FEM2021/11/30

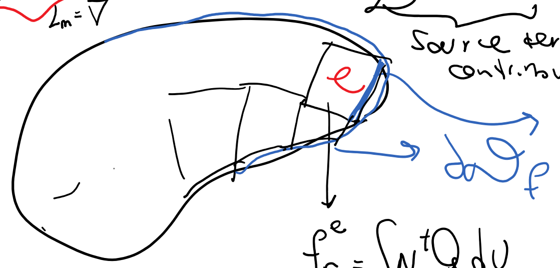
Tuesday, November 30, 2021 2:45 PM

2D and 3D higher order elements

As a sample application I consider thermal heat conduction (HW1) for 2D quad elements:

Weak statement

$$\underbrace{\int_D \underbrace{\nabla w}_{L_m(w)} \, \underbrace{k}_{2\times2 \text{ conductivity matrix}} \, \nabla T \, dV}_{L_m = \nabla} = \underbrace{\int_D w Q \, dV}_{\text{Source term contribution}} - \underbrace{\int_{\partial D_f} w \bar{q} \, ds}_{\text{Neumann boundary}}$$



$$f_r^e = \int_e N^T Q \, dV \qquad f_N^e = -\int_{\partial D_f} N^T \bar{q} \, ds$$

$$k^e = \int_e B^{e^T} D^e B^e \, dV \qquad D^e = \bar{\bar{k}}_{2\times2}$$

$$B^e = L_m(N^e) = \nabla N^e$$

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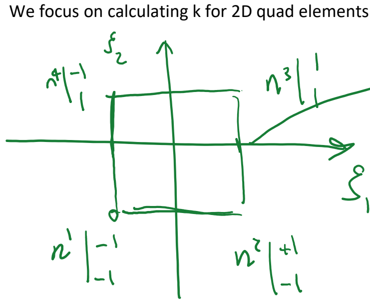
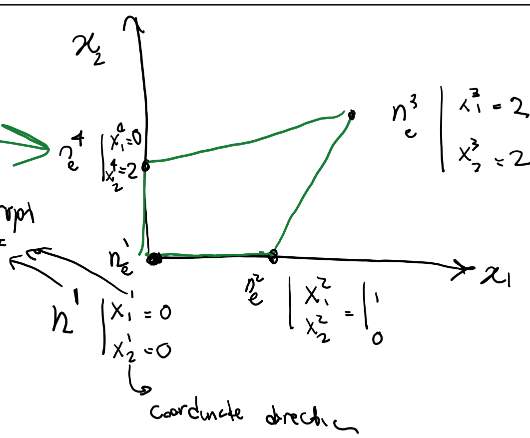
fig

We focus on calculating k for 2D quad elements today



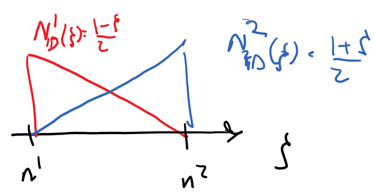
parent geometry

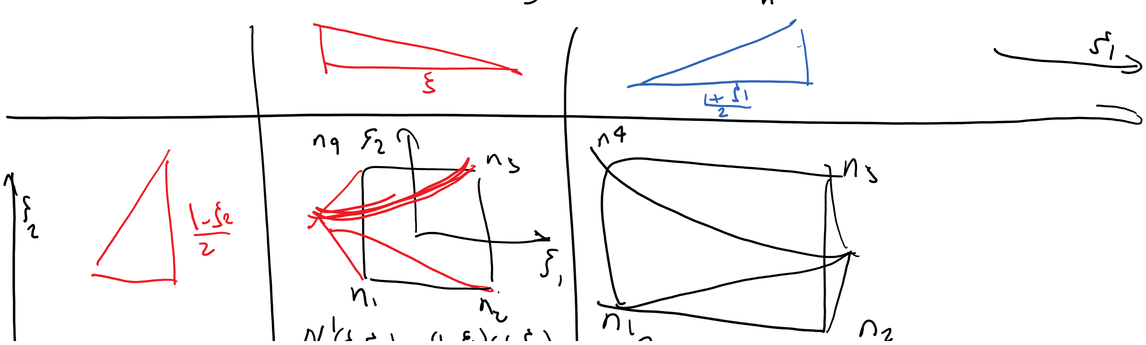


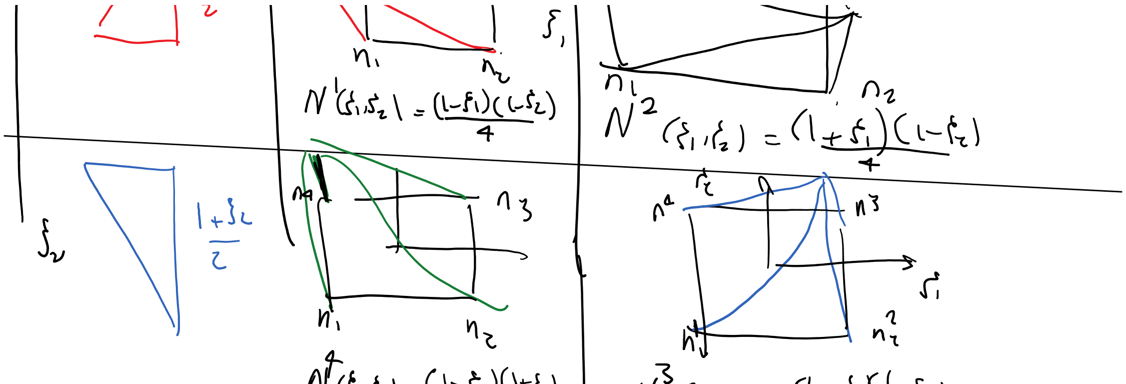
real geometry

Step 1: Formulate shape functions

Recall 1D p=1 element







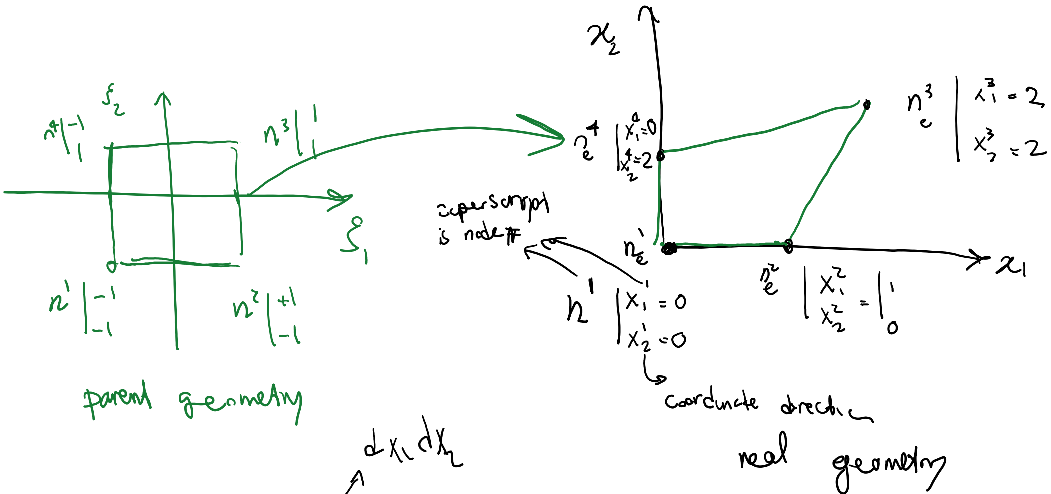
$$N^1(\xi_1,\xi_2) = \frac{(1-\xi_1)(1-\xi_2)}{4} \qquad N^2(\xi_1,\xi_2) = \frac{(1+\xi_1)(1-\xi_2)}{4}$$

$$N^4(\xi_1,\xi_2) = \frac{(1-\xi_1)(1+\xi_2)}{4} \qquad N^3(\xi_1,\xi_2) = \frac{(1+\xi_1)(1+\xi_2)}{4}$$

$$\boxed{N = [N^1, N^2, N^3, N^4] = \left[\frac{(1-\xi_1)(1-\xi_2)}{4}, \frac{(1+\xi_1)(1-\xi_2)}{4}, \frac{(1+\xi_1)(1+\xi_2)}{4}, \frac{(1-\xi_1)(1+\xi_2)}{4}\right]}$$

$N(\xi_1, \xi_2)$

ND



$dx_1 dx_2$

$$K^e = \int_e B^{e^T} D^e B^e \, dV \quad , \quad B^e = L_m(N) = \nabla [N^1, N^2, N^3, N^4]$$

$$B^e = \begin{bmatrix} \frac{\partial N^1}{\cancel{\partial x^1}} & \frac{\partial N^2}{\partial x^1} & \frac{\partial N^3}{\cancel{\partial x^1}} & \frac{\partial N^4}{\partial x^1} \\ \frac{\partial N^1}{\cancel{\partial x^2}} & \frac{\partial N^2}{\partial x^2} & \frac{\partial N^3}{\partial x^2} & \frac{\partial N^4}{\partial x^2} \end{bmatrix}$$

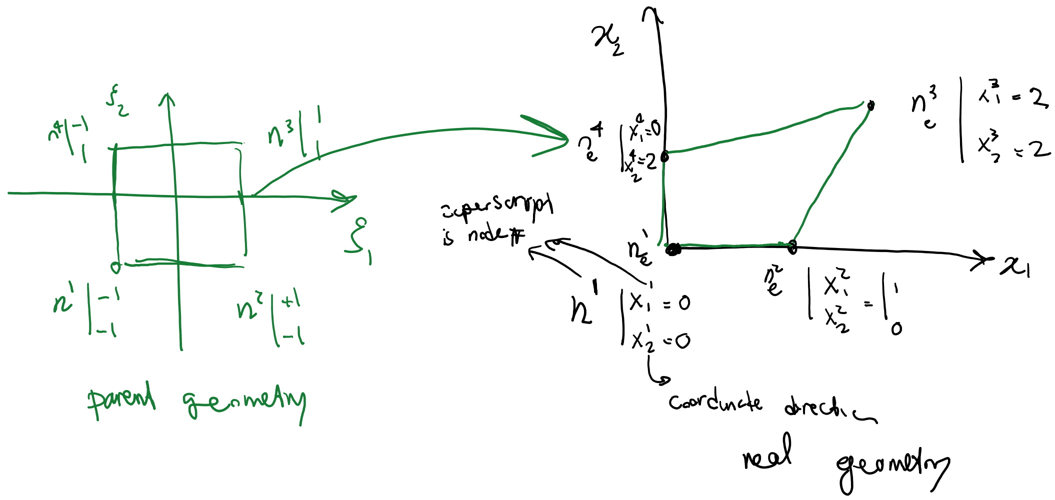
4x2 matrix

We can calculate B for ξ but not for x (challenge 1)

Integral needs to be pulled back to the parent element

$$dV_x = dx_1 dx_2 = (\,?\,)\, d\xi_1 d\xi_2$$

How to address challenge number one:



$$\xi \longrightarrow X \checkmark$$

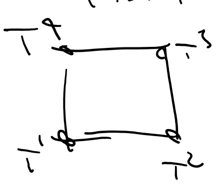
or

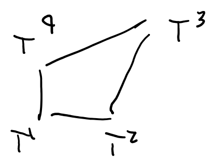
$$X \longrightarrow \xi$$

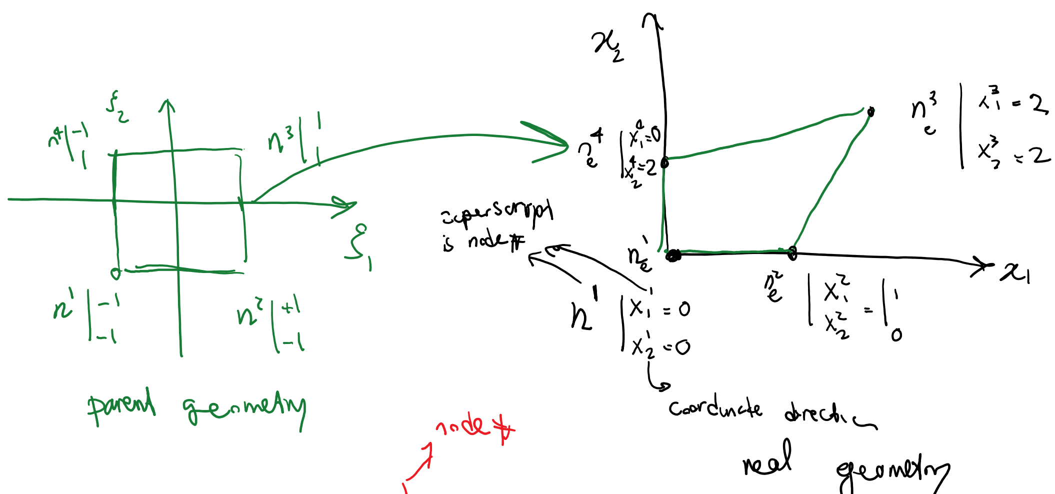
Again we use FEM shape function interpolation property:

$$T^h(\xi_1,\xi_2) = T^1 N^1(\xi_1,\xi_2) + T^2 N^2(\xi_1,\xi_2) + T^3 N^3(\xi_1,\xi_2) + T^4 N^4(\xi_1,\xi_2)$$

$T^1 \dots T^4$ are nodal (unknown) temperatures / dofs







node #

$$X_1(\xi_1,\xi_2) = X_1^1 N^1(\xi_1,\xi_2) + X_1^2 N^2(\xi_1,\xi_2) + X_1^3 N^3(\xi_1,\xi_2) + X_1^4 N^4(\xi_1,\xi_2)$$

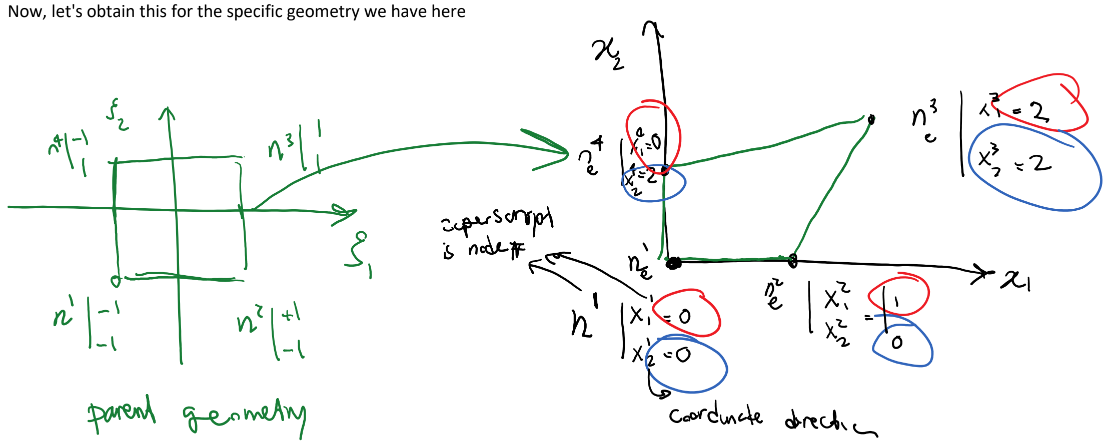
→ direction

This always gives $X_1(\xi_1,\xi_2)$ will be obtained

Similar process $X_1 \rightarrow X_2$ will provide $X_2(\xi_1, \xi_2)$

$$\boxed{\begin{matrix} X_1(\xi_1, \xi_2) \\ X_2(\xi_1, \xi_2) \end{matrix}}$$

Now, let's obtain this for the specific geometry we have here



$$X_1(\xi_1,\xi_2) = \overset{X_1^1}{0}\left[\overset{N^1}{\frac{(1-\xi_1)(1-\xi_2)}{4}}\right] + \overset{X_1^2}{1}\left[\overset{N^2}{\frac{(1+\xi_1)(1-\xi_2)}{4}}\right] + \overset{X_1^3}{2}\left[\overset{N^3}{\frac{(1+\xi_1)(1+\xi_2)}{4}}\right] + \overset{X_1^4}{0}\left[\overset{N^4}{\frac{(1-\xi_1)(1+\xi_2)}{4}}\right]$$

$$X_2(\xi_1,\xi_2) = 0 \quad '' \quad + 0 \quad '' \quad + 2 \quad '' \quad 2 \quad ''$$

simplify to get

$$\boxed{\begin{aligned} x_1(\xi_1,\xi_2) &= \frac{1}{4}(3 + 3\xi_1 + \xi_2 + \xi_1\xi_2) \\ x_2(\xi_1,\xi_2) &= \frac{1}{4}(3 + \xi_1 + 3\xi_2 + \xi_1\xi_2) \end{aligned}} \quad (2)$$

only for this geometry

$$B = \begin{bmatrix} \frac{\partial N^1}{\partial x^1} & N^2 & N^3 & N^4 \\ \frac{\partial N^1}{\partial x^2} & & & \end{bmatrix} \qquad B_\xi = \begin{bmatrix} \frac{\partial N^1}{\partial \xi_1} & \frac{\partial N^2}{\partial \xi_1} & & \\ \frac{\partial N^1}{\partial \xi_2} & \cdots & & \frac{\partial N^4}{\partial \xi_2} \end{bmatrix}$$

? $N^1(\xi_1,\xi_2)$ $\times$:(we can calculate :)

but we need B

$$K^e = \int_e B^T D B \, dV$$

relation between $B_\xi \longrightarrow B$

or to begin with $\begin{bmatrix} \dfrac{\partial N^i}{\partial \xi_1} \\ \dfrac{\partial N^i}{\partial \xi_2} \end{bmatrix} \to \begin{bmatrix} \dfrac{\partial N^i}{\partial x_1} \\ \dfrac{\partial N^i}{\partial x_2} \end{bmatrix}$ $i = 1, \ldots, 4$

want this

$$\frac{\partial N^i}{\partial x_1} = \frac{\partial N^i}{\partial \xi_1} \underbrace{\frac{\partial \xi_1}{\partial x_1} + \frac{\partial N^i}{\partial \xi_2} \frac{\partial \xi_2}{\partial x_1}}_{\text{need } \xi(x)}$$

need $\xi(x)$ but we have $X(\xi)$

Let's do the opposite

$$\frac{\partial N^i}{\partial \xi_1} = \frac{\partial N^i}{\partial x_1} \frac{\partial x_1}{\partial \xi_1} + \frac{\partial N^i}{\partial x_2} \frac{\partial x_2}{\partial \xi_1}$$

$$\frac{\partial N^i}{\partial \xi_2} = \frac{\partial N^i}{\partial x_1} \frac{\partial x_1}{\partial \xi_2} + \frac{\partial N^i}{\partial x_2} \frac{\partial x_2}{\partial \xi_2}$$

$$\boxed{\underbrace{\begin{bmatrix} \dfrac{\partial N^i}{\partial \xi_1} \\ \dfrac{\partial N^i}{\partial \xi_2} \end{bmatrix}}_{\text{can easily calculate this}} = \underbrace{\begin{bmatrix} \dfrac{\partial x_1}{\partial \xi_1} & \dfrac{\partial x_2}{\partial \xi_1} \\ \dfrac{\partial x_1}{\partial \xi_2} & \dfrac{\partial x_2}{\partial \xi_2} \end{bmatrix}}_{J \text{ Jacobian (matrix)}} \begin{bmatrix} \dfrac{\partial N^i}{\partial x_1} \\ \dfrac{\partial N^i}{\partial x_2} \end{bmatrix}}$$

② we want this

repeat this for $i = 1, \ldots, 4$

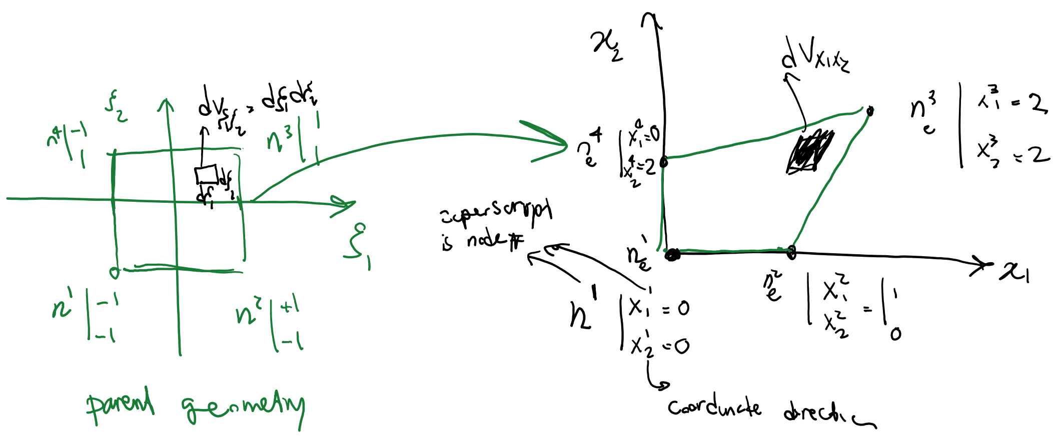
$$\begin{bmatrix} \dfrac{\partial N^1}{\partial \xi_1} & \dfrac{\partial N^2}{\partial \xi_1} & \dfrac{\partial N^3}{\partial \xi_1} & \dfrac{\partial N^4}{\partial \xi_1} \\ \dfrac{\partial N^1}{\partial \xi_2} & \dfrac{\partial N^2}{\partial \xi_2} & \dfrac{\partial N^3}{\partial \xi_2} & \dfrac{\partial N^4}{\partial \xi_2} \end{bmatrix} = J \underbrace{\begin{bmatrix} \dfrac{\partial N^1}{\partial x_1} & \dfrac{\partial N^2}{\partial x_1} & \dfrac{\partial N^3}{\partial x_1} & \dfrac{\partial N^4}{\partial x_1} \\ \dfrac{\partial N^1}{\partial x_2} & \dfrac{\partial N^2}{\partial x_2} & \dfrac{\partial N^3}{\partial x_2} & \dfrac{\partial N^4}{\partial x_2} \end{bmatrix}}_{B = \nabla_{x_1 x_2} N \;?}$$

4×2

$B_\xi = \nabla_{\xi_1 \xi_2} N$ ✓

$B = \nabla_{x_1 x_2} N$?

$$B = J^{-1} B_\xi$$



Challenge 2:

real geometry

$$\underbrace{\int_e B^T D B \, dV_{x_1x_2} = \int_{\xi_1=-1}^{1}\int_{\xi_2=-1}^{1} B^t D B \, |J| \, \underbrace{(d\xi_1 d\xi_2)}_{dV_{\xi_1\xi_2}}}$$

$$\frac{dV_{x_1x_2}}{dV_{\xi_1\xi_2}} = |J|$$

notation for $\det J$

for a valid change of coordinate $\det J > 0$ almost everywhere
a.e.

$$\boxed{K^e = \int_{\xi_1=-1}^{1}\int_{\xi_2=-1}^{1} B^t D \overset{\nabla N}{B} \, |J| \, d\xi_1 d\xi_2}$$

$$\boxed{\text{eq (3)} \quad B = J^{-1} B_\xi}$$

$$k^e = \int_{-1}^{1}\int_{-1}^{1} (J^{-1} B_\xi)^t \, \kappa_{2\times2} \, (J^{-1} B_\xi) \, |J| \, d\xi_1 d\xi_2$$

$$\boxed{(K^e)_{4\times4} = \int_{-1}^{1}\int_{-1}^{1} (B_\xi^t)_{4\times2} \left(J^{-t} \kappa J^{-1} |J|\right)_{2\times2} (B_\xi)_{2\times4} \, d\xi_1 d\xi_2} \quad (4a)$$

correct for any quad $p=1$ element

$$N = [N^1 \; N^2 \; N^3 \; N^4] = \left[\frac{(1-\xi_1)(1-\xi_2)}{4} \;\; \frac{(1+\xi_1)(1-\xi_2)}{4} \;\; \frac{(1+\xi_1)(1+\xi_2)}{4} \;\; \frac{(1-\xi_1)(1+\xi_2)}{4}\right]$$

$$N = [N^1 \; N^2 \; N^3 \; N^4] = \left[\frac{(1-\xi_1)(1-\xi_2)}{4} \quad \frac{(1+\xi_1)(1-\xi_2)}{4} \quad \frac{(1+\xi_1)(1+\xi_2)}{4} \quad \frac{(1-\xi_1)(1+\xi_2)}{4} \right]$$ quad $p=1$ element

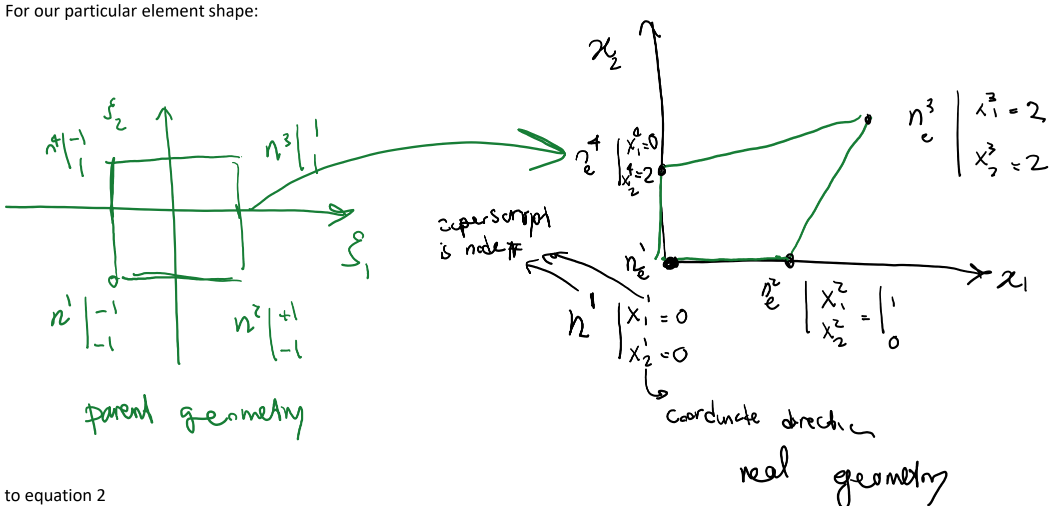
$$B_\xi = \nabla_{\xi_1,\xi_2} N = \begin{bmatrix} \frac{\partial N^1}{\partial \xi_1} & \frac{\partial N^2}{\partial \xi_1} & \cdots \\ \frac{\partial N^1}{\partial \xi_2} & \frac{\partial N^2}{\partial \xi_1} & \end{bmatrix}$$

$$N^2 = \frac{(1+\xi_1)(1-\xi_2)}{4} \longrightarrow \frac{\partial N_2}{\partial \xi_1} = \frac{1-\xi_2}{4} \qquad \frac{\partial N_2}{\partial \xi_2} = -\left(\frac{1+\xi_1}{4}\right)$$

$$\boxed{K^e = \int_{\xi_1=-1}^{1}\int_{\xi_2=-1}^{1} \underbrace{\begin{bmatrix} -\left(\frac{1-\xi_2}{4}\right) & -\left(\frac{1-\xi_1}{4}\right) \\ \frac{1-\xi_2}{4} & -\left(\frac{1+\xi_1}{4}\right) \\ \frac{1+\xi_2}{4} & \frac{1+\xi_1}{4} \\ -\left(\frac{1+\xi_2}{4}\right) & \left(\frac{1-\xi_1}{4}\right) \end{bmatrix}}_{B_\xi^t} J^{-t} \kappa J^{-1} |J| \underbrace{\begin{bmatrix} -\left(\frac{1-\xi_2}{4}\right) & \left(\frac{1-\xi_2}{4}\right) & \frac{1+\xi_2}{4} & -\left(\frac{1+\xi_2}{4}\right) \\ -\left(\frac{1-\xi_1}{4}\right) & -\left(\frac{1+\xi_1}{4}\right) & \frac{1+\xi_1}{4} & \frac{1-\xi_1}{4} \end{bmatrix}}_{B_\xi} d\xi_1 d\xi_2}$$

(4b) General for any quad $p=1$, thermal problem

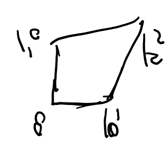
For our particular element shape:



Referring to equation 2

(2)

$$\boxed{\begin{aligned} x_1(\xi_1,\xi_2) &= \tfrac{1}{4}(3+3\xi_1+\xi_2+\xi_1\xi_2) \\ x_2(\xi_1,\xi_2) &= \tfrac{1}{4}(3+\xi_1+3\xi_2+\xi_1\xi_2) \end{aligned} \quad \text{only for this geometry}}$$



$$J^{-t} \kappa J^{-1} \longleftarrow J = \begin{bmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_1} \\ \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_2} \end{bmatrix} = \begin{bmatrix} \frac{3+\xi_2}{4} & \frac{1+\xi_2}{4} \\ \frac{1+\xi_1}{4} & \frac{3+\xi_1}{4} \end{bmatrix}$$

$$J_{ij} = \frac{\partial x_j}{\partial \xi_i}$$

$$|J| = \frac{(3+\xi_2)(3+\xi_1)}{16} - \frac{(1+\xi_2)(1+\xi_1)}{16} = \frac{4+\xi_1+\xi_2}{8}$$

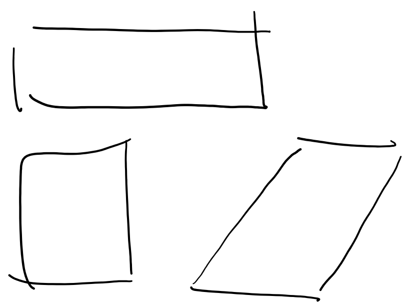
$$J = \nabla^t_{x/\xi}$$

$$|J| = \frac{(3+\xi_2)(3+\xi_1)}{16} - \frac{(1+\xi_2)(1+\xi_1)}{16} = \frac{4+\xi_1+\xi_2}{8} \qquad J = \nabla_{x/\xi}^{t}$$

(4b)

$$\boxed{K^e = \int_{\xi_1=-1}^{1}\int_{\xi_2=-1}^{1} \underbrace{\begin{bmatrix} -\left(\frac{1-\xi_2}{4}\right) & -\left(\frac{1-\xi_1}{4}\right) \\ \frac{1-\xi_2}{4} & -\left(\frac{1+\xi_1}{4}\right) \\ \frac{1+\xi_2}{4} & \frac{1+\xi_1}{4} \\ -\left(\frac{1+\xi_2}{4}\right) & \left(\frac{1-\xi_1}{4}\right) \end{bmatrix}}_{B_\xi^t} \underbrace{J^{-t}}_{=(J^{-1})^t} k J^{-1} |J| \underbrace{\begin{bmatrix} -\left(\frac{1-\xi_2}{4}\right) & \left(\frac{1-\xi_2}{4}\right) & \frac{1+\xi_2}{4} & -\left(\frac{1+\xi_2}{4}\right) \\ -\left(\frac{1-\xi_1}{4}\right) & -\left(\frac{1+\xi_1}{4}\right) & \frac{1+\xi_1}{4} & \frac{1-\xi_1}{4} \end{bmatrix}}_{B_\xi} d\xi_1 d\xi_2}$$

full integration order $\quad J$ = constant
assumes



"It ignores $J^{-t} k J^{-1} |J|$ term

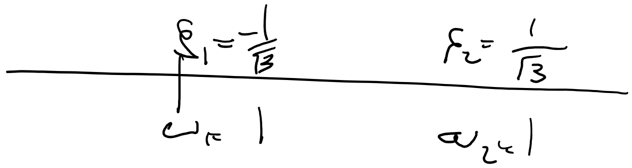
ignore

$$\underbrace{\boxed{K^e = \int_{\xi_1=-1}^{1}\int_{\xi_2=-1}^{1} \underbrace{\begin{bmatrix} -\left(\frac{1-\xi_2}{4}\right) & -\left(\frac{1-\xi_1}{4}\right) \\ \frac{1-\xi_2}{4} & -\left(\frac{1+\xi_1}{4}\right) \\ \frac{1+\xi_2}{4} & \frac{1+\xi_1}{4} \\ -\left(\frac{1+\xi_2}{4}\right) & \left(\frac{1-\xi_1}{4}\right) \end{bmatrix}}_{B_\xi^t} \underbrace{J^{-t}}_{=(J^{-1})^t} k J^{-1} |J| \underbrace{\begin{bmatrix} -\left(\frac{1-\xi_2}{4}\right) & \left(\frac{1-\xi_2}{4}\right) & \frac{1+\xi_2}{4} & -\left(\frac{1+\xi_2}{4}\right) \\ -\left(\frac{1-\xi_1}{4}\right) & -\left(\frac{1+\xi_1}{4}\right) & \frac{1+\xi_1}{4} & \frac{1-\xi_1}{4} \end{bmatrix}}_{B_\xi} d\xi_1 d\xi_2}}_{I(\xi_1,\xi_2)}$$

$$I_{11} = +\left(\frac{1-\xi_2}{4}\right)^2 + \left(\frac{1-\xi_1}{4}\right)^2 \qquad \text{order 2 in } \xi_1 \,\&\, \xi_2$$

Gauss Quadrature

order in $\xi_1 = 2 \;\rightarrow\; n\,gp_{\xi_1} = \text{ceil}\left(\frac{o_1+1}{2}\right) = \text{ceil}(1.5) = 2$



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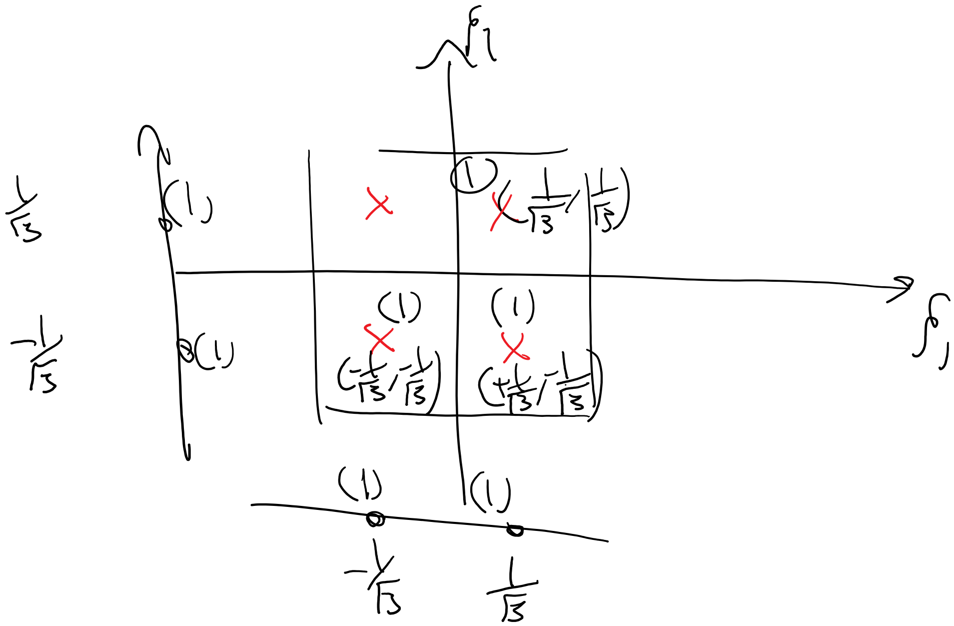
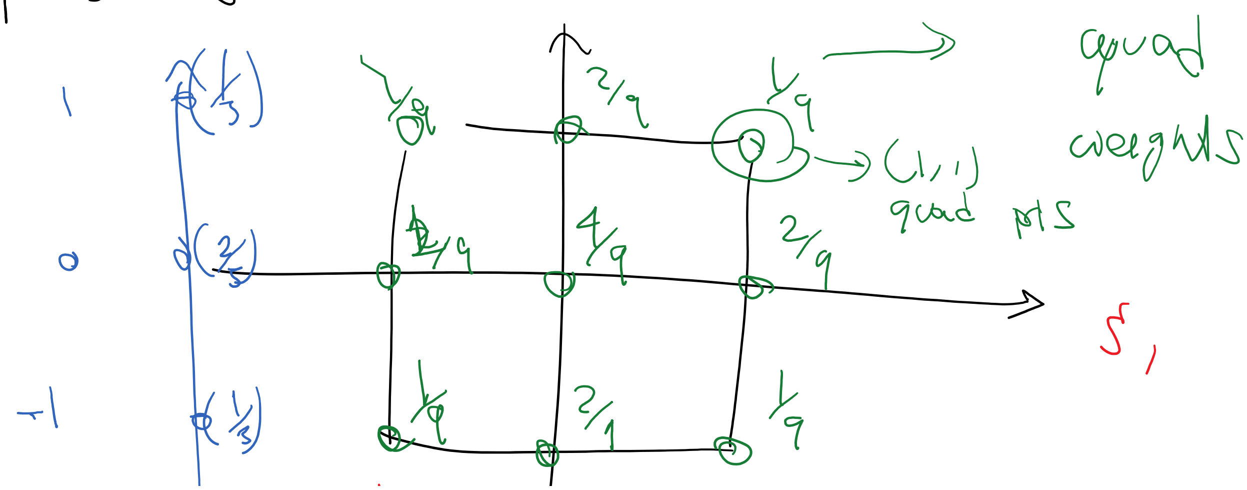
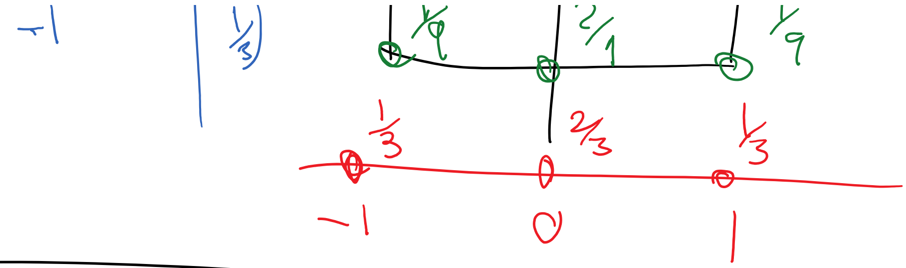


$$k^e = 1\, I\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) + 1\, I\left(+\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) + 1\, I\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) + 1\, I\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

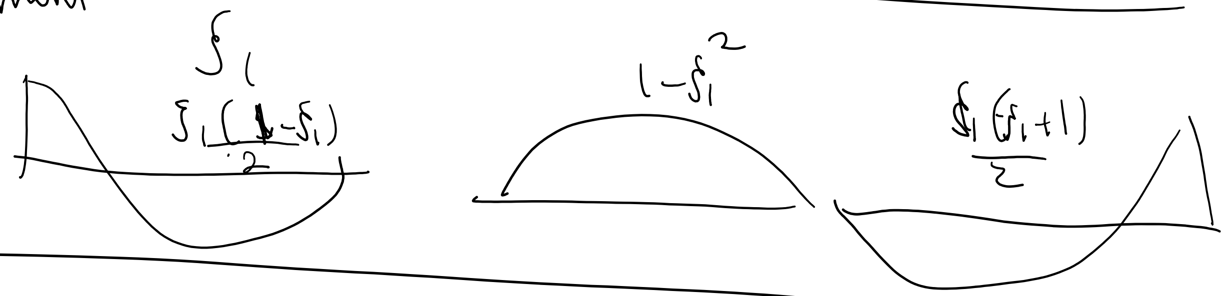
NCs order 2 $\rightarrow$ 3 NC points

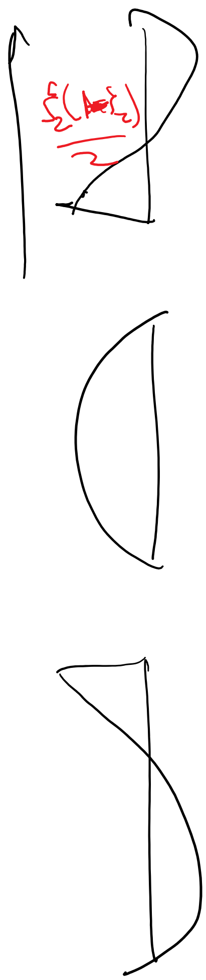
Simpson's rule

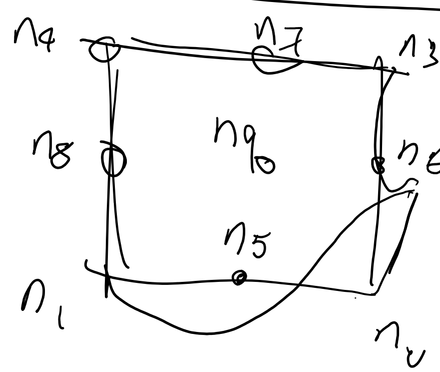




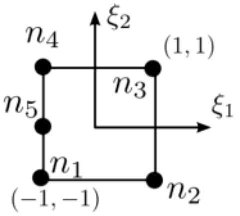
P=2 element



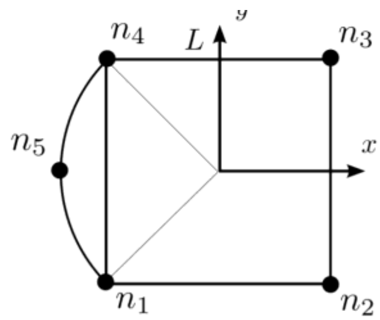




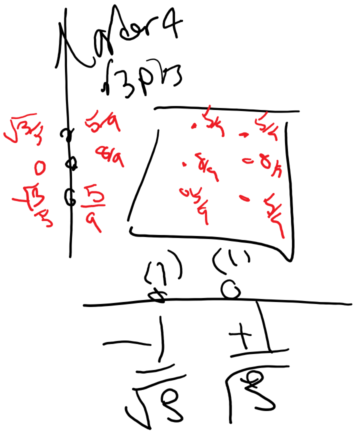
$$N^2 = \frac{\xi_2(1-\xi_2)}{2} \quad \frac{\xi_1(1+\xi_1)}{2}$$



n_4
ξ_2
$(1,1)$
n_3
ξ_1
n_5
n_1
$(-1,-1)$
n_2



n_4
L
y
n_3
n_5
x
n_1
n_2



order 4
3pts
5/9
8/9
5/9
0
(1)
(1)
$-\frac{1}{\sqrt{3}}$
$+\frac{1}{\sqrt{3}}$

← order 2