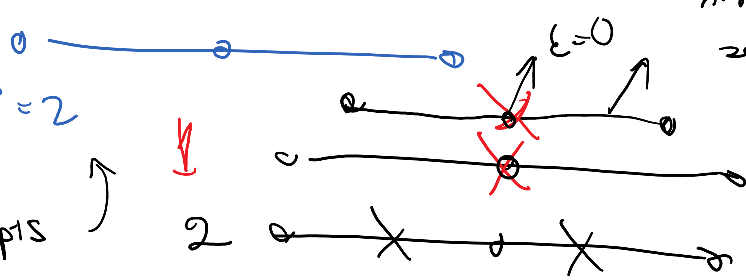
physically acceptable

correct rank = n − # zeros = 3 − 1 = 2

$K_{3\times3}$

HW

$P=2$



$\varepsilon=0$

nonphysical zero mode

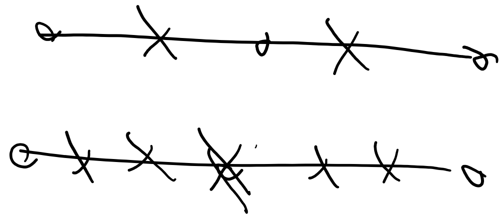
n Gauss pts $\quad$ 1 $\quad$ 2

rank=1

rank=2 :)

n Gauss pts / 2   rank = 2 :)

≥ 5   = 2

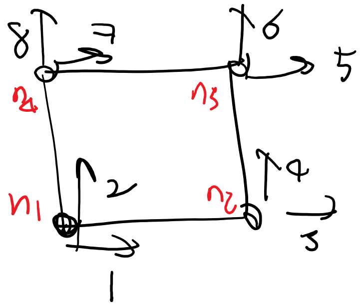


Another example:
2D elasticity

8 dofs

↓

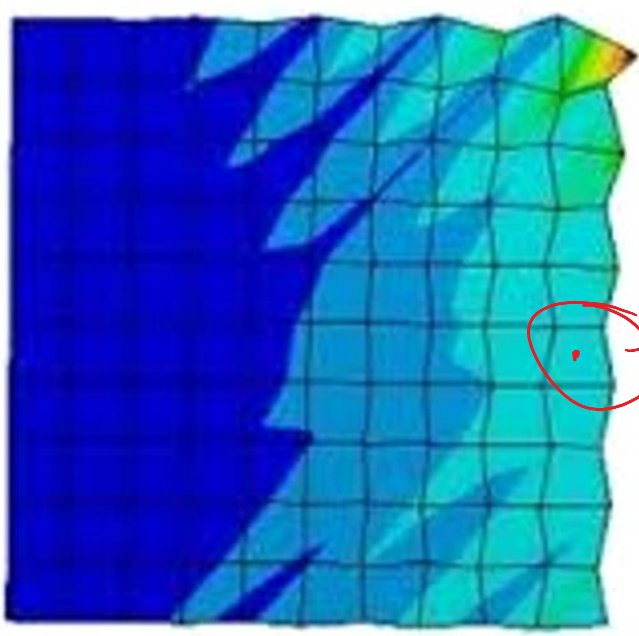
$K_{8\times8}$



→ how many zero modes
3

Correct rank = 8 − 3 = 5

Unacceptable reduced integration has rank < 5



physically unacceptable zero modes