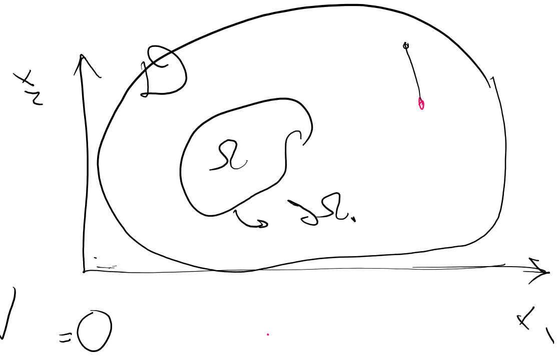
FEM20240904

Wednesday, September 4, 2024 12:40 PM

2D/3D elastostatics
Recall from last time



$$\forall \Omega \quad \int_{\partial\Omega} \sigma \cdot n \, ds + \int_{\Omega} \rho b \, dV = 0$$

Apply divergence theorem

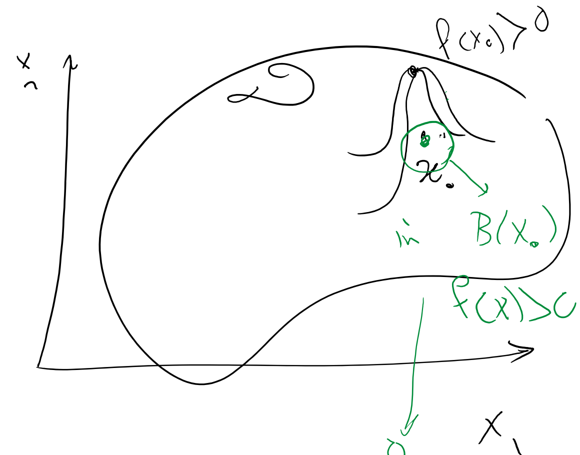
$$\int_{\Omega} \nabla \cdot \sigma \, dV + \int_{\Omega} \rho b \, dV = 0 \Rightarrow$$

$$\boxed{\forall \Omega \quad \int_{\Omega} \underbrace{(\nabla \cdot \sigma + \rho b)}_{f} \, dV = 0}$$

Localization theorem:

- Let f be continuous in $\mathcal{D}$
- $\forall \Omega \in \mathcal{D} \quad \int_{\Omega} f(x) \underbrace{dx}_{\text{like } dV} = 0$

$$\Rightarrow \quad \forall x \in \mathcal{D} \quad \underline{f(x) = 0}$$

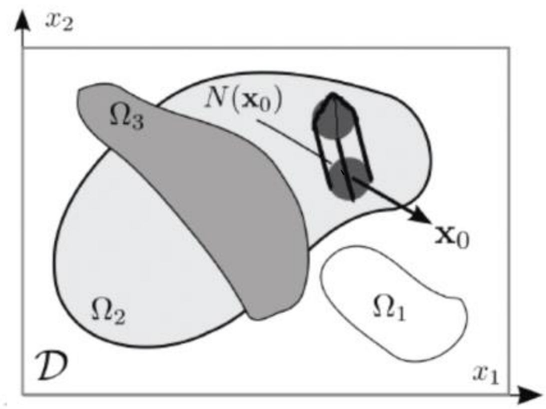


$$\int_{B(x_0)} f(x) \, dV > 0$$

Localization theorem

Localization theorem states that if the integral of a continuus function is zero for all subsets of $\mathcal{D}$, then the function is zero:

$$\boxed{\forall \Omega \subset \mathcal{D}: \int_\Omega \mathbf{g}(\mathbf{x})\,\mathrm{dv} = \mathbf{0} \quad \Rightarrow \quad \forall x \in \mathcal{D}:\; g(\mathbf{x}) = 0} \qquad (21)$$



Let's assume $g(x_0) \neq 0$ (*e.g.*, $g(x_0) > 0$). Since $g(\mathbf{x})$ is continuus, there is a neighborhood of $\mathbf{x}_0$ ($N(\mathbf{x}_0)$) that $g(x) > 0$. We choose an Ω that is only nonzero inside $N(\mathbf{x}_0)$. Then, $\int_\Omega g(\mathbf{x})\,\mathrm{d}V > 0$. Thus, $g(\mathbf{x}_0)$ cannot be nonzero and the function g is identically zero.

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we can apply this to our balance law

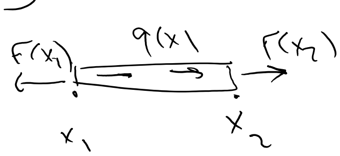
$$\forall \Omega \subseteq \mathcal{D} \quad \int_\Omega \underbrace{(\nabla\cdot\sigma + \rho b)}_{\text{our "f" (or ("g")) above}} dV = 0$$

$\Longrightarrow$

$$① \quad \boxed{\forall x \in \mathcal{D} \qquad \nabla\cdot\sigma + \rho b = 0}$$

Strong form (the) Partial Differential Equation (PDE)

1D



2D/3D

Balance law

$$\forall x_1, x_2 \quad F(x_2) - F(x_1) + \int_{x_1}^{x_2} q(x)\,dx = 0$$

$$\forall \Omega \quad \int \sigma\cdot n\,ds + \int \rho b\,dV = 0$$

Strong form

$$\frac{dF}{dx} + q(x) = 0$$

$$\nabla\cdot\sigma + \rho b = 0$$

$$\forall \Omega \quad \int_{\partial\Omega} \sigma\cdot n\, ds + \int_{\Omega} \rho b\, dV = 0 \qquad \nabla\cdot\sigma + \rho b = 0$$

$$-\nabla\cdot f_x + r = 0$$

Heat conduction,

T = temperature

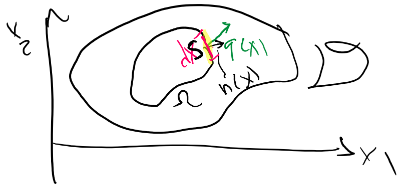
q : heat flux density

Q : heat source

Balance of energy

$$\mathcal{E} \quad -\int_{\partial\Omega} q\cdot n\, ds + \int_{\Omega} Q\, dV = 0 \qquad -\nabla\cdot q + Q = 0$$

$$-\nabla\cdot f_x + r = 0$$



energy flux through ds (surface differential)

$q\cdot n\, ds \longrightarrow$ energy that goes out

Most balance laws are in this form

outward Spatial flux density f_x : >0 it means the quantity (energy, mass, ...) decreases

Source term : r

$$\forall \Omega \quad -\int_{\partial\Omega} f_x\cdot n\, ds + \int_{\Omega} r\, dV = 0$$

balance law

$\Downarrow$

Divergence theorem

Divergence theorem

$$\int_{\Omega} \left(-\nabla \cdot f_x + r\right) dV = 0$$

$$\Downarrow$$

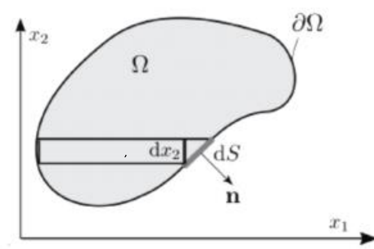
$$\boxed{\begin{array}{ll} \forall x & -\nabla \cdot f_x + r = 0 \\ \text{PDE} & \text{(strong form)} \end{array}}$$

(2)

Fun fact for you

Transfer of boundary to interior integral higher dimensions

- Ω is compact and closed.
- $\partial\Omega$ is piecewise smooth.
- Normal vector $\mathbf{n}$ is defined almost everywhere (a.e.) and is pointing outward.
- tensor field (scalar, vector, matrix, ...):
  - $\mathbf{F}_{,i} = \partial \mathbf{F} / \partial x_i$ exists everywhere and
  - is continuous.



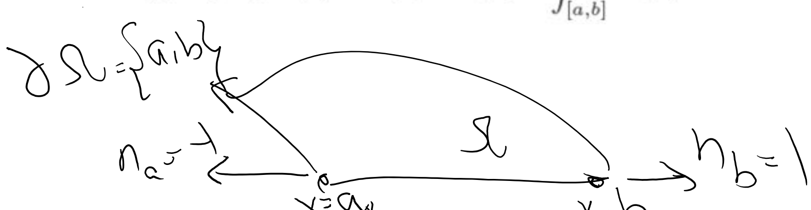
$$\boxed{\int_{\partial\Omega} \mathbf{F}.n_i \,\mathrm{d}S = \int_{\Omega} F_{,i} \,\mathrm{d}V} \quad (18)$$

$$\int_{\Omega} \frac{\partial F}{\partial x} dV = \int_{\partial\Omega} F n \, ds$$

$$\int_{\Omega} \nabla \cdot F \, dV = \int_{\partial\Omega} F \cdot \vec{n} \, ds$$

This is the generalization of the 1D version:

$$1.F(b) + (-1).F(a) = F(b) - F(a) = \int_{[a,b]} F'(x)\,\mathrm{d}x$$



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$$\int_{\Omega = (a,b)} F'(x)\,dx = \int_{\partial\Omega} F n \, ds = F(a) n_a + F(b) n_b$$

$$= F(b) \times 1 + F(a)(-1)$$

$$= F(b) - F(a)$$

$= F(b) - F(a)$

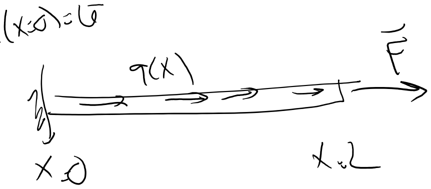
Balance law => Strong Form (PDE)
Can we solve the PDE now?

No

> We need to "close" the system by adding more equations:
> - Constitutive equations
> - Compatibility equations
> - ...

Example 1: Elastostatics

$u(x=0)=0$

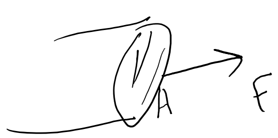


$F(L) = EAu'(L) = \bar{F}$

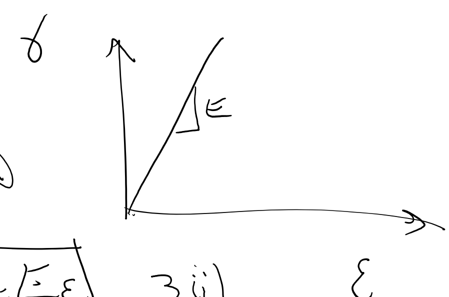
DE $\frac{dF(x)}{dx} + q(x) = 0$ (3i)

F
↓
σ
↓
ε
↓
u

$F = \sigma A$



linear material

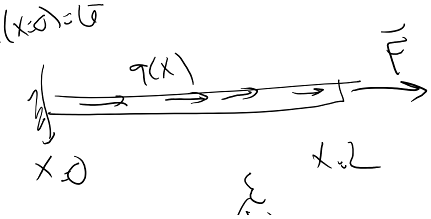


Constitutive equation $\boxed{\sigma = E\varepsilon}$ 3ii)

Compatibility equation $\varepsilon = \frac{du}{dx}$ 3iii)

3i — 3iii

$u(x=0)=0$



$F(L) = EAu'(L) = \bar{F}$

ε

u $x=0$ $x=L$ $u(L)$

$$\frac{df}{dx} = \frac{d\,A\sigma}{dx} = \frac{d\,AE\varepsilon}{dx} = \frac{d\,AE\overset{\varepsilon}{\left(\frac{du}{dx}\right)}}{dx}$$

$$\frac{df}{dx} + q = 0 \longrightarrow \quad (4)\ \boxed{\frac{d}{dx}\left(EA\frac{du}{dx}\right) + q = 0}$$

Differential Eq
Strong form

$$(\ )' = \frac{d}{dx}$$

$$\boxed{(EAu')' + q = 0}$$

order of differential eq $M = 2$

half order of DE $\boxed{m = \frac{M}{2} = 1}$

We also need to add the boundary conditions (BCs)

BCs

$u(x=0)=\bar{u}$

$\partial D_u = \{0\}$

$F(L) = EAu'(L) = \bar{F}$

$\partial D_F = \{L\}$

Neumann (Natural) BC

Dirichlet (Essential) BC : value of the function is specified

$u(x=0) = \bar{u}$

order 0 (u)

$F(L) = EA\,u'(L) = \bar{F}$

order 1

Neumann

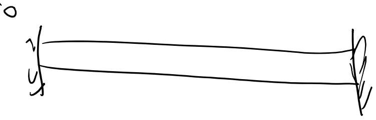
$M = 2 \qquad (EAu')' + q = 0$

BCs

lower order than M

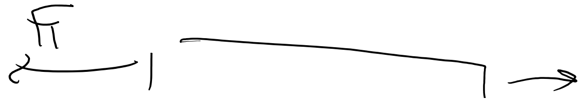
0	$u(x_0) = \bar{u}$
1	$f = EA\,u'(x_0) = \bar{F}$

$u(0) = \bar{u}_0 = 0.01 \qquad u = \bar{u}_1 = 0.02$



2 Dirichlet

$\bar{F}_1 \qquad \bar{F}_2$



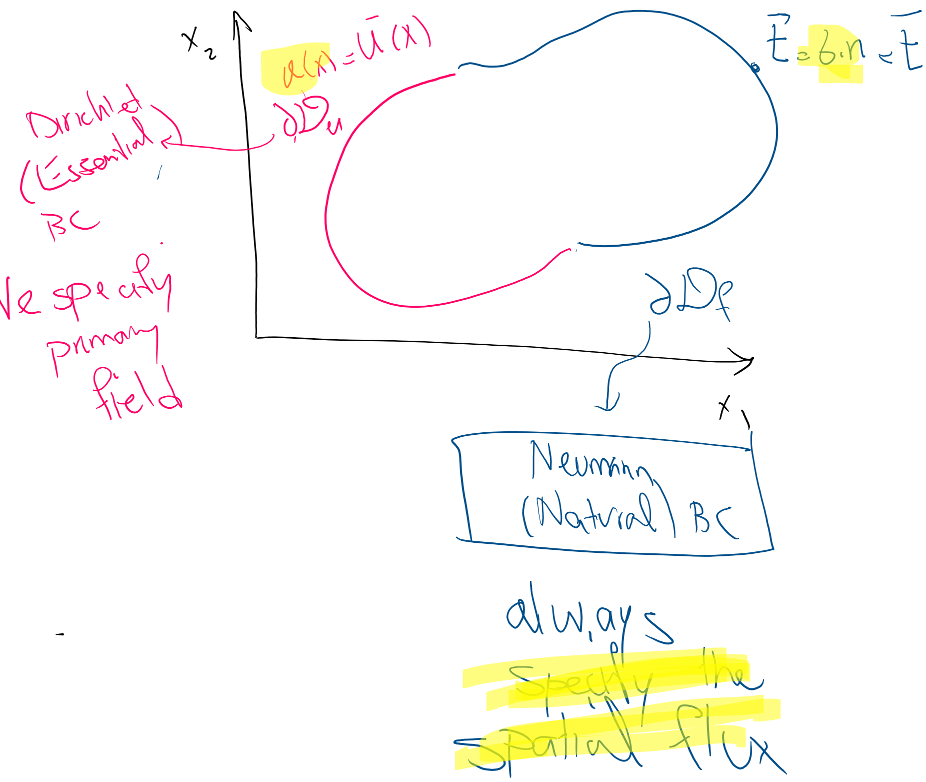
2 Neumann

2 Neumann

Not suitable for statics

2D/3D elastostatics

We already took care of BCs



Strong form

$$\nabla\cdot\sigma+\rho b=0$$

2D

$$\sigma=\begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{bmatrix}$$

symmetric matrix $\sigma_{21}=\sigma_{12}$

3 unknown stresses

2 equations

$\nabla\cdot\sigma+\rho b=0$ means

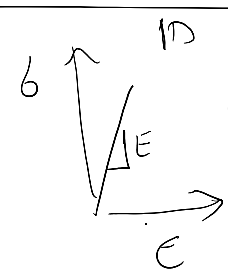
$$\begin{cases}\sigma_{11,1}+\sigma_{12,2}+\rho b_1=0\\ \underbrace{\sigma_{21,1}}_{\sigma_{12}}+\sigma_{22,2}+\rho b_2=0\end{cases}$$

$(\sigma_{ij,j}+\rho b_i=0)$

We lack eqns

Constitutive equation (Linear elastic solid)

1D



$\sigma=E\epsilon$

2D/3D

$$(3)\quad \sigma=\mathbb{C}\,\epsilon$$

σ → stress, $\mathbb{C}$ → elasticity stiffness, ϵ → strain

2D

$$\begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{bmatrix}=\Bigg(\qquad\qquad\Bigg)\begin{bmatrix}\epsilon_{11} & \epsilon_{12}\\ \epsilon_{21} & \epsilon_{22}\end{bmatrix}$$

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} = \left(\qquad \right) \begin{bmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{12} & \epsilon_{22} \end{bmatrix}$$

elasticity stiffness

C 4 index

16 values

C_{ijkl}

i 1 to 2, j, k, l 〃

$$\forall i,j \quad \sigma_{ij} = \sum_{k=1}^{2} \sum_{l=1}^{2} C_{ijkl}\, \epsilon_{kl}$$

Side note: there is an easier way to write:

$\epsilon \to \sigma$ relation (Voigt notation)

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} = C\, \varepsilon \quad \text{where} \quad \epsilon = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{12} & \epsilon_{22} \end{bmatrix}$$

$$S = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \tilde{C}_{3\times 3}\, \gamma \qquad \gamma = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{bmatrix}$$

Voigt stress — Voigt stiffness — Voigt (engineering) strain

$$\sigma \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{13} \\ \tilde{C}_{12} & \tilde{C}_{22} & \tilde{C}_{23} \\ \tilde{C}_{13} & \tilde{C}_{23} & \tilde{C}_{33} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{bmatrix}$$

symmetric

End of side note

$\nabla \cdot \sigma + \rho b = 0$

eq1 $\sigma_{11,1} + \sigma_{12,2} + \rho b_1 = 0$

eq2 $\sigma_{12,1} + \sigma_{22,2} + \rho b_2 = 0$

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \longrightarrow \text{3 unknowns}$$

$\sigma = C \epsilon$

3 eqn added: $\sigma_{11} = \tilde{C}_{11}\epsilon_{11} + \tilde{C}_{12}\epsilon_{22} + \tilde{C}_{13}\epsilon_{12}$

$$\sigma = C\,\varepsilon$$

3 eqn added

$$\begin{cases} \sigma_{11} = \tilde{C}_{11}\epsilon_{11} + \tilde{C}_{12}\epsilon_{22} + \tilde{C}_{13}\epsilon_{12} \\ \sigma_{22} = \text{〃} \\ \sigma_{12} = \text{〃} \end{cases}$$

$$\epsilon = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{12} & \epsilon_{22} \end{bmatrix}$$

3 unknowns added

We are still 1 equation behind

Compatibility equation

1D $\varepsilon = \dfrac{du}{dx}$

2D & 3D $\varepsilon = \dfrac{\nabla u + (\nabla u)^t}{2}$ ⟶ transpose

$f_{,1} = \dfrac{\partial f}{\partial x_1}$, $f_{,2} = \dfrac{\partial f}{\partial x_2}$

Expanded form

$$\epsilon = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} \\ & \epsilon_{22} \end{bmatrix} = \begin{bmatrix} u_{1,1} & \dfrac{u_{1,2}+u_{2,1}}{2} \\ & u_{2,2} \end{bmatrix}$$

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eq1) $\epsilon_{11} = u_{1,1}$

eq2) $\epsilon_{22} = u_{2,2}$

eq3) $\epsilon_{12} = \dfrac{(u_{1,2}+u_{2,1})}{2}$

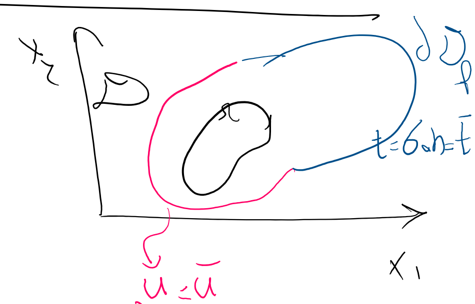
we just added u_1 & u_2

Now # eqns & # unknowns are the same ⟹

We can solve this

Summary of 2D/3D elasticity

Balance law $\forall$ $\int_{\partial\Omega} \sigma\cdot n\,ds + \int_{\Omega} \rho b = 0$ ⟹



Balance law^{ve} $\int_{\partial\Omega} \sigma \cdot n \, ds + \int_{\Omega} \rho b = 0 \Longrightarrow$

x_1

$u = \bar{u}$

$\partial\Omega_u$

Dirichlet

Strong form $\boxed{\forall x \quad \nabla \cdot \sigma + \rho b = 0}$ 2 eqns (2D)

Constitutive eqn $\boxed{\sigma = C\varepsilon}$

Compatibility eqn $\varepsilon = \dfrac{\nabla u + (\nabla u)^T}{2}$

$\Longrightarrow$

Strong form in terms of primary solution field u

$\nabla \cdot C\varepsilon + \rho b =$

$\underbrace{\nabla \cdot C\left(\frac{\nabla u + \nabla u^T}{2}\right)}_{\text{from symmetries of } C} + \rho b = 0 \Longrightarrow$

$$\boxed{\begin{array}{lll} (8)\ \nabla \cdot C\nabla u + \rho b = 0 & & \text{2D/3D} \\ \frac{d}{dx}\left(EA \frac{du}{dx}\right) + q = 0 & & \text{1D} \end{array}}$$

HW $-\nabla \cdot q + Q = 0$

$q = -k\nabla T$

- Order of PDE $M = m = 2 \quad (m = 1)$

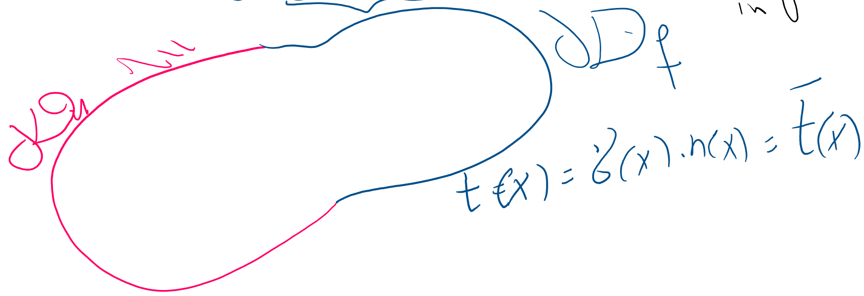
BCs

$M = 2$

0	$\forall x \in \partial\Omega_u \quad u(x) = \bar{u}(x)$
1	$\forall x \in \partial ... \quad t(x) = \sigma(x) \cdot n(x) = C\nabla u \cdot n = \bar{t}(x)$

1 | $\forall x \in \partial D_f \quad t(x) = \sigma(x) \cdot n(x) = C\nabla u \cdot n = \bar{t}(x)$ order 1 in u

$C\varepsilon = C\,\frac{\nabla u + \nabla u^T}{2} = C\nabla u$

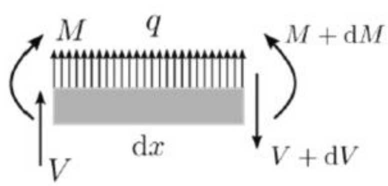


Sample Boundary value problems: Euler Bernoulli beam

$$\left.\begin{array}{rll} \nabla.\mathbf{F} - \mathbf{r} & = 0 & \text{Balance law} \\ \mathbf{F} & = \begin{bmatrix} M \\ V \end{bmatrix} & \text{Flux} \\ \mathbf{r} & = \begin{bmatrix} V \\ q \end{bmatrix} & \text{Source term} \\ M & = EI\kappa & \text{Constitutive equation} \\ \kappa & = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} & \text{Kinematic compatibility} \end{array}\right\} \Rightarrow \begin{array}{c} \boxed{\begin{array}{ll} \frac{\mathrm{d}M}{\mathrm{d}x} - V & = 0 \\ \frac{\mathrm{d}V}{\mathrm{d}x} - q & = 0 \end{array}} \\ \\ \boxed{\frac{\mathrm{d}^2 EI}{\mathrm{d}x^2}\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) - q = 0} \end{array} \quad (28)$$

where

M	= Momentum
V	= Shear force
q	= Distributed load
E	= Elastic modulus
I	= Second moment of area
κ	= Curvature
y	= Vertical displacement



Euler Bernoulli beam: BCs

Operator	Sample	elastostatics	operator order
$L_{2m}(\mathbf{u}) = \mathbf{r}$	$\frac{\mathrm{d}^2 EI}{\mathrm{d}x^2}\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) = q$	$\frac{\mathrm{d}^2 EI}{\mathrm{d}x^2}\left(\frac{\mathrm{d}^2 (.)}{\mathrm{d}x^2}\right) = q$	$m = 2(M = 4)$
$L_u(\mathbf{u}) = \mathfrak{u}$	$u = \begin{bmatrix} \theta \\ y \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{d}y}{\mathrm{d}x} \\ y \end{bmatrix} = \begin{bmatrix} \bar{\theta} \\ \bar{y} \end{bmatrix} = \bar{u}$	$L_u = \begin{bmatrix} \frac{\mathrm{d}(.)}{\mathrm{d}x} \\ (.) \end{bmatrix}$	$M_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$L_f(\mathbf{u}) = \bar{\mathbf{f}}$	$\begin{bmatrix} EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) \end{bmatrix} = \begin{bmatrix} \bar{M} \\ \bar{V} \end{bmatrix}$	$L_f = \begin{bmatrix} EI\frac{\mathrm{d}^2 (.)}{\mathrm{d}x^2} \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(EI\frac{\mathrm{d}^2 (.)}{\mathrm{d}x^2}\right) \end{bmatrix}$	$M_f = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

- One and only one of the pair M (Neumann) and θ (Dirichlet) is enforced at each end of the beam.
- One and only one of the pair V (Neumann) and y (Dirichlet) is enforced at each end of the beam.
- Neumann boundary conditions correspond to the flux terms (M and V).
- Neumann boundary conditions fall in the upper half of derivatives ($[m, 2m-1] = [2, 3]$).
- Dirichlet boundary conditions fall in the lower half of derivatives ($[0, m-1] = [0, 1]$).
- There are two boundary conditions at each end point (equal to $m = M/2$).
- $M_u + M_f = M - 1$.

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And slide 34

Read before the class last time