FEM20240911

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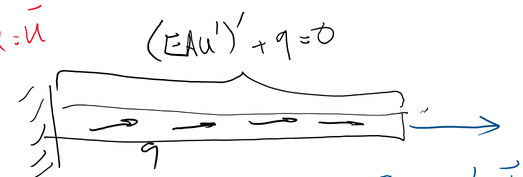

$\forall x \in D \quad R_i(u(x)) = (EAu'(x))' + q(x)$

$\forall x \in \partial D_f \quad R_f = \bar F - F = \bar F - EAu'\big|_{x=L}$

in this 1D example $\partial D_f = \{L\}$

$R_u(u) = \bar u - u \,\big|_{x=0}$

$\forall x \in \partial D_u$

$\partial D_u = \{0\}$

$\int_{\partial D_f} \to$ 2D, 3D

1D

$\int_{\partial D_u} \to$ 2D, 3D

L

$$WRS \quad \forall w \quad \int_D w \, R_i(u)\, dx + \underbrace{w(L)(\bar F - F)\big|_{x=L}}_{R_f} + \underbrace{w'(0)(\bar u - u)\big|_{x=0}}_{R_u} = 0$$

Exact sol'n: $WRS = 0$

$WRS = 0 \quad \forall w$ (weights) $\Longrightarrow$ u is the exact sol'n

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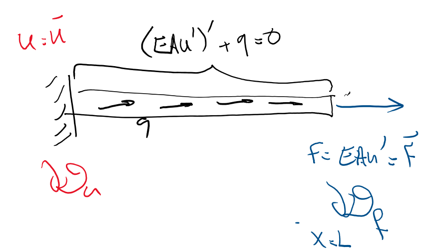
Find $u \ni$ (such that) WRS



(1) $\underbrace{\forall w}_{\text{for all weight functions}} \quad \int_0^L \underbrace{w(x)}_{\text{how many derivatives on } w?\ 0} \underbrace{\left[(EAu'(x))' + q(x) \right]}_{R_i \ \to \ 2 \text{ derivatives}} dx + w(L)\left(\bar F - EAu'(L)\right) + w'(0)\left(\bar u - u(0)\right) = 0$

finite element functions look like this

don't have u''

We need to transfer one derivative of u'' to w:

don't have u

We need to transfer one derivative of u'' to w:

① $$\int_0^L \underbrace{w}_{U} \underbrace{(EAu')'}_{V'} dx = -\int_0^L U'V\,dx + UV\Big|_0^L$$

$$\int w(EAu')'dx = -\int w'EAu'dx + w(EAu')\Big|_0^L$$

plug this in ①

eq

$$\int_0^L \underbrace{w(x)\underbrace{\left[(EAu'(x))' + q(x)\right]}_{}}_{R_i} dx + w(L)(\bar{F} - EAu'(L)) + w'(0)(\bar{u} - u(0)) = 0$$

x=L

$$\underbrace{\left[\int_0^L -w'EAu'dx + \cancel{w(L)EAu'(L)} - w(0)EAu'(0)\right]}_{\int_0^L w(EAu')'dx} + \int wq\,dx + w(L)\bar{F}$$

$$- \cancel{w(L)EAu'(L)} + w'(0)(\bar{u} - u(0)) = 0$$

② $$\boxed{\int_0^L w'EAu'dx = \int_0^L wq\,dx + w(L)\bar{F} + w'(0)(\bar{u} - u(0)) - w(0)EAu'(0) = 0}$$

half way to "weak" statement

$\forall x$: strong statement, e.g. $\forall x \; (EAu(x))' + q(x) = 0$

weak $\int$ weak

$$\int_\Omega w(x)\underbrace{\left((EAu'(x))' + q(x)\right)}dx = 0$$

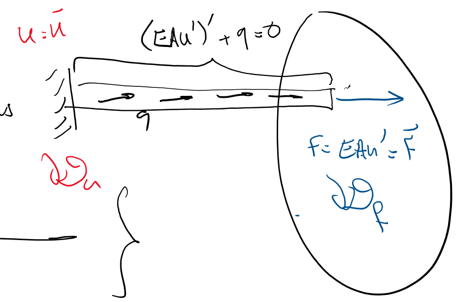
Summary & clean-up

option 1

WRS

Find $u \in V = \{ f \in C^2([0,L]) \mid \text{———} \}$

function, f, f', f'' exist & are continuous



$\ni \quad \forall w \in W = \{ f \in C^0([0,L]) \mid - \; - \}$

such that

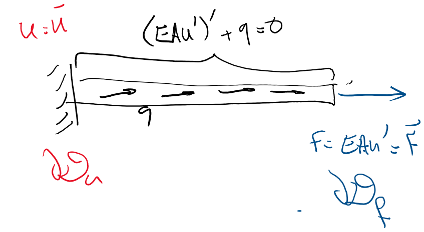
$$\int_0^L w(x) R_i(x)\,dx + w(L) R_f(L) + w'(0) R_u(0) = 0$$

$$\int_0^L w(x)\{(EA\,u'(x))' + q(x)\}\,dx + w(L)(\bar F - EAu'(L)) + w'(0)(\bar u - u(0)) = 0$$

The most common form of WRS:

In general, we satisfy the essential BC a priori, meaning that we only look for functions that from the beginning already satisfy the essential BCs:

WRS_2



Find $u \in V = \{ f \in C^2([0,L]) \mid f(0) = \bar u \}$

$\ni \quad \forall w \in W = \{ f \in C^0([0,L]) \mid \quad \}$

$$\int_0^L w((EAu')' + q)\,dx + w(L)(\bar F - EAu'(L)) + \cancel{w'(0)(\bar u - u(0))} = 0$$

are satisfy this a priori

a priori

How could we do this in practice

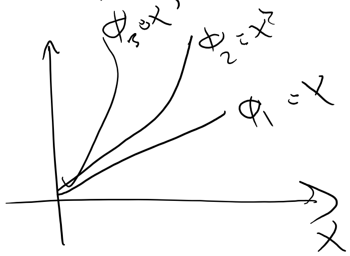
$\bar{u} = .5$

we want to form good u's that belong to V

unknowns

$$u(x) = 0.5 + \sum_{i=1}^{n} a_i \underbrace{\phi_i(x)}_{\text{shape function}} \qquad \phi_i(0) = 0$$

$$u(0) = 0.5 + \sum a_i \underbrace{\phi_i(0)}_{0} = \bar{u}$$



sin x ✓

cos x ✗

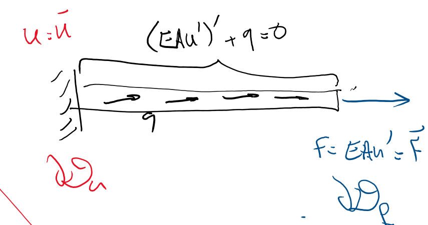
A less common WRS is one that satisfies ALL BCs a priori

WRS$_3$ (Not common) doing this part is difficult

Find $u \in V = \{ f \in C^2(0,L) \mid f(0) = \bar{u},\ EAf'(L) = \bar{F} \}$

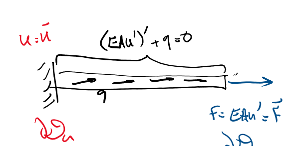
$\ni \forall w \in W = \{ f \in C^0(0,L) \mid \_\_\_\_\_\_ \}$

$$\int_0^L w((EAu')' + q)\,dx + w(L)\underbrace{(\bar{F} - EAu'(L))}_{=0} + \cancel{w'(0)(\bar{u} - u(0))} = 0$$



It is the WRS2 (where we satisfied the essential BC a prior) that will be used in deriving the weak statement.

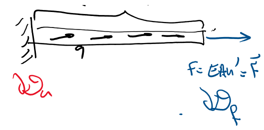
WRS$_2$ strongly satisfying ess. BC



WK 2

$$\forall\, u \in \mathcal{V} : \{ f \in C^{?}(0,L) \mid f(0) = \bar{u} \}$$

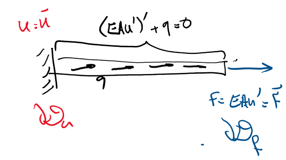
$$\Rightarrow \forall\, w \in \mathcal{W} : \{ f \in C^{?}(0,L) \mid \}$$



$$\int_0^L w\left((EAu')' + q\right)dx + w(L)\left(\bar{F} - EAu'(L)\right) = 0$$

Recall $\int_0^L w(EAu')' = -\int_0^L w' EAu' \, dx + w(L)EAu'(L) - w(0)EAu'(0)$

$$\Longrightarrow \int_0^L w' EAu' \, dx = \int_0^L wq \, dx + w(L)\bar{F} - w(0)\underbrace{EAu'(0)}_{F(0)} = 0$$



$$\forall\, u \in \mathcal{V} : \{ f \in C^{?}(0,L) \mid f(0) = \bar{u} \}$$

$$\forall\, w \in \mathcal{W} : \{ f \in C^{?}(0,L) \mid \}$$

we decide $w(0) = 0$ (in general $w = 0$ on $\partial\Omega_u$)

$$\int_0^L w' EAu' \, dx = \int_0^L wq \, dx + w(L)\bar{F} \cancel{- \underbrace{w(0)}_{0}\underbrace{EAu'(0)}_{F(0)}} = 0$$

Weak statement

Weak statement

Find $u \in V = \{ f \in C^1(0,L) \mid f(0) = \bar{u} \}$

$\ni \forall w \in W = \{ f \in C^1(0,L) \mid f(0) = 0 \}$

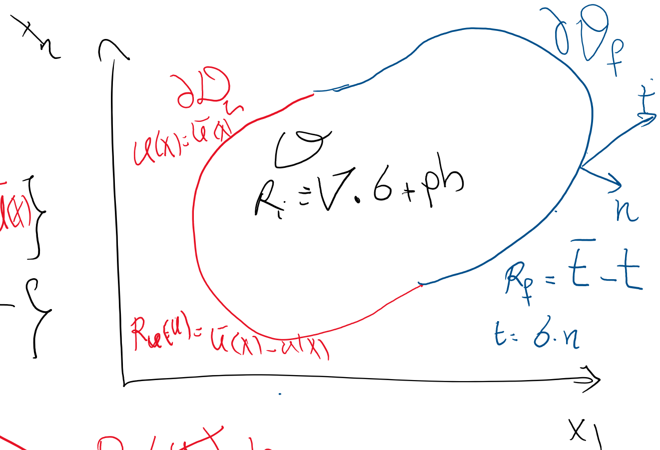
$$\int_0^L w' EA u' \, dx = \underbrace{\int_0^L w q \, dx}_{\text{source term contrib.}} + \underbrace{w(L)\bar{F}}_{\text{natural BC contrib.}}$$

2D elasticity

WRS(2) → essential BC satisfied strongly

Find $u \in V = \{ f \in C^2(\Omega) \mid \forall x \in \partial\Omega_u \; f(x) = \bar{u}(x) \}$

$\ni \forall w \in W = \{ f \in C^0(\Omega) \mid \text{———} \}$



$$\int_\Omega w R_i(u) \, dv + \int_{\partial\Omega_f} w R_f(u) \, ds + \cancel{\int_{\partial\Omega_u} w R_u(u) \, ds} = 0$$

$$\int_\Omega w (\nabla\cdot\sigma + \rho b) \, dv + \int_{\partial\Omega} w (\bar{t} - t) \, ds = 0$$

derivative order on u

$$\nabla\cdot\sigma = \nabla\cdot \underbrace{C\varepsilon}_{\sigma} = \nabla\cdot C\left(\frac{\nabla u + \nabla u^T}{2}\right) = \underbrace{\nabla\cdot C \nabla u}_{2 \text{ derivatives}}$$

$$\int_D \omega \, \nabla \cdot \underbrace{\sigma}_{\text{do not plug in formula of } \sigma \text{ now}} \, dV = -\int_D \nabla \omega \cdot \underbrace{\sigma}_{C \nabla u} + \text{BC terms}$$