FEM20240923

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Energy Method for Solid Mechanics

The total energy in solid mechanics is,

$$\Pi = (V - W) - T = \text{Total energy} \qquad (85a)$$

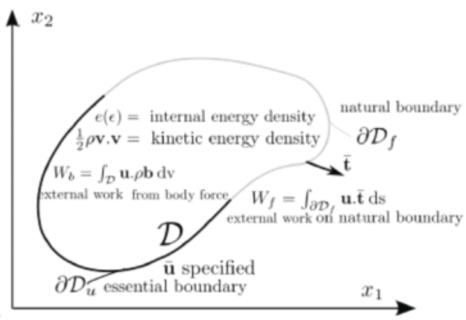
$$T = \int_{\mathcal{D}} \frac{1}{2}\rho \mathbf{v}.\mathbf{v}\, \mathrm{dv} = \text{Kinetic energy} \qquad (85b)$$

$$V = \int_{\mathcal{D}} e(\epsilon)\, \mathrm{dv} = \text{Internal energy} \qquad (85c)$$

$$W = W_b + W_f = \text{External work} \qquad (85d)$$

$$W_b = \int_{\mathcal{D}} \mathbf{u}.\rho\mathbf{b}\, \mathrm{dv} \qquad (85e)$$

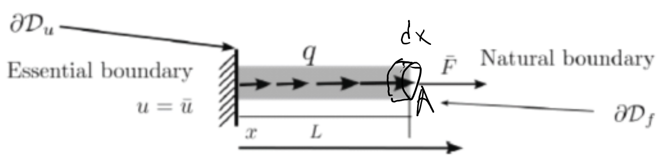
$$W_f = \int_{\partial\mathcal{D}_f} \mathbf{u}.\bar{\mathbf{t}}\, \mathrm{ds} \qquad (85f)$$

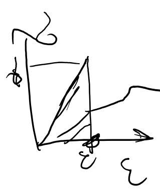


- For static problems $T = 0$.
- Internal energy density, $e(\epsilon) = \frac{1}{2}\epsilon : \sigma(\epsilon) = \frac{1}{2}C_{ijkl}\epsilon_{ij}\epsilon_{kl}$ for linear solid.
- Natural boundary forces are naturally incorporated into the energy (W_f).
- Essential boundary conditions are incorporated into function space:

$$\mathbf{u} \in \mathcal{V} = \{\mathbf{v} \mid \mathbf{v} \in C^1(\mathcal{D}) : \forall \mathbf{x} \in \partial\mathcal{D}_u \ \mathbf{v}(\mathbf{x}) = \bar{\mathbf{u}}(\mathbf{x})\}, \text{ is a solution if}$$
$$\forall \tilde{\mathbf{u}} \in \mathcal{V}, \quad \Pi(\mathbf{u}) \le \Pi(\tilde{\mathbf{u}}). \qquad (86)$$

Energy Method for 1D solid bar (statics)





$$T = 0 \text{ statics} \qquad (87a)$$

$$V = \int_0^L \frac{1}{2}\epsilon\sigma\, \mathrm{dv} = \int_0^L \frac{1}{2}\epsilon\overset{\sigma}{(E\epsilon)}\, \underbrace{(A\mathrm{d}x)}_{dA} = \int_0^L \frac{1}{2}EA\left(\overset{\varepsilon}{\frac{\mathrm{d}u}{\mathrm{d}x}}\right)^2 \mathrm{d}x \qquad (87b)$$

$$W_b = \int_0^L u(x)q(x)\, \mathrm{d}x \qquad (87c)$$

$$W_f = u(L)\bar{F} \qquad (87d)$$

where A is the cross section area and $q(x)$ is distributed load. Thus the total energy is,

$$\boxed{\Pi(u) = \int_0^L \frac{1}{2}EA\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 \mathrm{d}x - \int_0^L u(x)q(x)\, \mathrm{d}x - u(L)\bar{F}} \qquad (88)$$

Since $\partial\mathcal{D}_u = \{0\}$ with the essential boundary condition $u(0) = \bar{u}$, the energy statement is,

$$\text{Find } u \in \mathcal{V} = \{v \mid v \in C^1(\mathcal{D}) : v(0) = \bar{u}\}, \text{ such that}$$
$$\forall \tilde{u} \in \mathcal{V}, \quad \Pi(u) \le \Pi(\tilde{u}). \qquad (89)$$

We want to find the solution that ① minimizes (88)
satisfies essential BC

$$f(x_0 + \Delta x) = f(x_0) + \Delta x f'(x_0) + \frac{\Delta x^2}{2} f''(x_0) + \cdots$$

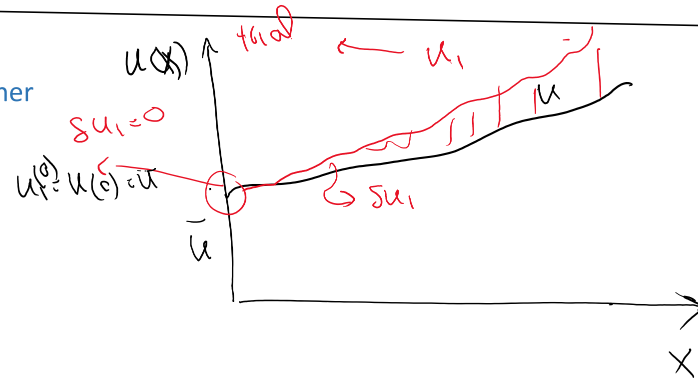
$$f(x_0+\Delta x) = f(x_0) + \Delta x f'(x_0) + \frac{\Delta x^2}{2} f''(x_0) + \cdots$$

Example

$f(x) = x^3$

$$f(x_0+\Delta x) = (x_0+\Delta x)^3 = \underbrace{x_0^3}_{f(x_0)} + \underbrace{\Delta x\underbrace{(3x_0^2)}_{f'(x_0)}}_{\delta^1 f} + \underbrace{\frac{\Delta x^2}{2}\underbrace{(6x_0)}_{f''(x_0)}}_{\delta^2 f} + \underbrace{\frac{(\Delta x)^3}{6}\underbrace{6}_{f'''(x_0)}}_{\delta^3 f}$$

I'm going to expand the energy functional in a similar manner to see what the first and second increments are:



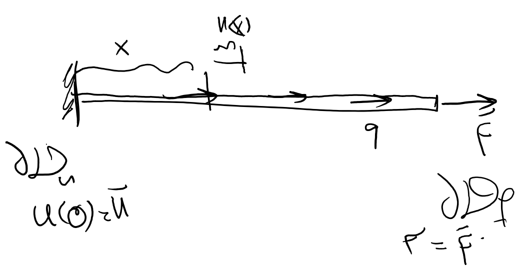
u_1 is a valid trial function

$u_1(0) = \bar{u}$

u is the exact solution

$$\Pi(u) \leq \Pi(u_1) = \Pi(u+\delta u_1)$$

$$\Pi(\tilde{u}) = \int_0^L \frac{1}{2} EA(\tilde{u}')^2 dx - \int_0^L \tilde{u} q\, dx - \tilde{u}(L)\bar{F}$$



$\tilde{u}$: any trial function

for $\tilde{u} = u_1 = u + \delta u_1$ we have

$$\Pi(u_1) = \int_0^L \frac{1}{2} EA\{(u+\delta u_1)'\}^2 dx - \int_0^L (u+\delta u_1) q\, dx - (u+\delta u_1)(x=L)\bar{F}$$

$$= \int_0^L \frac{1}{2} EA(u'+\delta u_1')^2 dx - \int_0^L uq\, dx - \int \delta u_1 q\, dx - u(L)\bar{F} - \delta u_1(L)\bar{F}$$

$$= \int_0^L \frac{1}{2} EA(\underline{u'^2} + \underline{2u'\delta u_1'} + \underline{\delta u_1'^2}) dx - \underline{\int_0^L uq\, dx} - \underline{u(L)\bar{F}}$$

$$- \underline{\int \delta u_1 q\, dx} - \underline{\delta u_1(L)\bar{F}} =$$

-0

$$-\int \delta u_1 q\,dx - \delta u_1(L)F =$$

$$\Rightarrow \Pi(u_1) = \Pi(u+\delta u_1) = \overbrace{\int_0^L \frac{1}{2} EAu'^2 dx - \int_0^L uq\,dx - u(L)\bar{F}}^{\Pi(u)}$$

$$\underbrace{+\int_0^L \delta u_1' EAu'\,dx - \int_0^L \delta u_1 q\,dx - \delta u_1(L)\bar{F}}_{\delta\Pi}$$

$$\underbrace{+\int_0^L \frac{1}{2} EA\,\delta u_1'^2\,dx}_{\delta\Pi^2}$$

Example

$f(x) = x^3$

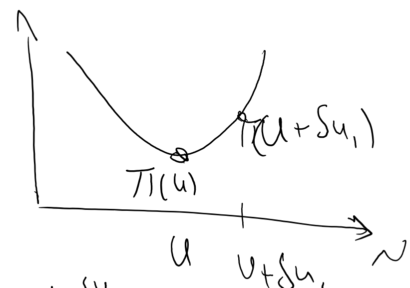
$$f(x_0+\Delta x) = (x_0+\Delta x)^3 = \underbrace{x_0^3}_{f(x_0)} + \underbrace{\Delta x \underbrace{(3x_0^2)}_{f'(x_0)}}_{\delta' f} + \underbrace{\frac{\Delta x^2}{2}\underbrace{(6x_0)}_{f''(x_0)}}_{\delta^2 f} + \underbrace{\frac{(\Delta x)^3}{6}\underbrace{6}_{f'''(x_0)}}_{\delta^3 f}$$

$$\Pi(u+\delta u_1) - \Pi(u) = \delta\Pi + \delta^2\Pi \geq 0$$

$$\delta\Pi = \int_0^L u' EA\,\delta u_1' - \int_0^L \delta u_1 q\,dx - \delta u_1(L)\bar{F}$$

$-\delta u_1'$ $\quad -\delta u_1$ $\quad -\delta u_1(L)$

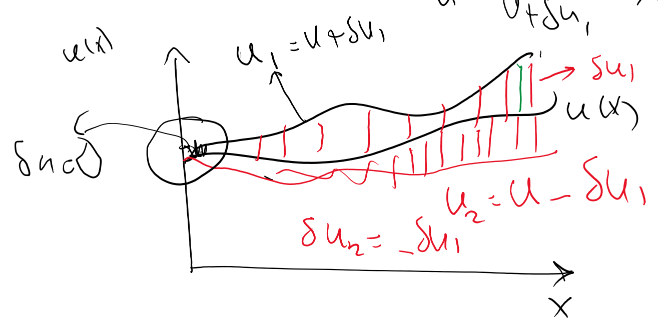
$$\delta^2\Pi = \int_0^L \frac{1}{2} EA(\delta u')^2\,dx$$



For minimum condition we must have:

$\delta\Pi = 0$

$\delta^2\Pi \geq 0$ already ≥ 0



$\delta\Pi = 0$ is

$$\delta\Pi = \int_0^L \delta u' EAu'\,dx - \int_0^L \delta u q - \delta u(L)\bar{F} = 0$$

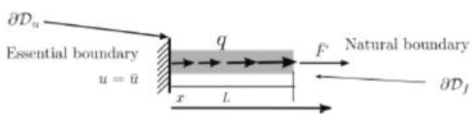
OR for all δu

$$\int_0^L \underbrace{\delta u'}_{w'} EA u' \, dx = \int_0^L \underbrace{\delta u}_{w} q + \underbrace{\delta u(L)}_{w(L)} \bar{F}$$

this is the weak statement

Conditions on $\delta u \in \mathcal{W} = \{ f \in C^1(0,L) \mid f(0) = 0 \}$

Example: 1D solid bar



Based on the condition $\delta \Pi = 0$, u the solution from the energy method is,

Find $u \in \mathcal{V} = \{ v \in C^1([0,L]) \mid v(0) = \bar{u} \}$, such that,
$\forall \delta u \in \mathcal{W} = \{ v \in C^1([0,L]) \mid v(0) = 0 \}$

$$\delta \Pi = \int_0^L \delta u'(x) EA u'(x) \, dx - \int_0^L \delta u(x) q(x) \, dx - \delta u(L) \bar{F} = 0 \tag{98}$$

- What would be the weak form for this problem based on the balance law/Differential Equation approach (*cf.* (73))?

Find $u \in \mathcal{V} = \{ v \in C^1([0,L]) \mid v(0) = \bar{u} \}$, such that,
$\forall w \in \mathcal{W} = \{ v \in C^1([0,L]) \mid v(0) = 0 \}$

$$\delta \Pi = \int_0^L w'(x) EA u'(x) \, dx - \int_0^L w(x) q(x) \, dx - w(L) \bar{F} = 0 \tag{99}$$

- What are the differences between this weak statement and the optimality condition obtained from energy method?

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Balance law

↓

PDE / DE

↓

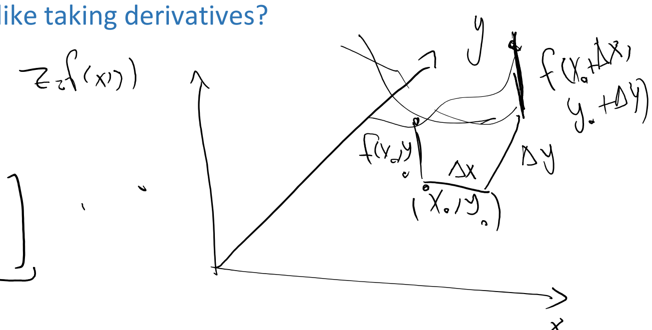
WRS

↓

Weak statement ← Energy method

Is there any easier way to calculate the first and second increments, like taking derivatives?

$$f(x_0+\Delta x, y_0+\Delta y) = f(x_0,y_0) + \underbrace{\left[\frac{\partial f}{\partial x}(x_0,y_0) \Delta x + \frac{\partial f}{\partial y} \Delta y \right]}_{\delta f} + \underbrace{\left[\frac{1}{2} \frac{\partial^2 f}{\partial x^2} \Delta x^2 + \frac{\partial^2 f}{\partial x \partial y} \Delta x \Delta y + \frac{\partial^2 f}{\partial y^2} \Delta y^2 \right]}_{\delta^2 \Pi} + \cdots$$

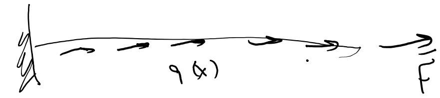


$$\delta \pi$$

$$\delta f(x,y) = \frac{\partial f}{\partial x}\underset{\delta}{\Delta} x + \frac{\partial f}{\partial y}\underset{\delta}{\Delta} y$$

Calculating functional increments is very similar

$$\Pi(\underbrace{u}_{\text{eg } x \text{ above}}, \underbrace{u'}_{\text{eg } y \text{ above}}) = \int_0^L \left(\tfrac{1}{2} EA u'^2\right) dx - \int_0^L q u\, dx - u(L)\bar{F} \qquad @x=0 \; u=\bar{u}$$



$$\delta \Pi(u,u') = \int_0^L \left(\cancel{\frac{\partial \frac{1}{2} EA u'^2}{\partial u}}\,\delta u + \frac{\partial \frac{1}{2} EA u'^2}{\partial u'}\,\delta(u') \right) dx \qquad \frac{\partial \frac{1}{2} EA y^2}{\partial y} = EAy$$

$$- \int \left(\frac{\partial q u}{\partial u}\,\delta u + \cancel{\frac{\partial q u}{\partial u'}}\,\delta(u') \right) dx$$

$$- \underbrace{\frac{\partial u}{\partial u}}_{1}(L)\,\delta u(L)\bar{F} - \underbrace{\cancel{\frac{\partial u}{\partial u'}}}_{0}(L)\,\delta(u'(L))\,\bar{F}$$

$$\delta \Pi = \int_0^L (EAu')\,\delta(u')\,dx - \int_0^L q\,\delta u\,dx - \delta u(L)\bar{F}$$

Does it match $\delta \pi$ below

$$\Pi(u+\delta u_1) - \Pi(u) = \delta \Pi + \delta^2 \Pi \geq 0$$

$$\delta \Pi = \int_0^L u' EA\,\delta u_1' - \int_0^L \delta u_1\, q\,dx - \delta u_1(L)\bar{F}$$

$$\delta^2 \Pi = \int_0^L \tfrac{1}{2} EA(\delta u')^2\,dx$$

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energy statement

$$\Pi = \int^L \tfrac{1}{2} EA u'^2 dx - \int^L u\, q\, dx - u(L)\bar{F}$$

energy statement

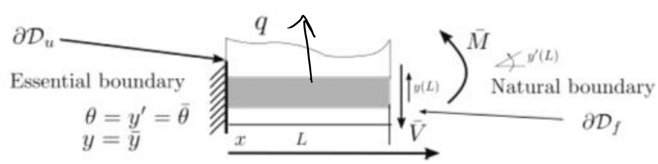
$$\Pi(u,u') = \int_0^L \underbrace{\tfrac{1}{2}EAu'^2}\,dx - \int_0^L u q\,dx - u(L)\bar{F}$$

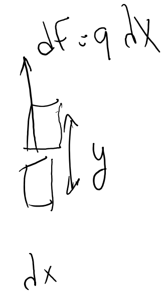
$$\delta\Pi = \int_0^L \frac{\partial \tfrac{1}{2}EAu'^2}{\partial u'}\,\delta(u')\,dx - \int_0^L \frac{\partial uq}{\partial u}\,\delta u\,dx - \delta u(L)\bar{F}$$

$$= \int (EAu')\,\delta(u')\,dx - \int q\,\delta u\,dx - \delta u(L)F$$

weak statement

Example: Euler Bernoulli beam





We determined the internal energy of the beam to be (*cf.* (85c)),

$$V = \int_{\mathcal{D}} \frac{1}{2}\epsilon\sigma\,\mathrm{dv} = \int_0^L \left(\int_A \frac{1}{2}\epsilon^2 E\,\mathrm{d}A\right)\mathrm{d}x = \int_0^L \left(\int_A \frac{1}{2}\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}z\right)^2 E\,\mathrm{d}A\right)\mathrm{d}x$$
$$= \int_0^L \frac{1}{2}E\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)^2 \underbrace{\left(\int_A z^2\,\mathrm{d}A\right)}_{I}\mathrm{d}x \Rightarrow$$

$$\boxed{V = \int_0^L \frac{1}{2}EI\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)^2\mathrm{d}x} \tag{100}$$

The external works are:

$$W_b = \int_0^L y(x)\overbrace{q(x)\,\mathrm{d}x}^{df} \tag{101a}$$

$$W_f = y(L)(-\bar{V}) + \frac{\mathrm{d}y}{\mathrm{d}x}(L)\bar{M} = -y(L)\bar{V} + \frac{\mathrm{d}y}{\mathrm{d}x}(L)\bar{M} \tag{101b}$$

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$$\Pi(y,y',y'') = V - W = \int_0^L \tfrac{1}{2}EI\,y''^2\,dx - \int_0^L y q\,dx + y(L)\bar{V} - y'(L)\bar{M}$$

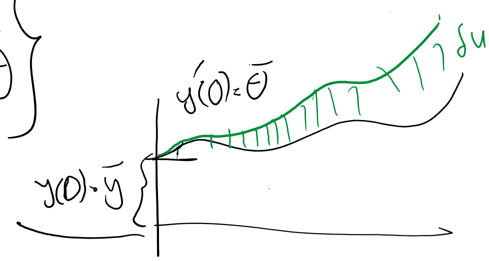
$$\delta\Pi = \int_0^L \frac{\partial \tfrac{1}{2}EI y''^2}{\partial y''}\,\underbrace{\delta(y'')}_{(\delta y)''}\,dx - \int_0^L (\delta y)\,q\,dx + \delta y(L)\bar{V} - \delta y'(L)\bar{M} = 0$$

$\int_0 (\delta y)'' \, EI y'' \quad \ldots \quad \delta y(L) \bar{M} = 0$

$$\int_0^L (\delta y)'' EI y'' \, dx = \int_0^L \delta y \, q \, dx - \delta y(L) \bar{V} + \delta y'(L) \bar{M}$$

$$\delta y \in W = \{ f \in C^2(0,L) \mid f(0) = 0, f'(0) = 0 \}$$

$$y \in V = \{ f \in C^2(0,L) \mid f(0) = \bar{y}, f'(0) = \bar{\theta} \}$$

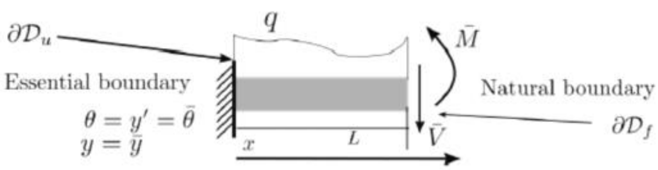


Other approach:
Balance law -> Strong form (DEs)

Des -> WRS

Weighted residual statement to Weak statement

To demonstrate the process of deriving the weak statement from the weighted residual statement consider the following problem:



The residuals for this problem are:

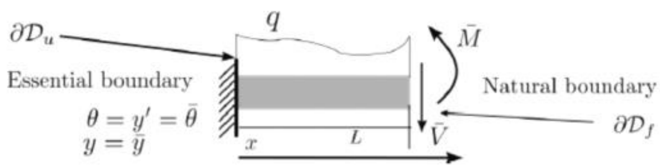
$$\begin{aligned}
\mathcal{R}_i &= \frac{d^2}{dx^2}\left(EI\frac{d^2 y}{dx^2}\right) - q && \text{Interior residual for} && \mathcal{D} = [0, L] \\
\mathcal{R}_f &= \begin{bmatrix} \bar{M} - M \\ \bar{V} - V \end{bmatrix} && \text{Natural BC residual for} && \partial\mathcal{D}_f = \{L\} \\
\mathcal{R}_u &= \begin{bmatrix} \bar{\theta} - \theta \\ \bar{y} - y \end{bmatrix} && \text{Essential BC residual for} && \partial\mathcal{D}_u = \{0\}
\end{aligned} \tag{53}$$

As mentioned previously, we want to drop the weighted residual term for essential boundary condition (why?). Accordingly, we need to strongly enforce the essential boundary condition (This is why this is called "essential" boundary condition). That is, we require:

$$\mathcal{R}_u = \begin{bmatrix} \bar{\theta} - \theta \\ \bar{y} - y \end{bmatrix} = 0 \quad \text{at } x = 0 \ (\partial\mathcal{D}_u). \tag{54}$$

Weighted residual statement to Weak statement

Weighted residual statement to Weak statement



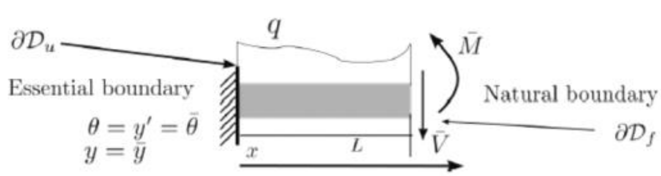
Since we strongly enforce the essential boundary condition, the weighted residual for this problem simplifies to:

$$\begin{aligned} 0 &= \int_{\mathcal{D}} w\mathcal{R}_i(y)\,\mathrm{d}v + \int_{\partial\mathcal{D}_f} \mathbf{w}_f\mathcal{R}_f(y)\,\mathrm{d}s \\ &= \int_0^L w\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)-q\right)\,\mathrm{d}x + \left[\begin{array}{c} -\frac{\mathrm{d}w}{\mathrm{d}x} \\ w \end{array}\right]\cdot\left[\begin{array}{c} \bar{M}-M \\ \bar{V}-V \end{array}\right]|_{x=L} \\ &= \int_0^L w\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)-q\right)\,\mathrm{d}x - \frac{\mathrm{d}w}{\mathrm{d}x}(\bar{M}-M(y))|_{x=L} + w(\bar{V}-V(y))|_{x=L} \end{aligned} \tag{55}$$

Next, we transfer derivatives from y to w (trial function to weight function). We note that

$$\begin{aligned} &\int_0^L w\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)\,\mathrm{d}x = \int_0^L \left[-\frac{\mathrm{d}w}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}EI\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)\right]\,\mathrm{d}x + \left[w\frac{\mathrm{d}}{\mathrm{d}x}\left(EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)\right]|_{x=0}^{x=L} \\ &= \int_0^L \left[\frac{\mathrm{d}^2w}{\mathrm{d}x^2}EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right]\,\mathrm{d}x + [wV(y)]|_{x=0}^{x=L} - \left[\frac{\mathrm{d}w}{\mathrm{d}x}\left(EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)\right]|_{x=0}^{x=L} \end{aligned} \tag{56}$$

Weighted residual statement to Weak statement



Plugging (55) in (56) yields,

$$\begin{aligned} 0 &= \int_0^L w\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)-q\right)\,\mathrm{d}x - \frac{\mathrm{d}w}{\mathrm{d}x}(\bar{M}-M(y))|_{x=L} + w(\bar{V}-V(y))|_{x=L} \\ &= \left\{\int_0^L \left[\frac{\mathrm{d}^2w}{\mathrm{d}x^2}EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - wq\right]\,\mathrm{d}x + \left[wV(y)] - \frac{\mathrm{d}w}{\mathrm{d}x}M(y)\right]|_{x=0}^{x=L}\right\} \\ &- \frac{\mathrm{d}w}{\mathrm{d}x}(\bar{M}-M(y))|_{x=L} + w(\bar{V}-V(y))|_{x=L} \\ &= \int_0^L \left[\frac{\mathrm{d}^2w}{\mathrm{d}x^2}EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - wq\right]\,\mathrm{d}x \\ &+ \left\{wV(y) - \frac{\mathrm{d}w}{\mathrm{d}x}M(y) - \frac{\mathrm{d}w}{\mathrm{d}x}(\bar{M}-M(y)) + w(\bar{V}-V(y))\right\}_{x=L} \\ &- \left\{wV(y) - \frac{\mathrm{d}w}{\mathrm{d}x}M(y)\right\}_{x=0} \end{aligned} \tag{57}$$

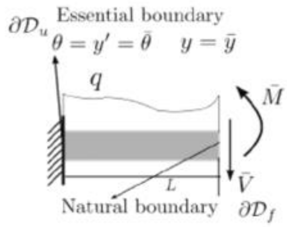
Weighted residual statement to Weak statement

This equation simplifies to

$$0 = \int_0^L \left[\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} EI \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - wq\right] \mathrm{d}x + \left\{-\frac{\mathrm{d}w}{\mathrm{d}x}\bar{M} + w\bar{V}\right\}_{x=L} \quad \text{(58a)}$$

$$+ \left\{w\left(V(y) - V(y)\right) - \frac{\mathrm{d}w}{\mathrm{d}x}\left(M(y) - M(y)\right)\right\}_{x=L} \quad \text{(58b)}$$

$$- \left\{wV(y) - \frac{\mathrm{d}w}{\mathrm{d}x}M(y)\right\}_{x=0} \quad \text{(58c)}$$



Noting that $M(y) = EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ and $V(y) = \frac{\mathrm{d}}{\mathrm{d}x}\left(EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)$ (second and third order derivatives in x):

- What is the maximum derivative order for y in the interior of the beam (58a)? 2.
- What is the maximum derivative order for w in the interior of the beam (58a)? 2.
- Are the remaining terms at $x = L$ (natural boundary) identically zero (58b)? Yes.
- What is the maximum derivative order for y at $x = 0$ (Essential boundary) (58c)? 3.
- What is the maximum derivative order for w at $x = 0$ (Essential boundary) (58c)? 1.

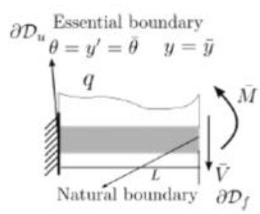
Summary

For the differential equation of order $M = 4$ ($m = 2$) we have been able to equally distribute differential orders between trial function (y) and weight function (w) (order $= m = 2$). The only term that violates this is the essential boundary where y has higher order derivative. We fix this problem by requiring w and $\frac{\mathrm{d}w}{\mathrm{d}x}$ to be identically zero at $x = 0$.

Essential boundary condition

We mentioned that the essential boundary condition is strongly enforced (That is, it is an "essential" condition). The essential conditions (54) require,

$$\mathcal{R}_u = \begin{bmatrix} \bar{\theta} - \theta \\ \bar{y} - y \end{bmatrix} = 0 \Rightarrow \boxed{\left\{ \begin{array}{c} \frac{\mathrm{d}y}{\mathrm{d}x} = \bar{\theta} \\ y = \bar{y} \end{array} \right\}, \text{ at } x = 0 \ (\partial\mathcal{D}_u)} \quad \text{(59)}$$



We discussed that to annihilate the high order derivatives of y in (58c):

$$- \left\{wV(y) - \frac{\mathrm{d}w}{\mathrm{d}x}M(y)\right\}_{x=0}$$

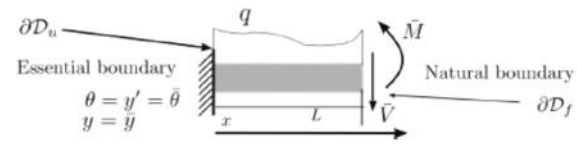
we set the corresponding weight functions identically zero:

$$\left\{ \begin{array}{c} \frac{\mathrm{d}w}{\mathrm{d}x} = 0 \\ w = 0 \end{array} \right\}, \quad \text{at } x = 0 \ (\partial\mathcal{D}_u) \quad \text{(60)}$$

Summary

1. Trial, y, (solution) functions exactly satisfy all essential boundary conditions.
2. Weight, w, functions exactly satisfy the homogeneous essential boundary conditions.
3. If both conditions are satisfied we can form a *weak statement* that requires *only half* the highest derivative order. In fact, this enlarged space of functions is the same as the space of the original balance law.

Weak Statement (WS)



The weak statement for the Euler Bernoulli problem and the BCs in the figure are:

$$\text{Find } y \in \mathcal{V} = \{u \in C^2(\mathcal{D}) \mid u(0) = \bar{y},\ \frac{\mathrm{d}u}{\mathrm{d}x}(0) = \bar{\theta}\}, \text{ such that,} \tag{62a}$$

$$\forall w \in \mathcal{W} = \{u \in C^2(\mathcal{D}) \mid u(0) = 0,\ \frac{\mathrm{d}u}{\mathrm{d}x}(0) = 0\} \tag{62b}$$

$$0 = \int_0^L \left[\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} EI \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - wq\right] \mathrm{d}x + \left\{-\frac{\mathrm{d}w}{\mathrm{d}x}\bar{M} + w\bar{V}\right\}_{x=L} \tag{62c}$$

$$\delta(u') = (\delta u)'$$

$$u_1 = u + \underbrace{\delta u}_{\text{increment of } u}$$

$$u_1' = u' + (\delta u)' \Longrightarrow$$

$$(\delta u)' = \underbrace{u_1' - u'}_{\text{diff of } (\ )'} = \delta(u')$$

$$\delta u'$$