FEM20240909

Monday, September 9, 2024 12:52 PM

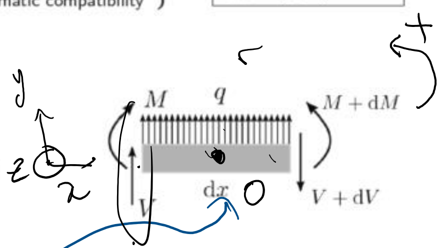
Boundary Conditions (BCs) continued

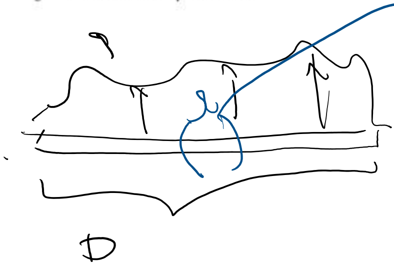
Sample Boundary value problems: Euler Bernoulli beam

$$\left.\begin{aligned}
\nabla.\mathbf{F}-\mathbf{r} &= 0 && \text{Balance law}\\
\mathbf{F} &= \begin{bmatrix} M \\ V \end{bmatrix} && \text{Flux}\\
\mathbf{r} &= \begin{bmatrix} V \\ q \end{bmatrix} && \text{Source term}\\
M &= EI\kappa && \text{Constitutive equation}\\
\kappa &= \frac{d^2y}{dx^2} && \text{Kinematic compatibility}
\end{aligned}\right\} \Rightarrow \begin{array}{l} \boxed{\begin{aligned}\frac{dM}{dx}-V &= 0\\ \frac{dV}{dx}-q &= 0\end{aligned}} \\[1em] \boxed{\frac{d^2EI}{dx^2}\left(\frac{d^2y}{dx^2}\right)-q=0} \end{array} \quad (28)$$

where

- M = Momentum
- V = Shear force
- q = Distributed load
- E = Elastic modulus
- I = Second moment of area
- κ = Curvature
- y = Vertical displacement





side note why no dV in moment

$-dV\left(\frac{dx}{2}\right)$

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Balance of linear momentum

$\sum F = 0 : \quad \sum F_y = 0$

$V - (V + dV) + q\,dx = 0$

$\Rightarrow -dV + q\,dx = 0 \Rightarrow \boxed{\frac{dV}{dx} = q}$ (B1)

Strong form

Balance of angular momentum

$\sum M = 0 : \quad \sum M_z^O = 0$ → around O

$\underbrace{-V\,dx}_{}$ $+ (M + dM) - M = 0$

couple moment from 2 V's at dist. dx

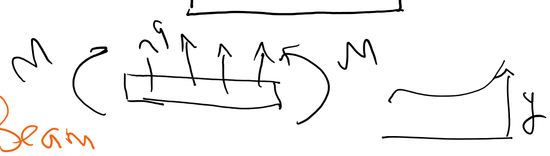
$\Rightarrow -V\,dx + dM = 0$

$\rightarrow \boxed{\frac{dM}{dx} = V}$ (B2)

Differential equations (B1, B2)

$\left.\begin{aligned}\frac{dV}{dx} &= q\\ \frac{dM}{dx} &= V\end{aligned}\right\} \Rightarrow \frac{d}{dx}\left(\frac{dM}{dx}\right) = q$

$\Rightarrow \boxed{\frac{d^2M}{dx^2} = q}$ (B3)



Beam

Constitutive eqn...

$\boxed{M = EI\kappa}$ (B4)

force like quantities

curvature

material section property

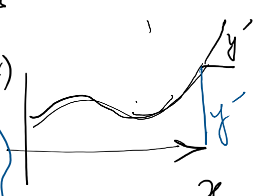
Kinematic quantity

Compatibility eqn

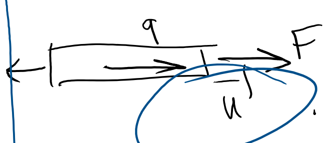
$\boxed{\kappa = y''}$ (B5)

Kinematic quantity

beam deformation



Compare with bar



$\frac{dF}{dx} = q$

$\boxed{F = EA\varepsilon}$

$\varepsilon = u'$

primary solution field

u

Bar $\frac{dEA\varepsilon}{dx} = \frac{d}{dx}\left(EA\frac{du}{dx}\right) = q$

$(EAu')' = q$

Bar $\boxed{M = 2,\ m = 1}$

Bar $\frac{d(EA\varepsilon)}{dx} = \frac{d}{dx}\left(EA\frac{du}{dx}\right) = q$ $\qquad (EAu')' = q$ $\qquad$ Bar $\boxed{M = 2, m = 1}$

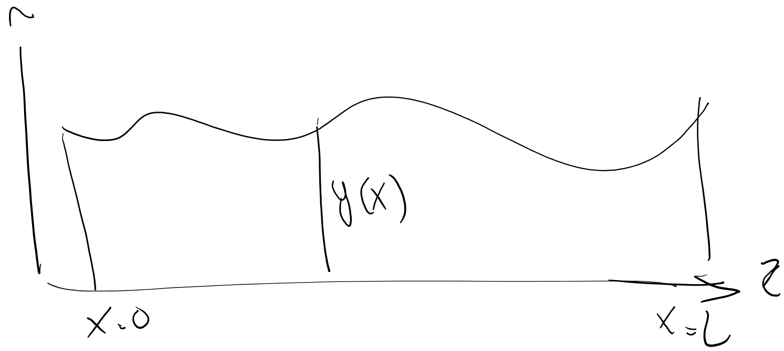
Beam $\frac{d^2}{dx^2}\left(EI\kappa\right) = \frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) = q$ $\qquad$ B6 $\qquad \boxed{(EIy'')'' = q \text{ beam}}$

Beam problem is a 4^{th} order Differential equation (bar was 2nd order)

$\forall x \in \Omega = [0, L]$

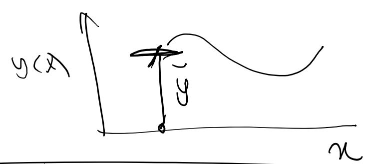
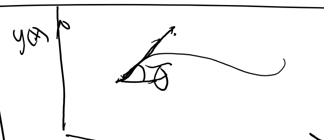
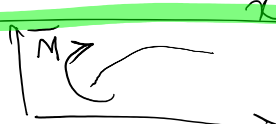
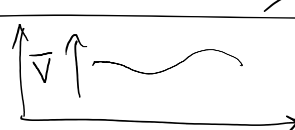
$R_i(y) = (EIy'')'' - q = 0$

residual



$$\underset{\text{order}}{M} = 2m = 4 \qquad \boxed{m = 2}$$

Boundary conditions for a beam problem $\qquad$ order $0 \ldots M-1$

BC order	what we specify	Example		
0	y		$y(0) = \bar{y}$	specify kinematic here (lower derivatives of primary field)
1	$\theta = y'$		$y'(0) = \theta(0) = \bar{\theta}$	Dirichlet (Essential)
2	$M = EIy''$		$M(0) = \bar{M}$	specify force-like
$M-1 = 3$ maximum	$V = M' = (EIy'')'$		$V(0) = (EIy'')'(x=0) = \bar{V}$	(specify spatial fluxes) Neumann (Natural)

lower half $\qquad$ upper half

0, 1 $\qquad$ 2, 3 $\qquad M-1 = 3$ $\qquad M = 4$

2	$M = EIy''$	$\overline{M}$	$M(0) = \overline{M}$	specify force-like
$M-1 = 3$	$V = M' = (EIy'')'$	$\overline{V}$	$V(0) = (EIy'')'(x=0) = \overline{V}$	(specify spatial fluxes) Nemann (Natural)

upper half

maximum

$M-1$

n des $\left.\begin{array}{l} M = EIk \\ k = y'' \end{array}\right\} \Rightarrow \boxed{M = EIy''}$ B7

B7 $V = \dfrac{dM}{dx} = (EIy'')'$

$V \cdot y$

energy dork pairs

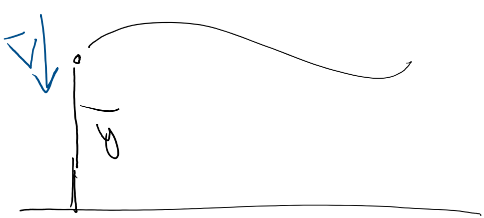
$\theta \cdot M$

same

1. Lower orders -> Dirichlet (0, M/2-1), upper orders -> Neumann (M/2, M-1).
2. Each Dirichlet BC has a Neumann pair, and their orders add to M - 1.
3. Can we specify both of Dirichlet and Neumann in a pair? No, we must specify ONE and only ONE of each pair.

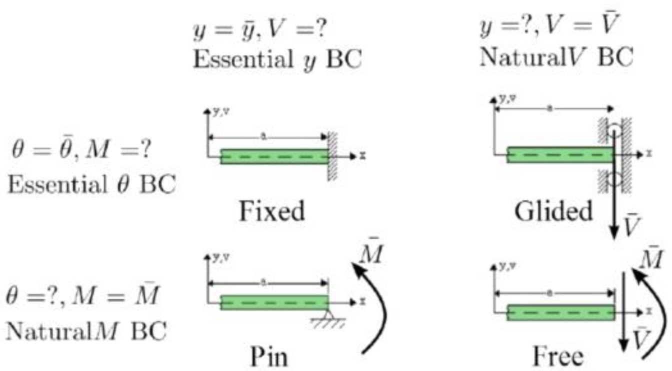
$y = \overline{y}$

$V = (EIy'')' = \overline{V}$



	$\theta = y'$	$M = EI y''$
y	Fixed $y = 0$, $y' = 0$ or more general $y = \bar{y}$ $\theta = \bar{\theta}$ $V = ?$ $M = ?$	pin $\bar{M}$ $y = \bar{y}$ $M = \bar{M}$ $V = ?$ $\theta = ?$
$V = (EI y'')'$	Glide Slide $\bar{V}$ $(V = \bar{V}) \rightarrow y = ?$ $\theta = \bar{\theta} \rightarrow M = ?$	Free $\bar{V}$ $\bar{M}$ $V(x=0) = \bar{V}$, $M = \bar{M}$ $y = ?$ $\theta = ?$

Euler Bernoulli beam: pairs of boundary conditions



- For each pair of Neumann and Dirichlet (Natural and Essential) boundary conditions, one and only one is specified.

Euler Bernoulli beam: BCs

Operator	Sample	elastostatics	operator order
$L_{2m}(\mathbf{u}) = \mathbf{r}$	$\frac{d^2 EI}{dx^2}\left(\frac{d^2 y}{dx^2}\right) = q$	$\frac{d^2 EI}{dx^2}\left(\frac{d^2 (.)}{dx^2}\right) = q$	$m = 2(M = 4)$
$L_u(\mathbf{u}) = \mathfrak{u}$	$u = \begin{bmatrix} \theta \\ y \end{bmatrix} = \begin{bmatrix} \frac{dy}{dx} \\ y \end{bmatrix} = \begin{bmatrix} \bar{\theta} \\ \bar{y} \end{bmatrix} = \bar{\mathfrak{u}}$	$L_u = \begin{bmatrix} \frac{d(.)}{dx} \\ (.) \end{bmatrix}$	$M_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$L_f(\mathbf{u}) = \bar{\mathfrak{f}}$	$\begin{bmatrix} EI\frac{d^2 y}{dx^2} \\ \frac{d}{dx}\left(EI\frac{d^2 y}{dx^2}\right) \end{bmatrix} = \begin{bmatrix} \bar{M} \\ \bar{V} \end{bmatrix}$	$L_f = \begin{bmatrix} EI\frac{d^2 (.)}{dx^2} \\ \frac{d}{dx}\left(EI\frac{d^2 (.)}{dx^2}\right) \end{bmatrix}$	$M_f = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

- One and only one of the pair M (Neumann) and θ (Dirichlet) is enforced at each end of the beam.
- One and only one of the pair V (Neumann) and y (Dirichlet) is enforced at each end of the beam.
- Neumann boundary conditions correspond to the flux terms (M and V).
- Neumann boundary conditions fall in the upper half of derivatives ($[m, 2m-1] = [2, 3]$).
- Dirichlet boundary conditions fall in the lower half of derivatives ($[0, m-1] = [0, 1]$).
- There are two boundary conditions at each end point (equal to $m = M/2$).
- $M_u + M_f = M - 1$.

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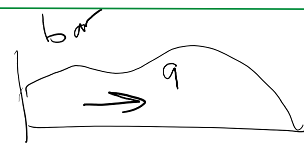
Weighted Residual Statement (WRS)

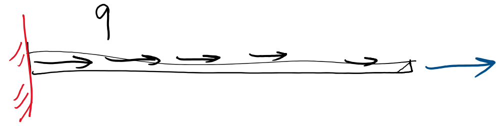
I draw this as

bar



Bar example

$\partial D_u = \{0\}$



$u(0) = \bar{u}$

$\bar{F}$

$EAu'(L) = \bar{F}$

∂D_u

At this point we have all the following:

1. ODE or PDE $\quad (EAu')' + q = 0$
2. Natural BCs $\quad F(x=L) = EAu'(x=L) = \bar{F}$
3. Essential BCs $\quad u(x=0) = \bar{u}$

$R_i(u) = (EAu')' + q \quad \forall x \in D \setminus \{0, L\}$

$R_f(u) = \bar{F} - F(u(L))$
$= \bar{F} - EAu'(L) \quad @\ \partial D_f\ (x=L)$

$R_u(u) = \bar{u} - u(0) \quad @\ \partial D_u\ (x=0)$

R = Residual

that's the error that if it's zero we have the exact solution

Ex: $\bar{u}=0$

$\bar{F}=1 \quad q=0$

$L=1 \quad EA=1$

$R_i = u''$

$R_i(u^{exact}) = 0$

Ex:

exact $u = x$ → exact soln

$R_i = u''$

$R_f = 1 - u'(1)$

$R_u = -u(0)$

$R_i(u^{exact}) = 0$

$R_f(u^{exact}) = 1 - 1 = 0$

$R_u(u^{exact}) = 0$

$\tilde{u} = x^2$

$R_i = 2$ ✗

$R_f = 0$ ✓

$R_u = 0$ ✓

$\tilde{u} = x^3$

$R_i = 6x$ ✗

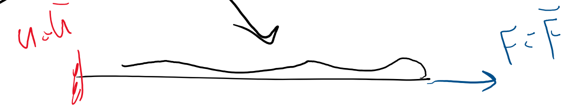
$R_f = 0$ ✓

$R_u = 0$ ✓

neither is the exact soln

Weight part of it:

Assume function $w(x)$ is an arbitrary func



$R_i(u(x)) = (EAu')' + q$

$R_f(u) = \bar{F} - F = \bar{F} - EAu'(x=L)$

$R_u(u) = \bar{u} - u(0)$

$$\text{WRS: } \forall_w \int_0^L \underbrace{w(x)}_{\text{weigh}} \underbrace{R_i(u(x))}_{\text{residual}} dx + \underline{w(L)\, R_f(u(L))} + \underline{w'(0)\, R_u(u(0))} = 0$$

for all weight func

If we have the exact solution, WRS = 0 .

Question: What is WRS = 0 for function u
then what can we say about u?

$$u_{exact} \begin{pmatrix} \forall x \;\; R_i(u)=0 \\ R_f(u)=0 \;\; \partial D_f \\ R_u(u)=0 \;\; \partial D_u \end{pmatrix} \Longrightarrow \forall w \;\; W S = 0$$

trivial

we have the exact solution

"

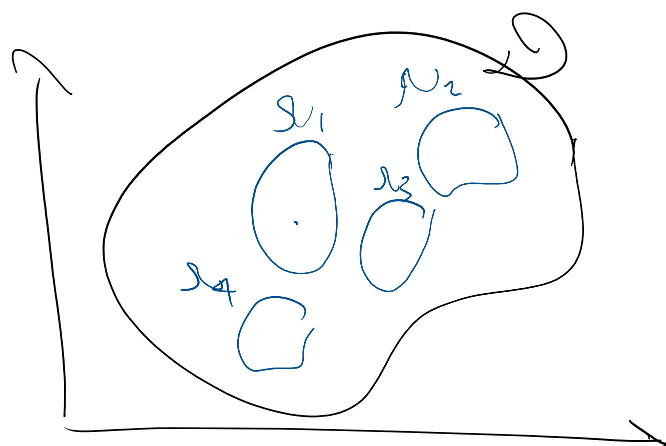
2D, 3D

$$u^h = a_1 \phi_1(x,y) + a_2 \phi_2(x,y) + a_{10} \phi_{10}$$

$\sin x \sin y$ $\quad$ $\sin 2x \sin 2y$

$a_1 \ldots a_{10}$ unknown

10 eqns



10 equations

Choose 10 Ω's

for i=1 ... 10 $\quad \int_{\partial\Omega_i} \sigma \cdot n \, ds + \int_{\Omega_i} \rho b = 0$

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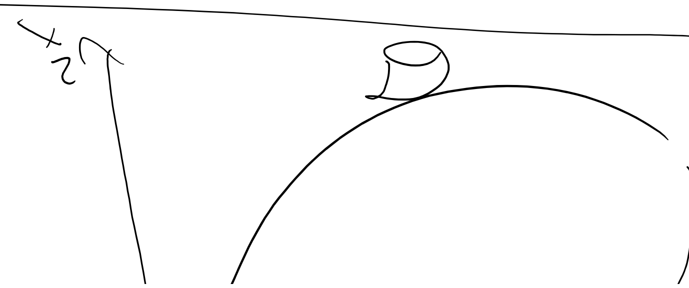
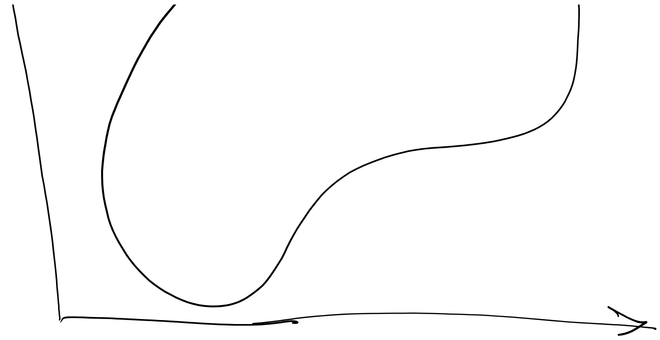
need 10 equations

W R S



WRS

for $i = 1, \ldots, 10$



$$\int_{\Omega} W_i \, R_{\text{inside}}(u(x,y)) + \frac{\quad}{R_f + R_i} = 0$$

$w_1 = \sin x$

$w_2 = \cos y$

So, the beauty of WRS is that we can integration over the SAME domain and what changes is just a weight function that is often much easier to handle than integrating on different omega's in the balance law.