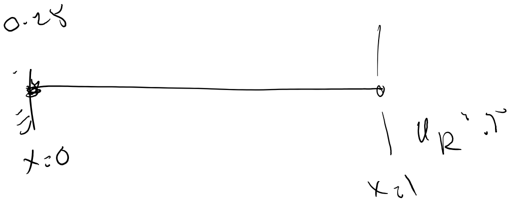


$$u^h = \phi_p(x) + a_1\phi_1(x) + a_2\phi_2(x) \ldots$$

$\partial\Omega_u : \{0, 1\}$

HW

$\phi_p(x) = .25(1+x)$ ✓

$\phi_1(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$ build from this

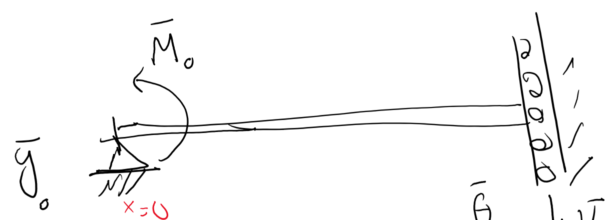
$\alpha_0 = 1$

$\phi_1(0) = 0$
$\phi_1(1) = 0$ $\Rightarrow \alpha_1, \alpha_2$

good basis funct.

$\phi_2(x) = x\phi_1(x)$ is a good choice

Another example



$x=1$

Essential BCs $\begin{cases} y(x=0) = \bar{y}_0 \\ y'(x=1) = \bar{\theta}_1 \end{cases}$

Natural BC $\begin{cases} M(0) = \bar{M}_0 \\ V(L) = \bar{V}_1 \end{cases} \rightarrow$ satisfied weakly by adding the

residuals $\bar{M}_0 - M(0)$, $\bar{V}_1 - V(L)$ to the WRS

should be satisfied a priori

$$y^h(x) = \phi_p(x) + \sum_{i=1}^{n} a_i \phi_i(x)$$

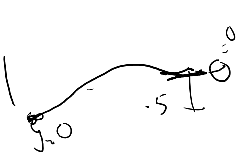
ϕ_p : conditions $\phi_p(0) = \bar{y}_0$ $\phi_p'(1) = \bar{\theta}_1$

ϕ_i' i from 1 to n $\phi_i(0) = 0$ $\phi_i'(1) = 0$

good $\phi_1 = x - \frac{1}{2}x^2$

$\phi_1(0) = 0$

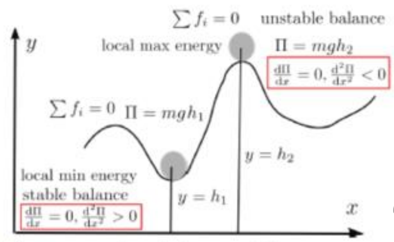
$\phi_1'(x) = 1 - x = 0$ @ $x = 1$



Energy Methods

Energy Methods: Motivation

For the gravitational potential shown, the position of the disc is determined by x. The potential energy $\Pi(x) = mgy(x)$ where $y(x)$ is the height function.



- Forces are in balance when $\frac{d\Pi}{dx} = 0$. That is, it corresponds to an extremum (minimum or maximum) or the energy function.
- The Balance is stable when the extremum is a local minimum. That is, $\frac{d\Pi}{dx} = 0$ and $\frac{d^2\Pi}{dx^2} > 0$ (If $\frac{d^2\Pi}{dx^2} = 0$ we need higher order derivatives).
- Energy methods seek states for which the energy of the system is minimum. These states are the solutions to the problem.
- Not all problems possess an energy function and balance law has more generality. For example while we cannot define a potential energy corresponding to friction, balance of linear momentum still works for frictional systems.
- In most practical applications, we only consider the extremum condition $\frac{d\Pi}{dx} = 0$, as the solutions we obtain are typically a local minimum $\frac{d^2\Pi}{dx^2} > 0$.

$\Pi(x) = mgy(x)$ weight

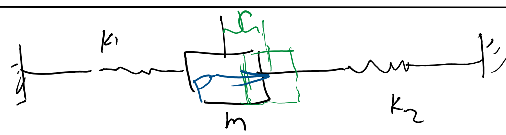
Balance of forces is when

$\frac{d\Pi(x)}{dx} = 0$ Equilibrium

+

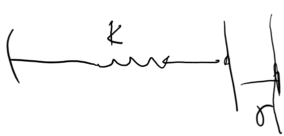
$\frac{d^2\Pi}{dx^2} > 0$ stable equilibrium

Ex. 1



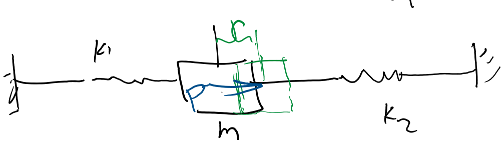
$\sum F_x = 0$ $\longrightarrow$ $r = \frac{P}{k_1+k_2}$ Force approach "balance law"

Energy



spring stretched by r

$V(r) = \frac{1}{2}kr^2$



$V(r) = \frac{1}{2}k_1 r^2 + \frac{1}{2}k_2(-r)^2$

$= \frac{1}{2}k_1 r^2 + \frac{1}{2}k_2 r^2$

$\Pi = \underbrace{V}_{\text{internal energy}} - \underbrace{W}_{\text{external work}}$, $W = Pr$

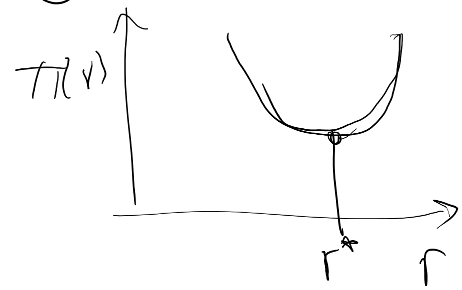
$\Pi(r) = \frac{1}{2}(k_1+k_2)r^2 - Pr$

equilibrium: $\frac{d\Pi}{dr} = (k_1+k_2)r - P = 0$ $r^* = \frac{P}{}$

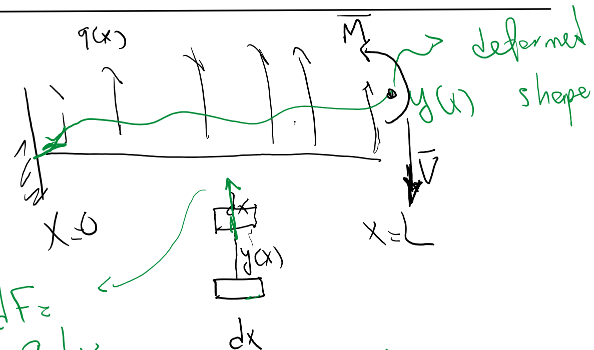
equilibrium: $\dfrac{d\Pi}{dr} = (k_1+k_2)r - P = 0$ $\qquad r^* = \dfrac{P}{k_1+k_2}$

$\hookrightarrow$ equil r

stable or not? $\qquad \dfrac{d^2\Pi}{dr^2} = k_1 + k_2 > 0$



Pot. Energy for a Beam



$dF = q\,dx$

$dW_r = (dF)\,y = (q\,dx)\,y$

$$\Pi = V - W$$

$$V = \int_0^L \frac{1}{2} EI y''^2 \, dx$$

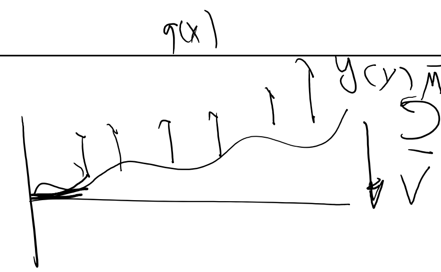
$$W = \underbrace{W_r}_{\text{source term}} + \underbrace{W_f}_{\text{natural boundary}}$$

$$W_r = \int_0^L dW_r = \int_0^L (q\,dx)\,y = \int_0^L q y \, dx$$

$$W_f = -\bar{V} y(L) + \bar{M} y'(L)$$

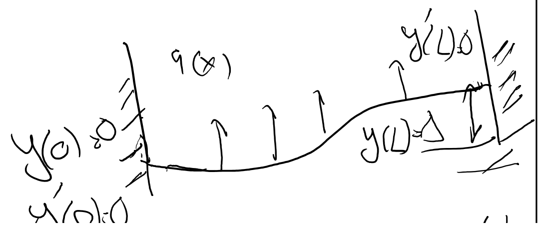
Prob 1

Beam $\quad \Pi(y) = \int_0^L \frac{1}{2} EI y''^2 \, dx - \left(-\bar{V} y(L) + \bar{M} y'(L)\right) - \int_0^L y q \, dx$



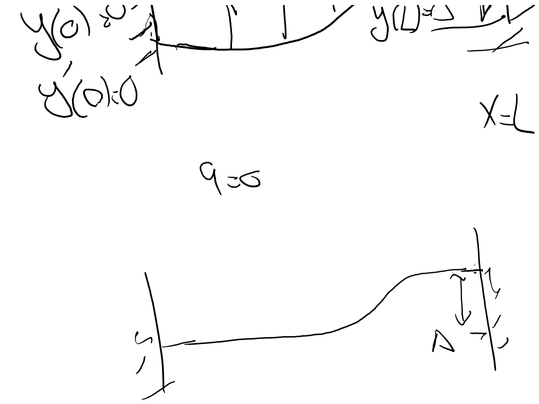
Prob 2a

$$\Pi(y) = V - W = \int_0^L \frac{1}{2} EI y''^2 \, dx - \int_0^L y q \, dx$$



Let $q = 0$

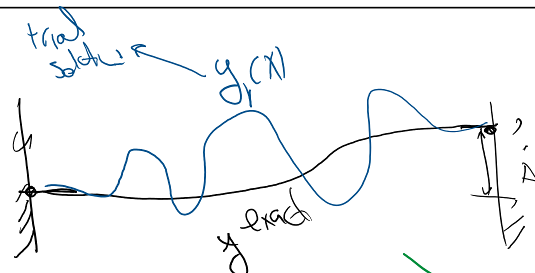
$$\Pi(y) = V - W = \int_0^L \frac{1}{2} EI y''^2 \, dx$$



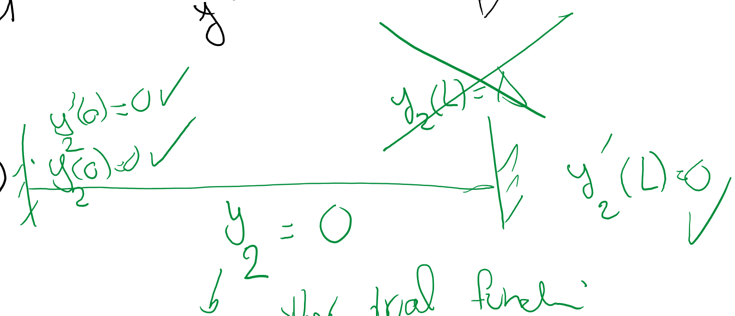
$$\Pi(y^{exact}) \le \Pi(y_1)$$

Exact solution

Any possible solution



$$\Pi(y^{exact}) = \int_0^L \frac{1}{2} EI y''^2_{exact} \, dx > 0$$



$$\Pi(y_2) = \int_0^L \frac{1}{2} EI y_2''^2 \, dx = 0$$

$$\Pi(y^{exact}) > \Pi(y_2)$$

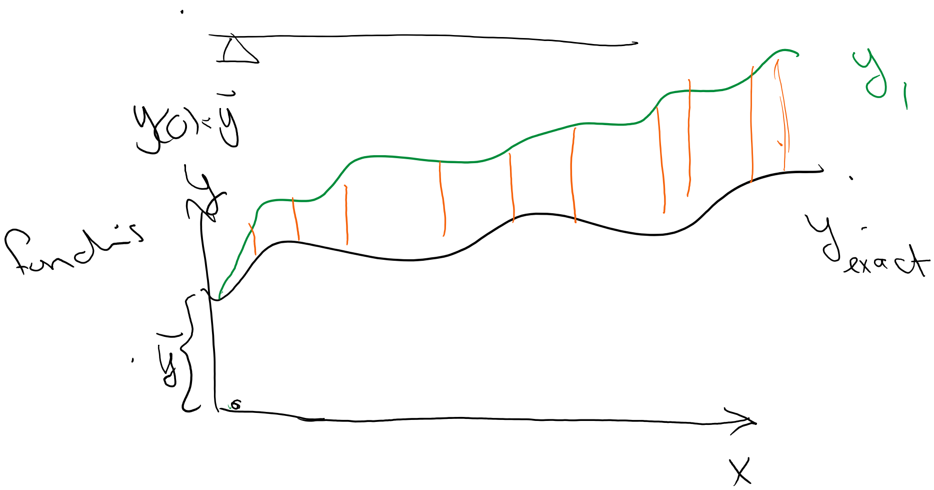
y2 is not a valid trial function because it does not satisfy ALL ESSENTIAL BCs.

A trial function, is a function that satisfies all essential BCs

y is a trial function

y_{exact} " exact solution

$$\Pi(y_{exact}) \leq \Pi(y)$$



Valid trial function y_1

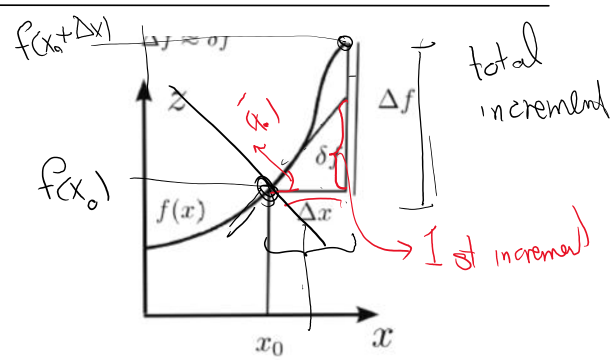
$$\delta y_1 = y_1 - y_{exact}$$

$$\delta y_1(x=0) = y_1(0) - y_{exact}(0) = \bar{y} - \bar{y} = 0$$

Trial function satisfies the essential BC
Increment of trial function satisfies the homogeneous (that is zero value) essential BC

$$f(x_0+\Delta x) = f(x_0) + \Delta x f'(x_0) + \frac{1}{2}\Delta x^2 f''(x_0) + \dots$$

$$+ \frac{1}{n!}\Delta x^n f^{(n)}(x_0) \Longrightarrow$$



$$\underbrace{\Delta f}_{\text{total increment}} = f(x_0+\Delta x) - f(x_0) = \underbrace{\Delta x f'(x_0)}_{\delta f} + \underbrace{\frac{1}{2}\Delta x^2 f''(x_0)}_{\delta^2 f} + \dots + \underbrace{\frac{1}{n!}\Delta x^n f^{(n)}(x_0)}$$

total increment

δf — 1st increment

$\delta^2 f$ — Second increment

n'th increment

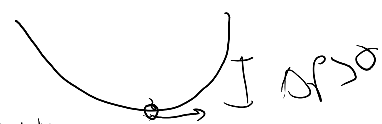
For minimum condition or max we have $\delta f = 0 \iff f'(x_0) = 0$

now ($\delta f = 0 \iff f'(x_0) = 0$) we have

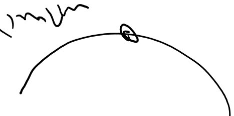
$$\Delta f = \delta^2 f + \delta^3 f + \dots$$

$$= \frac{1}{2} f''(x_0)\,\Delta x^2 + \frac{1}{3!} f'''(x_0)\,\Delta x^3$$

$$\delta^2 f > 0 \equiv f''(x_0) > 0$$

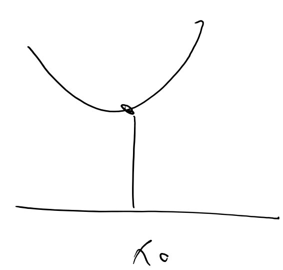


$$\delta^2 f < 0 \equiv f''(x_0) < 0$$



Minimum condition for a function

$$\left.\begin{array}{l} f'(x_0) = 0 \\ f''(x_0) > 0 \end{array}\right. \iff \left.\begin{array}{l} \delta f = 0 \\ \underbrace{\delta^2 f > 0}_{\text{this is the language we have for function}} \end{array}\right.$$

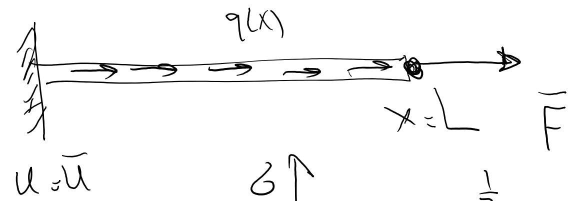


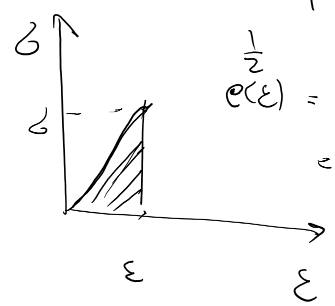
Example of increment calculation for functionals

solution $u(x)$

$$\Pi = V - W$$

$$V = \int_{D} \underbrace{e(\varepsilon)}_{\text{strain energy}} \, dV$$





$$e(\varepsilon) = \tfrac{1}{2}\sigma\varepsilon = \tfrac{1}{2}(E\varepsilon)\varepsilon = \tfrac{1}{2}E\varepsilon^2$$

$$V = \int_0^L \tfrac{1}{2}E\varepsilon^2 \, (\overbrace{A\,dx}^{dx}) = \int_0^L \tfrac{1}{2} EA \, \overbrace{\varepsilon^2}^{u'} \, dx$$

$$W = \underbrace{W_r}_{\text{or I note it as } W_b \text{ in the notes}} + W_f = \int q u \, dx + u(L)\bar{F}$$

$$\boxed{\Pi(u) = \frac{1}{2}\int_0^L E A u'^2 \, dx - \int_0^L q u \, dx - u(L)\bar{F}}$$