Consider the isoparametric 5-node element shown in the figure for a stationary thermal conduction problem with material conductivity $\kappa = k\mathbf{I}$ (isotropic conductivity), $k = 2$. In order to capture the curved domain boundary geometry, the extra node 5 is inserted on the left boundary. For $L = 4$ the coordinates of the element in Cartesian system are:

$$\mathbf{X} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 2 & -2 & -2\sqrt{2} \\ -2 & -2 & 2 & 2 & 0 \end{bmatrix} \tag{1}$$

The questions for this element are: Starting from Q4 bilinear shape functions, one can show that the

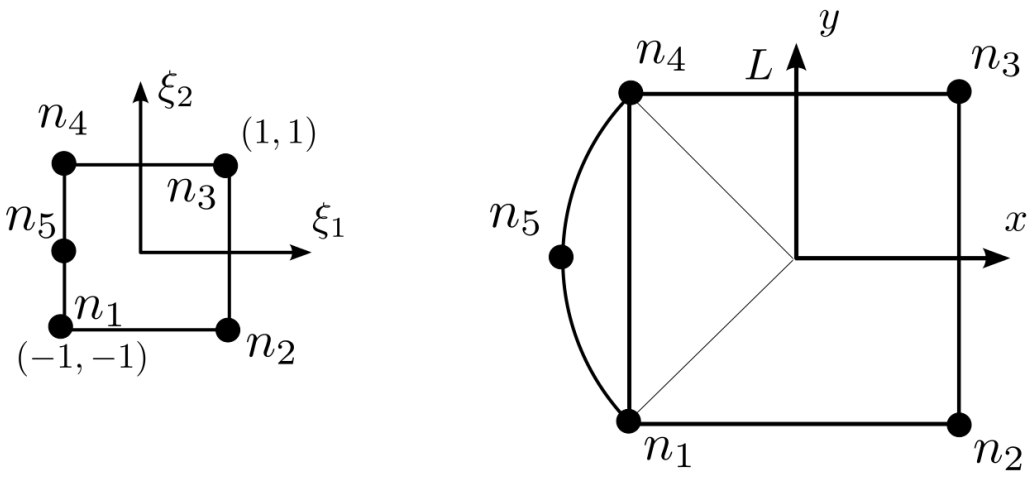

Figure 1: Parent element in (ξ_1, ξ_2) and the actual element.

shape functions for the element are:

$$\begin{aligned} \mathbf{N} &= \begin{bmatrix} N_1 & N_2 & N_3 & N_4 & N_5 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{(1-\xi_1)\xi_2(1-\xi_2)}{4} & \frac{(1+\xi_1)(1-\xi_2)}{4} & \frac{(1+\xi_1)(1+\xi_2)}{4} & \frac{(1-\xi_1)\xi_2(1+\xi_2)}{4} & \frac{(1-\xi_1)(1-\xi_2^2)}{2} \end{bmatrix} \end{aligned} \tag{2}$$

1. **40 Points** **Background:** Using $x = sum_{i=1}^{5} x_i N_i$, $y = sum_{i=1}^{5} y_i N_i$ one can show:

$$x = (1-\xi_1)\left(-\sqrt{2} + (\sqrt{2}-1)\xi_2^2\right) + (1+\xi_1) \tag{3a}$$

$$y = 2\xi_2 \tag{3b}$$

Questions: Given that,

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi_1} & \frac{\partial x}{\partial \xi_2} \\ \frac{\partial y}{\partial \xi_1} & \frac{\partial y}{\partial \xi_2} \end{bmatrix} \tag{4}$$

express $\mathbf{J}, |J|, \mathbf{B}_\xi, \mathbf{B}$ in terms of ξ_1, ξ_2. **: No need to explicitly compute $\mathbf{J}^{-1}$ in your expressions.**

For the point $(\xi_1, \xi_2) = (0, 0)$ compute 1) (x, y), 2) $\mathbf{N}$, 3) $\mathbf{B} = \nabla\mathbf{N}$. Assuming that the left boundary is on essential boundary condition with $\bar{T}(x, y) = y^2$ and the FEM solutions for nodes 2 and 3 given by $T_2 = 1, T_3 = 3$ compute 4) T and 5) $\mathbf{q} = -\kappa\nabla T$ at the same point. Note that $\kappa = 2\mathbf{I}$.

Some values to verify your results: $B(1,5) = -0.207, B(2,4) = 0.125, T = 1, q_2 = -0.5$.

2. **50 Points** **Background:** Using $k^e = \int_e \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\,\mathrm{dA}$ and the values of $\mathbf{B}$ and dA one can show that:

$$k^e = k\int_{-1}^{1}\int_{-1}^{1}\begin{bmatrix} \frac{\xi_2(1-\xi_2)}{4} & \frac{(1-\xi_1)(2\xi_2-1)}{4} \\ \frac{1-\xi_2}{4} & -\frac{1+\xi_1}{4} \\ \frac{1+\xi_2}{4} & \frac{1+\xi_1}{4} \\ -\frac{\xi_2(1+\xi_2)}{4} & \frac{(1-\xi_1)(2\xi_2+1)}{4} \\ -\frac{1-\xi_2^2}{2} & -\xi_2(1-\xi_1) \end{bmatrix}\left\{\mathbf{J}^{-1}\mathbf{J}^{-\mathrm{T}}|\mathbf{J}|\right\}\begin{bmatrix} \frac{\xi_2(1-\xi_2)}{4} & \frac{1-\xi_2}{4} & \frac{1+\xi_2}{4} & -\frac{\xi_2(1+\xi_2)}{4} & -\frac{1-\xi_2^2}{2} \\ \frac{(1-\xi_1)(2\xi_2-1)}{4} & -\frac{1+\xi_1}{4} & \frac{1+\xi_1}{4} & \frac{(1-\xi_1)(2\xi_2+1)}{4} & -\xi_2(1-\xi_1) \end{bmatrix}\mathrm{d}\xi_1\mathrm{d}\xi_2 \tag{5}$$

This equation in fact holds for any 5 node thermal element with parent geometry shown before.

Questions:

(a) Assuming that J is constant what are the maximum orders of integrand for ξ_1 ξ_2.

(b) Based on the maximum orders of integrand for constant J list the 1) coordinates of quadrature points (ξ_1, ξ_2) 2) with their corresponding weight values; 3) schematically show these points in the parent geometry.

(c) If we use Newton-Cotes to integrate k^e how many points are needed in ξ_1 and ξ_2 directions? Schematically, draw these quadrature points and compare them with Gauss quadrature points.

(d) Can a 2×3 Gauss quadrature stencil integrate k^e exactly for the geometry shown in the figure? If not, can any order of Gauss quadrature integrate it exactly? What is a full integration order?

(e) If the conductivity matrix is integrated exactly, what would it rank be? In other words, how many independent zero eigenvalues does the matrix possess? Comment on rank of conductivity matrix for a 1×1 Gauss integration scheme and its influence on FEM results.

3. **85 Points** **Static condensation (continuation of HW5)**: In course notes pages 331-334 we discussed how prescribed dofs can be eliminated at the global assembly stage if all dofs (free + prescribed) are assembled into the global system. We compared that approach with common approach of only assembling only the free dofs. The same decomposition of the matrix can be used in "condensing" the internal dofs of higher order elements (those that have the so-called "bubble shape functions") at the element level and only assembling the dofs that are shared between the element level at the global stage. This can substantially reduce the total number of dof and result in much smaller global stiffness matrix to solve. An example can be seen in fig. 2 where the average dof per element (assuming there is one dof per node) reduces by about a factor 2 (below)! The stiffness matrix at the element level can be written as,

$$\mathbf{k}\mathbf{a} = \mathbf{f} \quad \text{where } \mathbf{k} = \left[\begin{array}{c|c} \mathbf{k}_{ee} & \mathbf{k}_{ei} \\ \hline \mathbf{k}_{ie} & \mathbf{k}_{ii} \end{array}\right] \quad \mathbf{f} = \left[\begin{array}{c} \mathbf{f}_e \\ \hline \mathbf{f}_i \end{array}\right], \quad \mathbf{a} = \left[\begin{array}{c} \mathbf{a}_e \\ \hline \mathbf{a}_i \end{array}\right] \tag{6}$$

The unknown part that we want to condense and not assemble to the global system is $\mathbf{a}_i$. The equation (6) can be written as

$$\mathbf{f}_e = \mathbf{k}_{ee}\mathbf{a}_e + \mathbf{k}_{ei}\mathbf{a}_i \tag{7a}$$
$$\mathbf{f}_i = \mathbf{k}_{ie}\mathbf{a}_e + \mathbf{k}_{ii}\mathbf{a}_i \tag{7b}$$

By solving for $\mathbf{a}_i$ in (7b) and plugging it into (7a) obtain the "effective" element stiffness matrix $\tilde{\mathbf{k}}$ and force vector $\tilde{\mathbf{f}}$:

$$\tilde{\mathbf{f}} = \tilde{\mathbf{k}}\mathbf{a}_e \tag{8}$$

Note that static condensation does not reduce the order of accuracy of the method; at local element level we solve these small system of equations that can reduce the global number of

unknowns. Also $\mathbf{f}$ is the local elemental forces from all sources of forcing such as natural and essential boundary conditions (only external), source term, and nodal concentrated forces (only internal).

- **15 Points** Show the average dofs per element for the 3rd order rectangle in fig. 2 before condensation is 9 and after condensation the average reduces to 5. For this type of analysis we ignore the effect of dofs on the boundaries of the domain (*i.e.,* domain is infinite). This approximation is correct when number of elements in each direction is (very) high.
- **25 Points** Obtain $\tilde{\mathbf{k}}$ and $\tilde{\mathbf{f}}$ in terms of $\mathbf{k}_{ee}, \mathbf{k}_{ei}, \mathbf{k}_{ie}, \mathbf{k}_{ii}$ and $\mathbf{f}_e, \mathbf{f}_i$.
- **45 Points** For second order bar element with constant E, A and the internal node n_3 being in the center we have (*cf.* fig. 3),

$$\mathbf{k} = \frac{AE}{3L}\begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix} \tag{9}$$

using static condensation approach write the expressions for $\tilde{\mathbf{f}}$ and $\tilde{\mathbf{k}}$ when the internal dof u_3 is condensed. Compare $\tilde{\mathbf{k}}$ with the stiffness matrix for a linear bar element and comment on how this effective system is still more accurate given the similarities after condensation to a linear bar element.

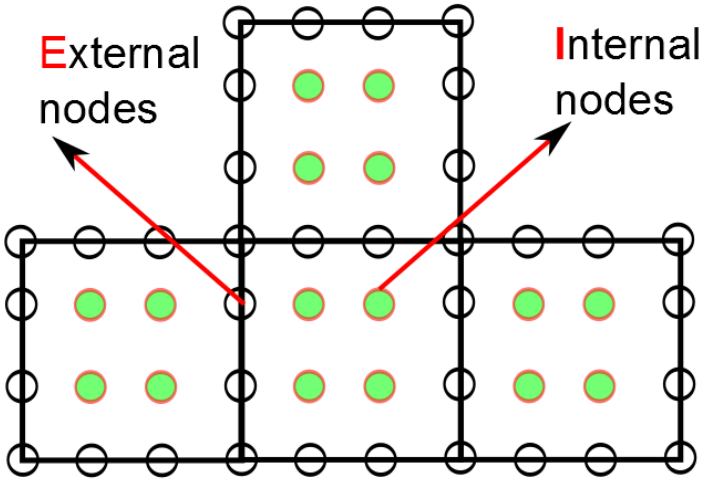

Figure 2: Internal nodes that can be condensed at the element level before global assembly

4. **30 Points** In FE method a priori error estimates generally take the form $\mathcal{E} = Ch^{ap+b}$ [1] where p is the element (polynomial) order C is an unknown constant, $\mathcal{E}$ is an error, and $a > 0$, b, are constants depending on the error type. That is, error convergence is linear with the rate $ap + b$: $(\log\mathcal{E} = \log C + (ap + b)\log h$. The purpose of these estimates are to understand how much the error is expected to decrease when from a current FEM solution we do h-refinement by decreasing element sizes or p-enrichment by increasing element order.[2] Generally, the equation is used for

[1] If the solution is not regular enough p must be replaced by $\min(p, s)$ where s is the regularity, *i.e.,* order of continuity C^s, of exact solution for the given error $\mathcal{E}$. For example if exact solution displacement for a problem is C^4 and we calculate strain energy error $\mathcal{E} = \int_D (\epsilon^h - \epsilon) : \boldsymbol{C} : (\epsilon^h - \epsilon) dV$ since $\epsilon = \frac{1}{2}(\nabla u + \nabla^{\mathrm{T}} u)$, s would be 4 - 1 = 3 (exact ϵ is C^3).

[2] If the elements are too large, these estimates generally do not hold as we are not in asymptotic error convergence range. For example by refining elements the error may reduce much less than estimated.

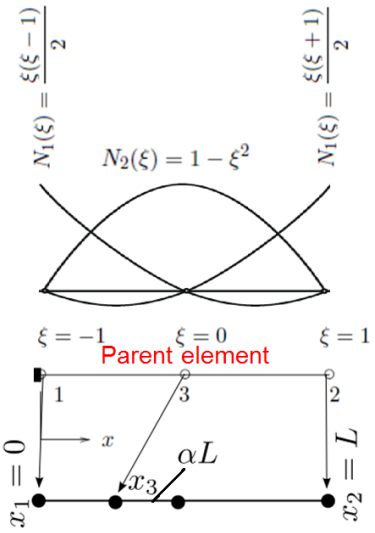

Figure 3: Second order element for 1D problems.

Gauss Points ($\pm x_i$)	Weights (w_i)
n = 2	
0.57735 02691 89626	1.00000 00000 00000
n = 3	
0.00000 00000 00000	0.88888 88888 88888
0.77459 66692 41483	0.55555 55555 55555
n = 4	
0.33998 10435 84856	0.65214 51548 62546
0.86113 63115 94053	0.34785 48451 37454
n = 5	
0.00000 00000 00000	0.56888 88888 88889
0.53846 93101 05683	0.47862 86704 99366
0.90617 98459 38664	0.23692 68850 56189

Intervals, i	No. of Points, n	C_0	C_1	C_2	C_3	C_4	C_5	C_6
1	2	1/2	1/2			(trapezoid rule)		
2	3	1/6	4/6	1/6		(Simpson s 1/3 rule)		
3	4	1/8	3/8	3/8	1/8	(Simpson s 3/8 rule)		
4	5	7/90	32/90	12/90	32/90	7/90		
5	6	19/288	75/288	50/288	50/288	75/288	19/288	
6	7	41/840	216/840	27/840	272/840	27/840	216/840	41/840

Figure 4: Gauss and Newton-Cotes quadrature points.

uniform meshes (same element size) and uniform element order. Assume for a simulation a given error $\mathcal{E}$, *e.g.*, numerical energy dissipation, takes the value of $\mathcal{E}_1 = 10^{-4}$ for $h_1 = 0.1$ and $p_1 = 1$ (linear element). Constants a and b are 2 and -1; that is, $\mathcal{E} = Ch^{2p-1}$. We want to estimate how much smaller the element sizes or how larger the element polynomial should be to reduce the error by 4 orders of magnitude. That is to achieve $\mathcal{E}_2 = 10^{-8}$ estimate,

(a) h-refinement: If we keep p fixed ($p_2 = p_1 = 1$) what the new element size h_2 is estimated to achieve the target error of 10^{-8}? How much smaller is the new element size (h_2/h_1)? If the problem is 2D how many more elements is estimated to require for achieve the smaller error ($(n_e)_2/(n_e)_1$)?

(b) p-enrichment: If we keep the element size fixed ($h_2 = h_1 = 0.1$) what is the smallest polynomial that we expect to meet the more stringent error requirement?

(c) Which scheme do you think requires fewer dofs to achieve the raget error, h-refinement or p-refinement? Note that having fewer dofs (with the same error value) for globally coupled static problems generally translates to higher efficiency.

Hint: You can either directly obtain the value C from $\mathcal{E}_1 = Ch^{2p_1-1}$ or divide the equations for 1 and 2 states to eliminate C in calculating h_2 or p_2.