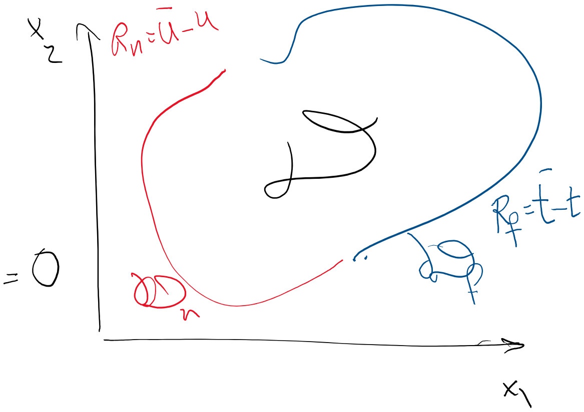
Continue on deriving the Weak statement for elastostatics



WRS last time more derivative

$$\int_{\Omega} w\left(\underbrace{\nabla\cdot\sigma + \rho b}_{R_i}\right) dV + \int_{\partial\Omega_t} w\left(\bar{t} - t\right) dS = 0$$

Goal

$$w\,\nabla\cdot\sigma \longrightarrow \nabla w\,\sigma$$

$$\boxed{w\,\nabla\cdot\sigma = \nabla\cdot w\sigma - \nabla w : \sigma}$$

(1) we will use this identity

In your HW I have given a similar identity for heat conduction

Side note: if we wanted to prove (1) Slide 60

$$\underbrace{w}_{\text{vector}}\left(\underbrace{\nabla\cdot\sigma}_{\text{vector}}\right) = w_i\,(\nabla\cdot\sigma)_i = w_i\left(\underbrace{\sigma_{ij,j}}_{\sum_{j=1}^{2}\sigma_{ij,j}}\right)$$

$$w_i\,(\nabla\cdot\sigma)_i = \sum_{i=1}^{d=2} w_i\,(\nabla\cdot\sigma)_i$$

$$\vec{a}\cdot\vec{b} = a_1 b_1 + a_2 b_2 = a_i b_i$$

$$w(\nabla\cdot\sigma) = \overbrace{w_i}^{A}\,\overbrace{\sigma_{ij,j}}^{B'} = \overbrace{\left(w_i\sigma_{ij}\right)_{,j}}^{(AB)'} - \overbrace{w_{i,j}\,\sigma_{ij}}^{A'B}$$

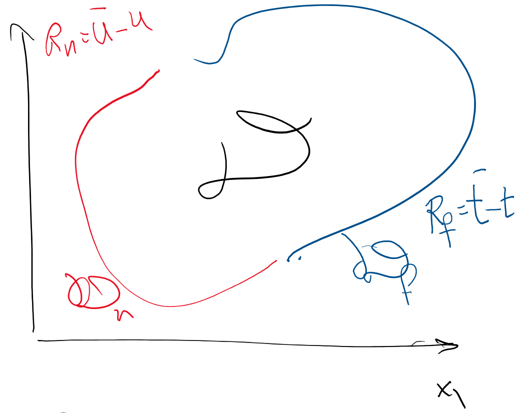
$$\boxed{AB' = (AB)' - A'B}$$

that is

$$\text{"} = \sum_{i=1}^{2}\sum_{j=1}^{2} w_i\,\sigma_{ij,j}$$

$$= (w\sigma)_{j,j} - (\nabla w)_{ij}\,\sigma_{ij} = \nabla\cdot w\sigma - \nabla w : \sigma$$

$$\int_{\mathcal{D}} \omega(\underbrace{\nabla\cdot\sigma + \rho b}_{R_i})\,dV + \int_{\partial\mathcal{D}_f} \omega(\bar{t} - t)\,dS = 0$$

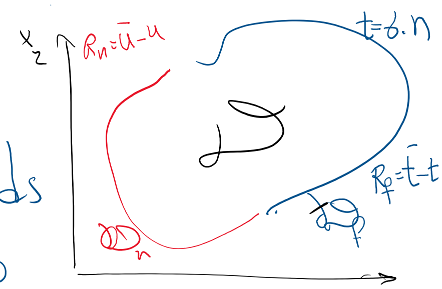


(1) $\omega\nabla\cdot\sigma = \nabla\cdot(\omega\sigma) - \nabla\omega : \sigma$

$$\int_{\mathcal{D}} \Big\{ \big[\overbrace{\nabla\cdot(\omega\sigma) - \nabla\omega : \sigma}^{\omega\nabla\cdot\sigma}\big] + \omega\rho b \Big\}\,dV + \int_{\partial\mathcal{D}_f} \omega(\bar{t} - t)\,ds = 0$$

$$\int_{\mathcal{D}} \nabla\cdot\underbrace{\omega\sigma}_{f}\,dV - \int_{\mathcal{D}} \nabla\omega : \sigma\,dV + \int_{\mathcal{D}} \omega\rho b\,dV + \int_{\partial\mathcal{D}_f} (\omega\bar{t} - \omega t)\,ds = 0 \Longrightarrow$$

$$* \int_{\partial\mathcal{D} = \partial\mathcal{D}_f \cup \partial\mathcal{D}_u} (\omega\sigma)n\,ds - \int_{\mathcal{D}} \nabla\omega : \sigma\,dV + \int_{\mathcal{D}} \omega\rho b\,dV + \int_{\partial\mathcal{D}_f} \omega\bar{t}\,ds - \int_{\partial\mathcal{D}_f} \omega\sigma n\,ds = 0.$$



$$\overbrace{\left(\int_{\partial\mathcal{D}_f} \omega\sigma.n\,ds + \underbrace{\cancel{\int_{\partial\mathcal{D}_u} \omega\sigma n\,ds}}\right)}^{*\ \text{term}} - \int_{\mathcal{D}} \nabla\omega : \sigma\,dV + \int_{\mathcal{D}} \omega\rho b\,dV + \int_{\partial\mathcal{D}_f} \omega\bar{t}\,ds - \int_{\partial\mathcal{D}_f} \omega\sigma.n\,ds = 0$$

Now, we choose weight functions that are zero on the essential BCs. So the term in red disappears

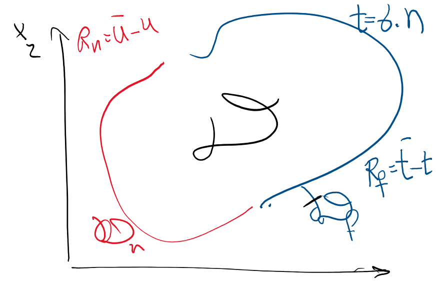
Find $u \in \mathcal{V} = \{ f \in C^1(\mathcal{D}) \mid \forall x \in \partial\mathcal{D}_u \;\; f(x) = \bar{u} \}$

$\ni \forall w \in \mathcal{W} = \{ f \in C^1(\mathcal{D}) \mid \forall x \in \partial\mathcal{D}_u \;\; f(x) = 0 \}$

$$\int_{\mathcal{D}} \nabla w : \sigma \, dV = \int_{\mathcal{D}} w \rho b \, dV + \int_{\partial\mathcal{D}_f} w \bar{t} \, ds$$

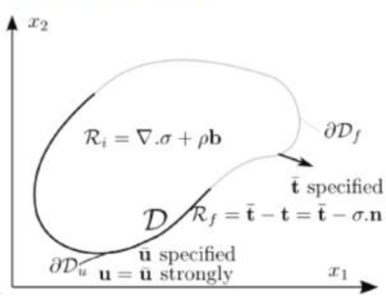
$$\sigma = C\varepsilon = C\Big(\underbrace{\frac{\nabla u + \nabla u^T}{2}}_{\varepsilon}\Big) = C \nabla u$$

$$\int_{\mathcal{D}} \nabla w : (C \nabla u) \, dV = \int w \rho b \, dV + \int_{\partial\mathcal{D}_f} w \bar{t} \, ds$$



Elastostatics: Weighted Residual Statement

Again to form the weighted residual statement, we take the common approach and strongly enforce the essential boundary conditions, while weakly enforcing the natural boundary conditions and the partial differential equation (strong form)



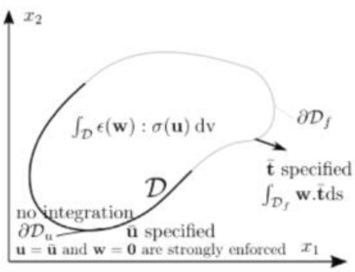
The Weighted Residual Statement reads as,

$$\text{Find } \mathbf{u} \in \mathcal{V}^{\text{WRS}} = \{\mathbf{v} \in C^2(\mathcal{D}) \mid \forall \mathbf{x} \in \partial\mathcal{D}_u \;\; \mathbf{v}(\mathbf{x}) = \bar{\mathbf{u}}\}, \text{ such that,} \tag{66a}$$

$$\forall w \in \mathcal{W}^{\text{WRS}} = C^0(\mathcal{D}) \text{ no need to enforce the homogeneous essential BCs for WRS} \tag{66b}$$

$$0 = \int_{\mathcal{D}} \mathbf{w}.(\underbrace{\nabla.\sigma}_{C_{ijkl}\,u_{k,lj}} + \rho\mathbf{b}) \, \mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathbf{w}.(\bar{\mathbf{t}} - \mathbf{t}) \, \mathrm{ds} \tag{66c}$$

Elastostatics: Weak Statement



The weak statement for elastostatics and the boundary conditions are:

$$\text{Find } \mathbf{u} \in \mathcal{V} = \{v \in C^1(\mathcal{D}) \mid \forall \mathbf{x} \in \partial\mathcal{D}_u \ \mathbf{v}(\mathbf{x}) = \bar{\mathbf{u}}(\mathbf{x})\}, \text{ such that,} \tag{72a}$$

$$\forall \mathbf{w} \in \mathcal{W} = \{v \in C^1(\mathcal{D}) \mid \forall \mathbf{x} \in \partial\mathcal{D}_u \ \mathbf{v}(\mathbf{x}) = 0\} \tag{72b}$$

$$\int_{\mathcal{D}} \epsilon(\mathbf{w}) : \sigma(\mathbf{u})\, \mathrm{dv} = \int_{\mathcal{D}} \mathbf{w}.\rho\mathbf{b}\, \mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathbf{w}.\bar{\mathbf{t}}\, \mathrm{ds} \tag{72c}$$

- Both $\mathcal{V}$ and $\mathcal{W}$ have the same regularity ($C^m(\mathcal{D})$): $m = M/2$, $M = 2$ is the order of the differential equation.
- The less demanding regularity conditions for the solution compared to the weighted residual statement ($C^M(\mathcal{D}) \rightarrow C^m(\mathcal{D})$) takes us to the same function space needed for the balance law (highest derivative is for $\sigma(\mathbf{u}) = C_{ijkl}\, u_{k,l}$ is 1).
- Both $\mathcal{V}$ and $\mathcal{W}$ exactly enforce the essential boundary conditions, with the difference that $\mathcal{W}$ satisfies the homogeneous version.

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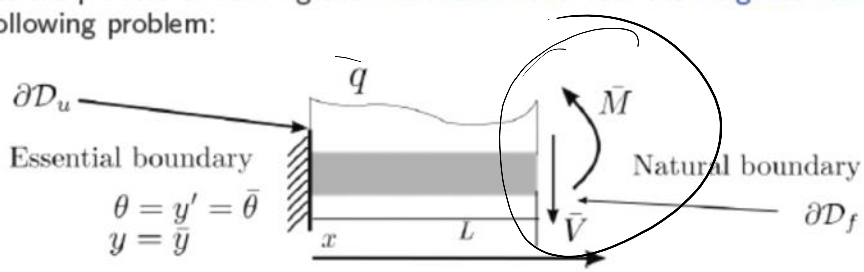
Weighted residual statement to Weak statement

To demonstrate the process of deriving the weak statement from the weighted residual statement consider the following problem:



The residuals for this problem are:

need to take 2 these

$$\begin{aligned}
\mathcal{R}_i &= \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) - q && \text{Interior residual for} && \mathcal{D} = [0, L] \\
\mathcal{R}_f &= \begin{bmatrix} \bar{M} - M \\ \bar{V} - V \end{bmatrix} && \text{Natural BC residual for} && \partial\mathcal{D}_f = \{L\} \\
\mathcal{R}_u &= \begin{bmatrix} \bar{\theta} - \theta \\ \bar{y} - y \end{bmatrix} && \text{Essential BC residual for} && \partial\mathcal{D}_u = \{0\}
\end{aligned} \tag{53}$$

As mentioned previously, we want to drop the weighted residual term for essential boundary condition (why?). Accordingly, we need to strongly enforce the essential boundary condition (This is why this is called "essential" boundary condition). That is, we require:

$$\mathcal{R}_u = \begin{bmatrix} \bar{\theta} - \theta \\ \bar{y} - y \end{bmatrix} = 0 \quad \text{at } x = 0 \ (\partial\mathcal{D}_u). \tag{54}$$

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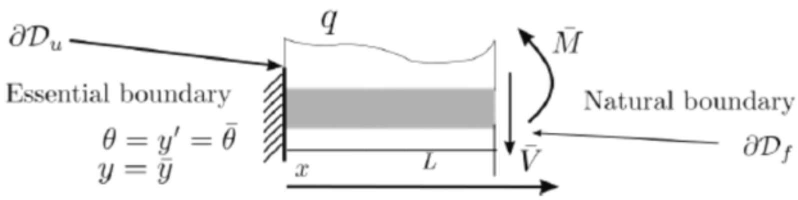
Weighted residual statement to Weak statement



Since we strongly enforce the essential boundary condition, the weighted residual for this problem simplifies to:

$$
\begin{aligned}
0 &= \int_{\mathcal{D}} w\mathcal{R}_i(y)\,\mathrm{d}v + \int_{\partial\mathcal{D}_f} \mathbf{w}_f\mathcal{R}_f(y)\,\mathrm{d}s \\
&= \int_0^L w\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) - q\right)\mathrm{d}x + \begin{bmatrix} -\frac{\mathrm{d}w}{\mathrm{d}x} \\ w \end{bmatrix}\cdot\begin{bmatrix} \bar{M} - M \\ \bar{V} - V \end{bmatrix}\Big|_{x=L} \\
&= \int_0^L w\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) - q\right)\mathrm{d}x - \frac{\mathrm{d}w}{\mathrm{d}x}(\bar{M} - M(y))|_{x=L} + w(\bar{V} - V(y))|_{x=L}
\end{aligned}
\tag{55}
$$

4th

Next, we transfer derivatives from y to w (trial function to weight function). We note that

$$
\begin{aligned}
&\int_0^L w\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)\mathrm{d}x = \int_0^L\left[-\frac{\mathrm{d}w}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}EI\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)\right]\mathrm{d}x + \left[w\frac{\mathrm{d}}{\mathrm{d}x}\left(EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)\right]\Big|_{x=0}^{x=L} \\
&= \int_0^L\left[\frac{\mathrm{d}^2w}{\mathrm{d}x^2}EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right]\mathrm{d}x + [wV(y)]\Big|_{x=0}^{x=L} - \left[\frac{\mathrm{d}w}{\mathrm{d}x}\left(EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)\right]\Big|_{x=0}^{x=L}
\end{aligned}
\tag{56}
$$

2nd order

$M = EIy''$

$V = \frac{dM}{dx} =$

$(EIy'')'$

3rd order

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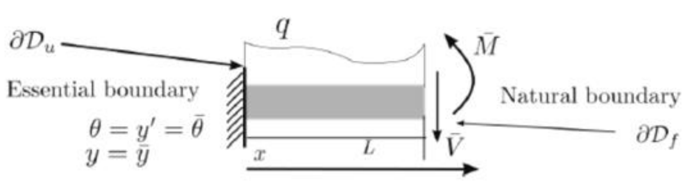
Weighted residual statement to Weak statement



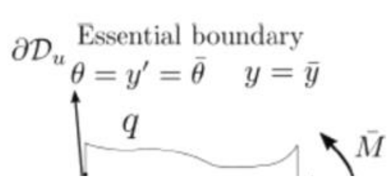
Plugging (55) in (56) yields,

$$
\begin{aligned}
0 &= \int_0^L w\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) - q\right)\mathrm{d}x - \frac{\mathrm{d}w}{\mathrm{d}x}(\bar{M} - M(y))|_{x=L} + w(\bar{V} - V(y))|_{x=L} \\
&= \left\{\int_0^L\left[\frac{\mathrm{d}^2w}{\mathrm{d}x^2}EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - wq\right]\mathrm{d}x + \left[wV(y)\right] - \frac{\mathrm{d}w}{\mathrm{d}x}M(y)\right]\Big|_{x=0}^{x=L}\right\} \\
&- \frac{\mathrm{d}w}{\mathrm{d}x}(\bar{M} - M(y))|_{x=L} + w(\bar{V} - V(y))|_{x=L} \\
&= \int_0^L\left[\frac{\mathrm{d}^2w}{\mathrm{d}x^2}EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - wq\right]\mathrm{d}x \\
&+ \left\{wV(y) - \frac{\mathrm{d}w}{\mathrm{d}x}M(y) - \frac{\mathrm{d}w}{\mathrm{d}x}(\bar{M} - M(y)) + w(\bar{V} - V(y))\right\}_{x=L} \\
&- \left\{wV(y) - \frac{\mathrm{d}w}{\mathrm{d}x}M(y)\right\}_{x=0}
\end{aligned}
\tag{57}
$$

Weighted residual statement to Weak statement

This equation simplifies to

$$
0 = \int_0^L\left[\frac{\mathrm{d}^2w}{\mathrm{d}x^2}EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - wq\right]\mathrm{d}x + \left\{-\frac{\mathrm{d}w}{\mathrm{d}x}\bar{M} + w\bar{V}\right\}_{x=L}
\tag{58a}
$$



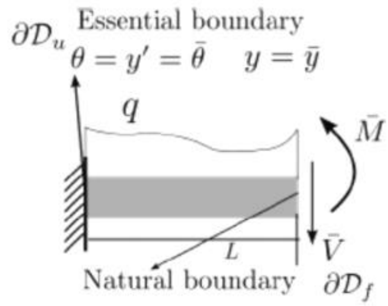
Weighted residual statement to Weak statement

This equation simplifies to

$$0 = \int_0^L \left[\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} EI \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - wq \right] \mathrm{d}x + \left\{ -\frac{\mathrm{d}w}{\mathrm{d}x} \bar{M} + w\bar{V} \right\}_{x=L} \quad \text{(58a)}$$

$$+ \left\{ w \left(V(y) - V(y) \right) - \frac{\mathrm{d}w}{\mathrm{d}x} \left(M(y) - M(y) \right) \right\}_{x=L} \quad \text{(58b)}$$

$$- \left\{ wV(y) - \frac{\mathrm{d}w}{\mathrm{d}x} M(y) \right\}_{x=0} \quad \text{(58c)}$$



we choose these to be zero

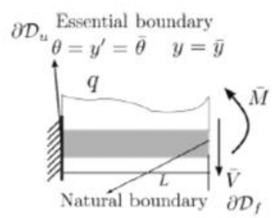
Essential boundary condition

We mentioned that the essential boundary condition is strongly enforced (That is, it is an "essential" condition). The essential conditions (54) require,

$$\mathcal{R}_u = \begin{bmatrix} \bar{\theta} - \theta \\ \bar{y} - y \end{bmatrix} = 0 \Rightarrow \boxed{\left\{ \begin{array}{c} \frac{\mathrm{d}y}{\mathrm{d}x} = \bar{\theta} \\ y = \bar{y} \end{array} \right\}, \text{ at } x = 0 \ (\partial\mathcal{D}_u)} \quad (59)$$

We discussed that to annihilate the high order derivatives of y in (58c):

$$- \left\{ wV(y) - \frac{\mathrm{d}w}{\mathrm{d}x} M(y) \right\}_{x=0}$$



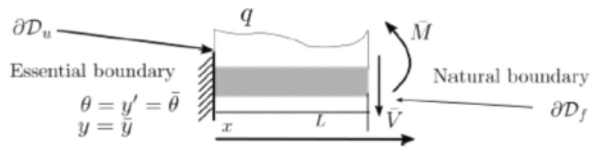
we set the corresponding weight functions identically zero:

$$\left\{ \begin{array}{c} \frac{\mathrm{d}w}{\mathrm{d}x} = 0 \\ w = 0 \end{array} \right\}, \quad \text{at } x = 0 \ (\partial\mathcal{D}_u) \quad (60)$$

Summary

1. Trial, y, (solution) functions exactly satisfy all essential boundary conditions.
2. Weight, w, functions exactly satisfy the homogeneous essential boundary conditions.
3. If both conditions are satisfied we can form a *weak statement* that requires *only half* the highest derivative order. In fact, this enlarged space of functions is the same as the space of the original balance law.

Weak Statement (WS)



The weak statement for the Euler Bernoulli problem and the BCs in the figure are:

essential BC

$$\text{Find } y \in \mathcal{V} = \{u \in C^2(\mathcal{D}) \mid u(0) = \bar{y},\ \frac{\mathrm{d}u}{\mathrm{d}x}(0) = \bar{\theta}\},\ \text{such that,} \tag{62a}$$

$$\forall w \in \mathcal{W} = \{u \in C^2(\mathcal{D}) \mid u(0) = 0,\ \frac{\mathrm{d}u}{\mathrm{d}x}(0) = 0\} \tag{62b}$$

$$0 = \int_0^L \left[\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} EI \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - wq\right] \mathrm{d}x + \left\{-\frac{\mathrm{d}w}{\mathrm{d}x}\bar{M} + w\bar{V}\right\}_{x=L} \tag{62c}$$

0 th version (homogeneous) essential BC

Summary

- Both $\mathcal{V}$ and $\mathcal{W}$ have the same regularity ($C^m(\mathcal{D})$): $m = M/2$, $M = 4$ is the order of the differential equation.
- The less demanding regularity conditions for the solution compared to the weighted residual statement ($C^M(\mathcal{D}) \rightarrow C^m(\mathcal{D})$) takes us to the same function space needed for the balance law (balance of linear and angular momentum for Euler Bernoulli beam.
- Both $\mathcal{V}$ and $\mathcal{W}$ exactly enforce the essential boundary conditions, with the difference that $\mathcal{W}$ satisfies the homogeneous version.

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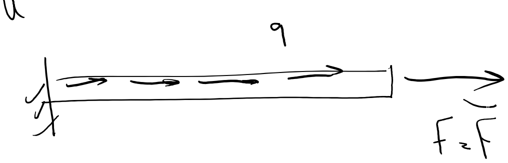
note $\nabla w : \sigma$

$$\text{matrix} \longleftarrow A : B = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} : \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$A_{ij} B_{ij} = A_{11} B_{11} + A_{12} B_{12} + A_{21} B_{21} + A_{22} B_{22}$$

Discretization of solution

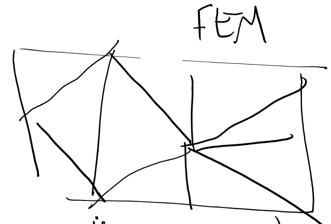
$$\text{WK: } \forall w \quad \int_0^L w' E A u' \, dx = \int_0^L w q \, dx + w(L) \bar{F}$$



Now, we need to ensure that the solution u, satisfies the essential BC.

We discretize the solution:
Going from infinite unknowns (continuum statement) to a finite number of unknowns

FEM



is one example

u: exact solution (u^{exact})

$$u^h(x) = \Phi(x) + a_1 \phi_1(x) + a_2 \phi_2(x) + \cdots + a_n \phi_n(x)$$

discretized solution

Φ: a function that satisfies the essential BCs

must satisfy the homog. ess. BC

$\Phi(x)$ ... $\phi_i(x)$ ... = are functions : we choose them solution: a function that satisfies the essential BCs homog. ess. BC

$\phi_1(x) \ldots \phi_n(x)$: shape functions : we choose them

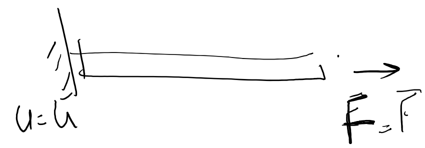
$x, x^2, x^3,$

$\sin x, \sin 2x, \sin 3x$

$a_1, \ldots, a_n$ unknowns of the problem

∞ unknowns $\longrightarrow$	n unknowns $a_1, \ldots, a_n$
u	u^h

$u^h(0) = \bar{u}$



$\partial\Omega_u: \{0\}$

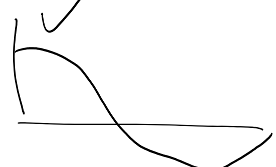
$\underbrace{\phi_p(0)}_{\bar{u}} + a_1 \underbrace{\phi_1(0)}_{0} + a_2 \underbrace{\phi_2(0)}_{0} + \cdots + a_n \underbrace{\phi_n(0)}_{0} = \bar{u}$

what if

Give me good ϕ_p's $\qquad \bar{u} = 1$

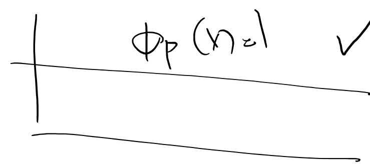
$\phi_p(0) = \bar{u} = 1$

$\cos x$



e^x

1



$\phi_i(0) = 0$

$\Phi = \left\{ \sin \frac{x}{L}, \sin 2\frac{x}{L}, \sin 3\frac{x}{L}, \ldots \right\}$

$$\phi = \{x, x^2, x^3, x^4, \ldots\}$$

$n = 4$ $$u^h(x) = \phi_p + \underbrace{\sum_{i=1}^{4} a_i \phi_i}_{a_i \phi_i} = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

WK:

$$\forall w \quad \int_0^L w' EA u' dx = \int_0^L w q \, dx + w(L)\bar{F} \quad \text{Continuum}$$

u: exact solution

Discrete

Find $a_1, \ldots, a_4$ ∋

~~$\forall w$~~ for a_1, a_2, a_3, a_4

$$\int_0^L EA u^{h\prime} dx = \int_0^L w q \, dx + w(L)\bar{F}$$

$n=4$ $u^h = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$

essential BC

$$w_i(0) = 0$$

ϕ_i's are perfect weights for weak statement as they are already zero on $\partial\Omega_u$ ($x=0$ here)

---

we'll solve this in next section

$$\phi = \{x, x^2, x^3, x^4\}$$

Find $a_1, \ldots, a_4$ ∋

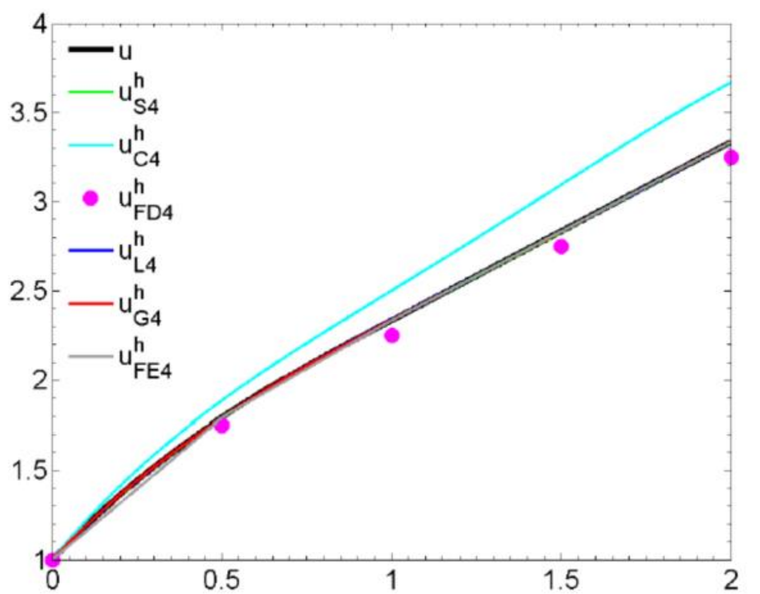
Find $a_1, \ldots, a_4$?

$$\forall i = 1, \ldots, 4 \quad \int \underbrace{\phi_i'}_{w_i'} \, EA \Big(\underbrace{\sum_{j=1}^{4} a_j \phi_j + \phi_p}_{u^h} \Big)' = \int_0^L \phi_i \, q \, dx + \bar{F} \phi_i(L)$$

$$\begin{bmatrix} & & \\ & K & \\ & & \end{bmatrix}_{4\times4} \begin{bmatrix} a_1 \\ \vdots \\ a_4 \end{bmatrix} = \begin{bmatrix} F_1 \\ \vdots \\ F_4 \end{bmatrix}$$

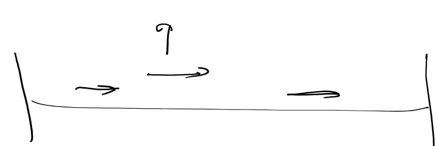
Find $a_1, \ldots, a_4$

Bar example, $n = 4$, Comparison of solutions



$\partial D_u = \{0, 1\}$

$\phi_p(x) = .5 + .5x$ ✓



$u(x=0) = 0.5$ $\qquad$ $u(x=1) = 1$

$\phi_p(x) = .5 + .5x$ ✓ $u(x=0)=0.5$ $u(x=1)=1$

ϕ_i's $\phi_i(0) = 0$ $\phi_i(1) = 0$

$\sin \pi x$, $\sin 2\pi x$, $\sin 3\pi x$

Make a function out of 1, x, x^2

$\phi_1 = \alpha_0 + \alpha_1 x + \alpha_2 x^2$. that is a good

ϕ $(\phi_1(0) = 0$, $\phi_1(1) = 0)$