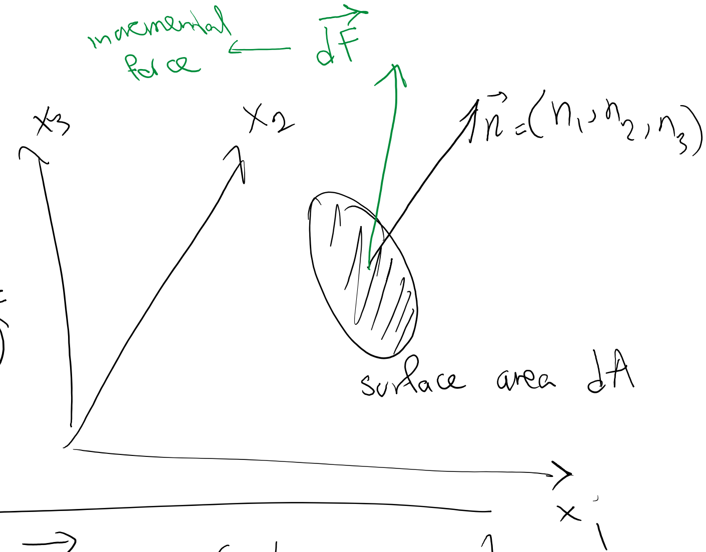
FEM20240828

Wednesday, August 28, 2024 12:54 PM

Continuing on the balance laws:

$\vec{t}$ = intensity of force = force per area $= \dfrac{d\vec{F}}{dA}$

traction



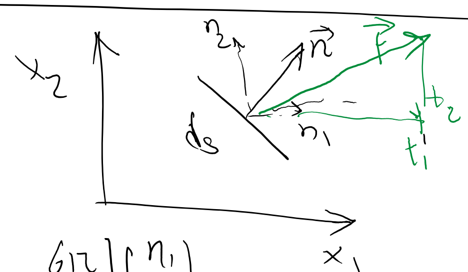
$$\boxed{\vec{t} = (\sigma)\vec{n}} \quad (1)$$

σ = 2nd stress tensor

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}$$

sym matrix

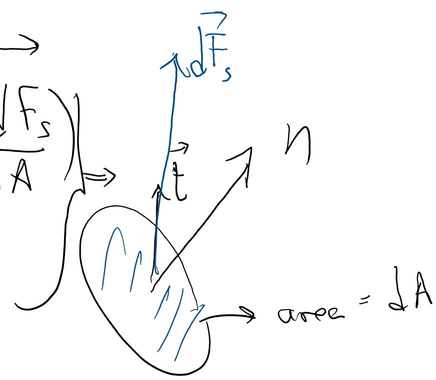
2D elasticity



$$\begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

$d\vec{F}_s = \vec{t}\, dA$ (why? $\vec{t} = \dfrac{d\vec{F}_s}{dA}$)

$\vec{t} = \sigma \vec{n}$

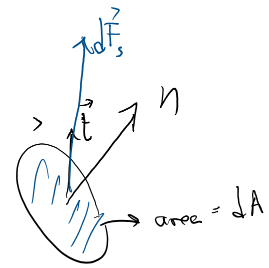


(2)

$$\boxed{\begin{aligned} d\vec{F}_s &= \vec{t}\, dA = \sigma \vec{n}\, dA \\ &= \sigma\, d\vec{A} \end{aligned}}$$

(some people use $d\vec{A} = \vec{n}\, dA$)

$d\vec{F}_s$



$$d\vec{A} = \vec{n}\,dA$$

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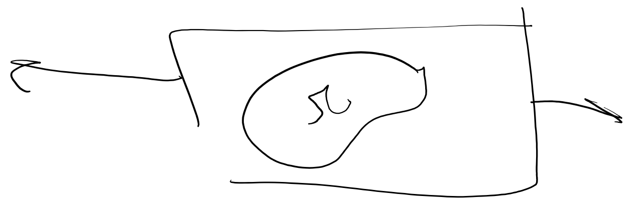
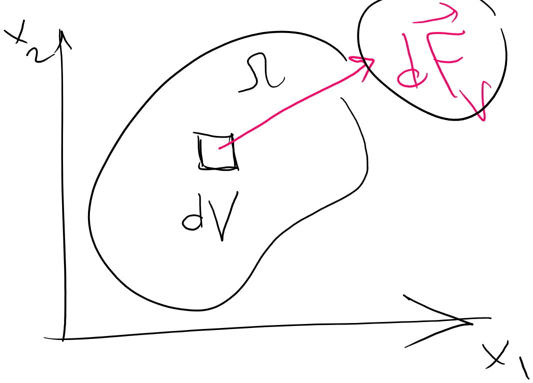
$\vec{t}$ acting on the boundary of Ω $(\partial\Omega)$



What is the total force from traction

$$\vec{F}_s = \int_{\partial\Omega} \underbrace{\vec{t}\,dA}_{d\vec{F}_s} = \int_{\partial\Omega} \underbrace{(\sigma\vec{n})}_{\vec{t}}\,dA$$

(3) $$\boxed{\vec{F}_s = \int_{\partial\Omega} \sigma\vec{n}\,dA}$$



$$d\vec{F}_V = \underbrace{(\rho\,dV)}_{dm}\,\vec{b}$$

ρ → mass density

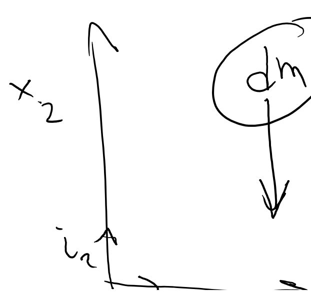
$dV = \underbrace{dx_1\,dx_2}_{\text{in 2D}}$

dm → differential mass

$\vec{b}$ → unit force/mass

$\vec{b}$ = body force

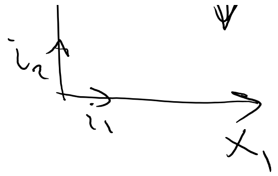
gravity



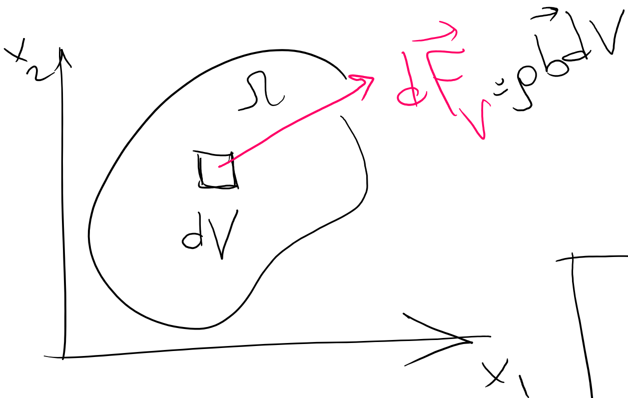
dm

$$\vec{W} = -g\,dm\,i_2 = (-g\,i_2)\,dm$$

$\vec W = -g\,dm\,i_2 = (-g\,i_2)\ \ldots$



$$\boxed{\vec b = -g\,i_2}$$



$$\vec F_V = \int_\Omega d\vec f_V \Rightarrow$$

$$\boxed{\vec F_V = \int_\Omega \rho \vec b\, dV} \quad (4)$$

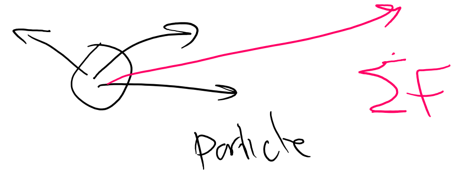
(3) $F_s = \int_{\partial\Omega} \sigma \vec n\, ds$

$$\underline{\sum F = F_s + F_V} = \int_{\partial\Omega} \sigma \cdot \vec n\, ds + \int_\Omega \rho \vec b\, dV = 0$$

static?

Dynamics

$$\sum F = ma$$

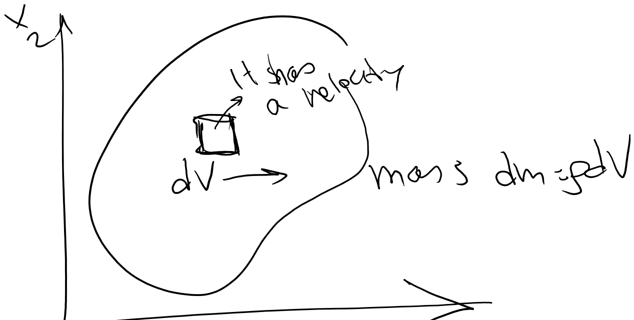


$$= m\frac{dV}{dt}$$

$$= \frac{d}{dt}(\underbrace{mV}_{P})$$

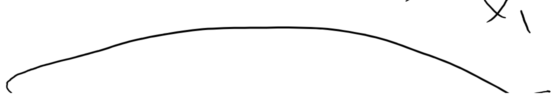
P linear momentum (or m in fluids)

$$\boxed{\sum F = \frac{d}{dt}P \qquad P = m\vec U}$$



$$\underbrace{d\vec p}_{\text{increment of linear momentum}} = \underbrace{(dm)}_{\text{mass}}\ \underbrace{\vec V}_{\text{velocity}}$$

$$= (\rho\, dV)\,\vec v = \underbrace{(\rho \vec v)}\, dV$$



$= (\rho\, dV)\vec{v} = \underbrace{(\rho\vec{v})}\, dV$

linear momentum density $\longleftarrow p:$

$$\boxed{\vec{P} = \int_\Omega d\vec{P} = \int_\Omega (\underbrace{\rho\vec{v}}_{p})\, dV} \quad (5)$$

Dynamics $\sum\vec{F} = \vec{F}_s + \vec{F}_v = \dfrac{d\vec{P}}{dt}$

$$\forall\Omega:\ \int_{\partial\Omega} \sigma\cdot\vec{n}\, dS + \int_\Omega \rho\vec{b}\, dV = \frac{dP}{dt} = \underbrace{\frac{d}{dt}\int_\Omega \vec{p}\, dV}_{P}$$

dynamics

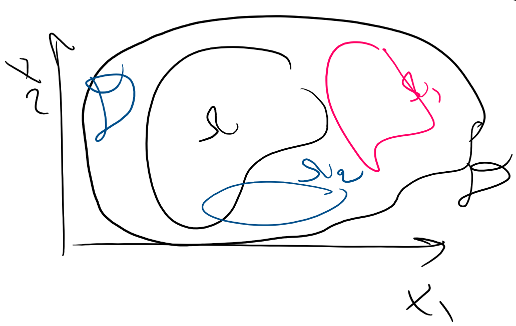
$\vec{p} = \rho\vec{v}$ ↓ linear momentum density

balance of linear momentum

$$\sum F = \frac{dP}{dt} \quad (6)$$

statics (this course)

$$\sum F = 0 \quad \forall\Omega \qquad \int_{\partial\Omega} \sigma\cdot n\, dS + \int_\Omega \rho\vec{b}\, dV = 0$$



For balance of
- 1 integral inside (Ω)
- " " on boundary ($\partial\Omega$)

mass (fluids)
energy " "
linear momentum (solid/fluid)

What's the use of balance laws?

use 1 Partial Differential equations (PDE) use 1 → Partial Differential equations (PDE)

(6) $\left(\nabla \cdot \sigma + \rho b = \rho \frac{dV}{dt} \right.$ dynamic

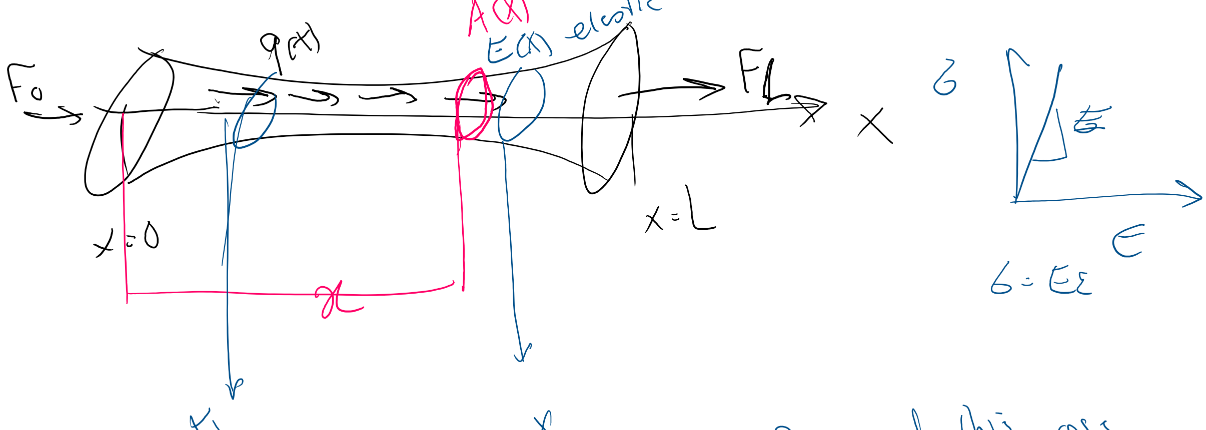
$\quad \quad \quad \quad \quad \quad \quad \quad \quad = 0$ static

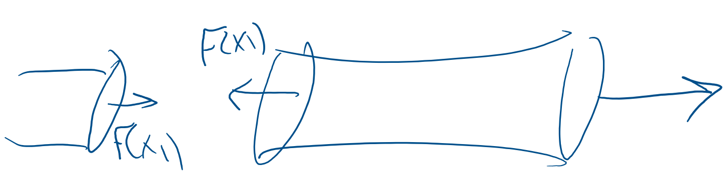
→ Solve PDE in this case (

FYI, use 2 Balance laws gives jump conditions (shocks in fluids, ...)

Not our business here

1D version of balance law





force at this cross section

$F(x_2) = \underbrace{\sigma(x_2)}\, A(x_2)$

$E(x_2)\,\varepsilon(x_2)\,A(x_2)$

$= A(x_2)\,E(x_2)\,\varepsilon(x_2)$

$x = x_1$ $\qquad\qquad x = x_2$

$$= A(x_2)\,E(x_2)\,\varepsilon(x_2)$$

$$\varepsilon = \frac{du}{dx} \quad u \rightarrow \text{displacement}$$

we'll use this later

$$\boxed{\text{for any } x \qquad F(x) = A(x)\,E(x)\,\frac{du}{dx}(x)} \qquad (7)$$

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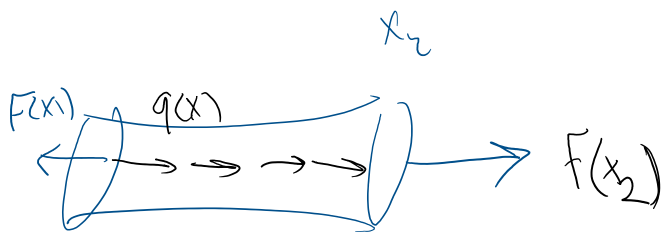
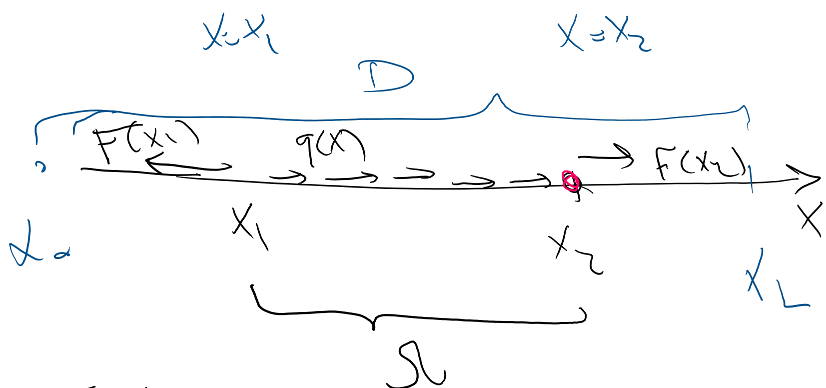
we just want to write the balance law





$$\partial\Omega = \{x_1, x_2\}$$

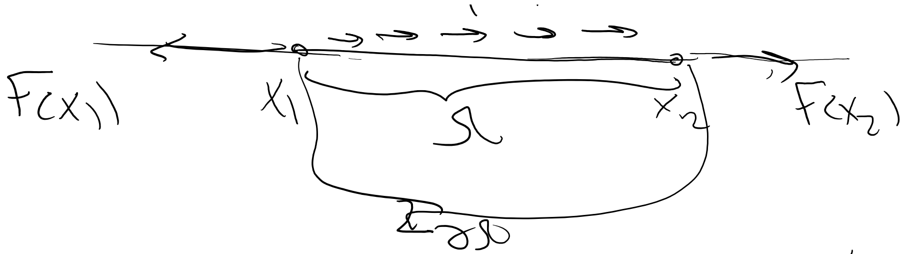
$$\Sigma F = F_S + F_V$$

$$F_S = F(x_2) - F(x_1)$$

$$F_V = \int_{\Omega} (\;)\,dV = \int_{x_1}^{x_2} q(x)\,dx$$

$\Omega \rightarrow x_1 \rightarrow x_2$ $\qquad dV \rightarrow dx$

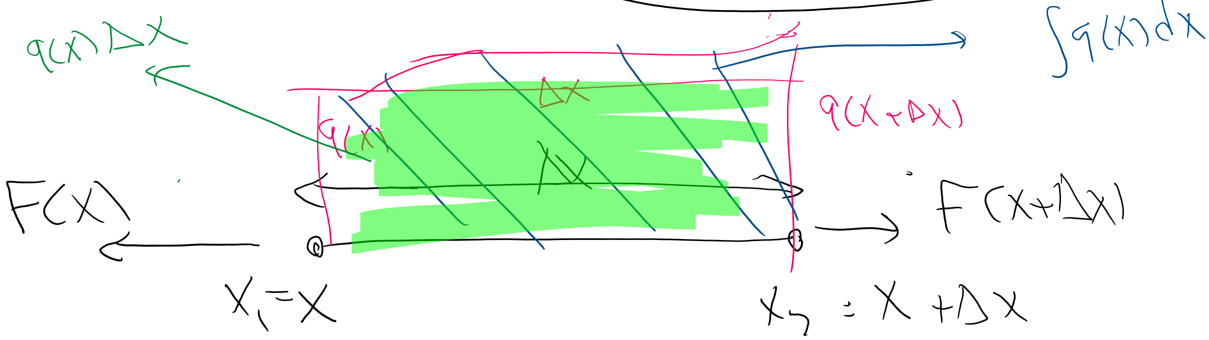
$q(x) \rightarrow$ force per length $\approx \rho\vec{b}$ in 2D/3D



$$\sum F = F_s + F_r = F(x_2) - F(x_1) + \int_{x_1}^{x_2} q(x)\,dx = 0 \qquad (8)$$

Balance of linear momentum of 1D solid static
($\sum F = 0$) elastostatic

Next step — Balance law $\rightarrow$ PDE $\rightarrow$ WRS $\rightarrow$ WF



Balance law is eq 8

$$F(x_2) - F(x_1) + \int_{x_1}^{x_2} q(x)\,dx = 0$$

$$F(x+\Delta x) - F(x) + \int_{x}^{x+\Delta x} q(x)\,dx = 0 \quad \text{let } \Delta x \to 0$$

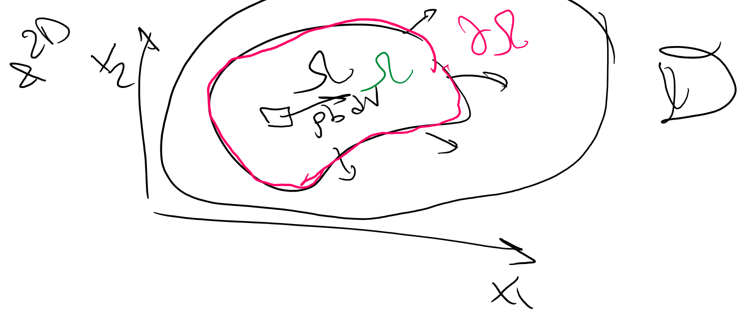
divide by Δx

$$\lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{1}{\Delta x}\int_{x}^{x+\Delta x} q(x)\,dx = 0$$

$$(9) \quad \forall x \in [0, L]: \quad \frac{dF}{dx} + q(x) = 0 \qquad \forall x$$

(I) $\boxed{\forall x \in \underbrace{[0,L]}_{\text{length of the bar}} : \frac{d\,T}{dx} + f(x) = 0}$ $\quad \forall x$

Differential Equation for 1D bar / Strong form



$$\forall \Omega \quad \int_{\partial\Omega} \sigma \cdot n\, dS + \int_{\Omega} \rho b\, dV = 0$$

two integrals are different!

we need to turn one to the other kind

(II) Short answer

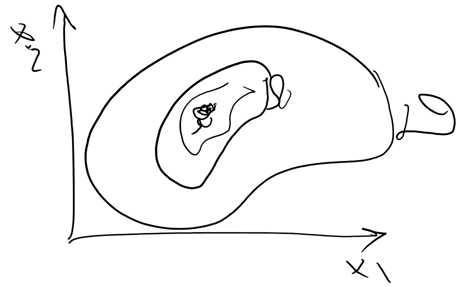
$$\int_{\partial\Omega} \sigma \cdot n\, dS = \int_{\Omega} \nabla \cdot \sigma\, dV$$

Divergence theorem, Gauss

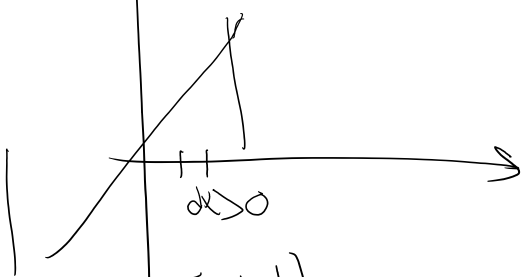
$$\forall \Omega \quad \int_{\partial\Omega} \sigma \cdot n\, dS + \int_{\Omega} \rho b\, dV = 0$$

(II) $$\int_{\Omega} \nabla \cdot \sigma\, dV + \int_{\Omega} \rho b\, dV = 0$$

$$\forall \Omega : \int_{\Omega} (\nabla \cdot \sigma + \rho b)\, dV = 0$$



$f(x) = x$



$\Omega = [-1, 1]$

$$\int_{\Omega} f(x)\, dx = 0 \qquad \text{but } f(x) \neq 0$$