FEM20240925

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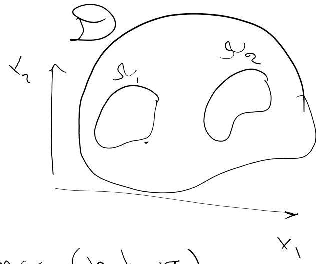
Two approaches: example problem -> elastostatics

(A) Balance law



(1) $\forall \Omega$ $\displaystyle\int_{\partial\Omega} \sigma \cdot n \, ds + \int_{\Omega} \rho b \, dV = 0$

→ we only need stress (not $\nabla\sigma$)

(2) Strong form

Div $\displaystyle\int_{\Omega} \nabla\cdot\sigma \, dV + \int_{\Omega} \rho b \, dV = 0$

$\nabla\cdot\sigma$ must exist & be continuous $\boxed{\sigma \in C^1}$

Localize: $\forall \Omega$ $\displaystyle\int (\nabla\cdot\sigma + \rho b) \, dV = 0$ $\Longrightarrow$ PDE $\nabla\cdot\sigma + \rho b = 0$

& must be continuous

+ Essential & Natural BC's

(3) WRS

$R_i = \nabla\cdot\sigma + \rho b$

$R_f = \bar{t} - \sigma\cdot n$

Find $u \in V = \{ f \in C^2 \mid \forall x \in \partial D_u \;\; u(x) = \bar{u}(x) \}$

$\ni \forall w \in W = \{ f \in C^0 \mid \_\_\_\_ \}$

$$\int_{D} w \underbrace{(\nabla\cdot\sigma + \rho b)}_{R_i} dV + \int_{\partial D_f} w \underbrace{(\bar{t} - \sigma\cdot n)}_{R_f} ds = 0$$

$\sigma = C\nabla u$

σ must be C^1

(4) Weak statement

$\int w \, \nabla\cdot\sigma$

$u \in V = \{ f \in C^1(D) \mid \forall x \in \partial D_u \;\; u(x) = \bar{u}(x) \}$

$w \in W = \{ f \in C^1(D) \mid \forall x \in \partial D_u \;\; u(x) = 0 \}$

$$\int_{D} \underbrace{\nabla w}_{\text{or } \varepsilon(w)} : \underbrace{\sigma(u)}_{C\varepsilon(u) = C\nabla u} dV = \int_{\partial D_f} w \bar{t} + \int_{D} w \rho b$$

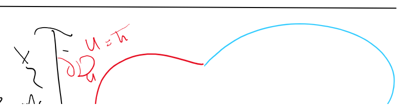
Eventually like the balance law in the weak statement we don't have the derivative of sigma

Catch: We can solve the weak statement with FEM (WRS doesn't work with FEM)

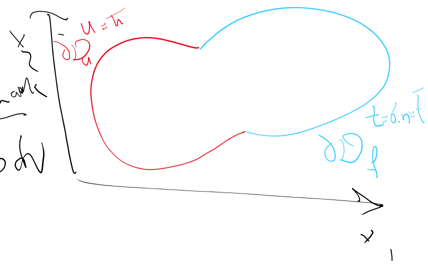
(B) Energy

Π

W_f → natural BC side



(B) Energy



$$\Pi = V - W$$

W_f → natural BC role

source term role

$$= \int_D e(\varepsilon)\, dV - \int_{\partial D_f} u \bar{t}\, ds - \int_D u \rho b\, dV$$

$$= \int_D \frac{1}{2} \varepsilon(u) : \sigma(u) \qquad //$$

$$\delta\left(\frac{1}{2} c x^2\right) = \delta x\, c\, x, \quad x = \nabla u$$

$$\Pi = -\int_D \frac{1}{2} \nabla u\, \underbrace{C \nabla u}_{\sigma}\, dV - \int_{\partial D_f} u \bar{t}\, ds - \int_D u \rho b\, dV$$

$$\delta \Pi = \int_D \nabla \delta u : \underbrace{C \nabla u}_{\sigma(u)}\, dV - \int_{\partial D_f} \delta u\, \bar{t}\, ds - \int_D \delta u\, \rho b\, dV$$

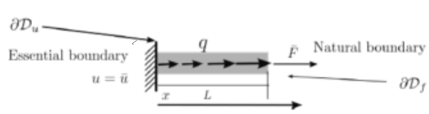
$$\Longrightarrow \int_D \varepsilon(\delta u) : \sigma(u)\, dV = \int_{\partial D_f} \delta u\, \bar{t}\, ds + \int_D \delta u\, \rho b\, dV$$

$$u \in \mathcal{V} = \{ f \in C^1(D) \mid \forall x \in \partial D_u\ u(x) = \bar{u}(x) \}$$

$$\delta u = w \in W = \{ f \in C^1(D) \mid \forall x \in \partial D_u\ u(0) = 0 \}$$

Energy method directly gives the Weak statement and in no step we dealt with a derivative of stress (spatial flux in general)

Discretization



Recall (1) W S: $\forall w \quad \int_0^L w\, R_i(u)\, dx + w_f R_f(u) \Big|_{x=L} = 0$

in least square the weight on R_f is different from weight on R_i

$$u^h = \phi_p(x) + \sum_{i=1}^{n} a_i \phi_i(x)$$

(2) $u^h = \phi_p + [\phi_1, \ldots, \phi_n] \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

$$= \phi_p + \Phi a$$

n unknowns $\Rightarrow$

n equations $\Rightarrow$ n weight

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}, \quad \mathbf{w}_f = \begin{bmatrix} w_{f_1} \\ \vdots \\ w_{f_n} \end{bmatrix}$$

$$R_i(u) = \underbrace{\mathcal{L}_M(u)}_{\text{Differential operator on } u} - \underbrace{r}_{\text{source term}} = \underbrace{(EAu')' + q}$$

$$\mathcal{L}_M(\cdot) = (EA(\cdot)')'$$

$$r = -q$$

Now $\mathcal{L}_M$ is a linear differential operator

I'll solve this problem in class for $n = 2$, $\phi_1 = 1$

$' = -q$

I'll solve this problem in class for $\underline{n=2}$ $\phi_p = 1$

$\phi_1 = x$, $\phi_2 = x^2$

$$u^h = \phi_p + [\phi_1 \cdots \phi_n]\begin{bmatrix} a_1 \\ | \\ | \\ a_n \end{bmatrix} \quad \omega = \begin{bmatrix} \omega_1 \\ | \\ | \\ \omega_n \end{bmatrix} \quad \omega_f = \begin{bmatrix} \omega_{f_1} \\ | \\ | \\ \omega_{f_n} \end{bmatrix}$$

WRS

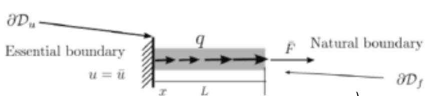
$$\int_0^L \omega \, \underbrace{R_i(u^h)}\, dx + \omega_f \, \underbrace{R_f(u^h)}_{\bar{F}-F}\Big|_{x=0} = 0$$

$$④ \quad \boxed{\int_0^L \begin{bmatrix} \omega_1 \\ \omega_2 \\ | \\ \omega_n \end{bmatrix} \underbrace{\Big(L_M\big(\overbrace{\phi_p + [\phi_1 \cdots \phi_n]\begin{bmatrix} a_1 \\ | \\ a_n \end{bmatrix}}^{u^h \,\phi_p + \sum_{i=1}^n a_i\phi_i}\big)\Big)}_{*} + \begin{bmatrix} \omega_{f_1} \\ \vdots \\ \omega_{f_n} \end{bmatrix}\Big(\bar{F} - \underbrace{L_f\big(\overbrace{\phi_p + [\phi_1 \cdots \phi_n]\begin{bmatrix} a_1 \\ | \\ a_n \end{bmatrix}}^{u^h}\big)}_{F}\Big)\Big|_{x=L} = 0}$$

note:

$$F = \sigma A = E\varepsilon A = (EA)\frac{du}{dx} = L_f(u)$$

$$L_f(\,) = EA(\,)'$$



$$③ \quad \boxed{\begin{aligned} L_M &= (EA(\,)')' \\ L_f &= EA(\,)' \end{aligned}}$$

$$\bar{F} - F =$$

$$\bar{F} - EA(\underline{\phi_p'} + \sum a_i\phi_i')\Big|_{x=L}$$

$$L_M(\overline{\phi_p + \sum a_i\phi_i}^{u^h}) = \Big(EA(\phi_p + \sum a_i\phi_i)'\Big)'$$

$$= \underline{EA(\phi_p + \sum a_i\phi_i)''}$$

assume EA is constant

$$(a_1f_1 - a_2f_2)'' = a_1f_1'' + a_2f_2''$$

for Linear problems this L_M is linear & we can open it:

$$L_M(u^h) = \underline{EA}\phi_p'' + \sum a_i \underline{EA}\phi_i'' = L_M(\phi_p) + \sum_{i=1}^{n} a_i L_M(\phi_i)$$

$$\boxed{L_M(u^h) = L_M(\phi_p) + [L_M(\phi_1) \cdots L_M(\phi_n)]\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}} \quad ⑤$$

plug ⑤ in ④

$$\int^L \begin{bmatrix} \omega_1 \\ \vdots \end{bmatrix} \Big(\Big(\qquad \Big)\begin{bmatrix} a_1 \\ \end{bmatrix}$$

$$\int_0^L \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \left\{ \left(L_M(\phi_p) + [L_M(\phi_1) \cdots L_M(\phi_n)] \right) \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + q(x) \right\} dx$$

unknowns

$$\begin{bmatrix} w_{f_1} \\ \vdots \\ w_{f_n} \end{bmatrix} \left(\bar{F} - \left(L_f(\phi_p) + [L_f(\phi_1) \cdots L_f(\phi_n)] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right) \right) \Bigg|_{x=L} = 0$$

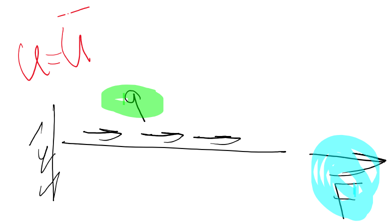
$$\left(\int_0^L \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} [L_M(\phi_1) \cdots L_M(\phi_n)]\, dx \right) \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} - \left(\begin{bmatrix} w_{f_1}(L) \\ \vdots \\ w_{f_n}(L) \end{bmatrix} [L_f(\phi_1) \cdots L_f(\phi_n)](x=L) \right) \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} =$$

$$- \int_0^L \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} q\, dx - \int_0^L \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} L_M(\phi_p)\, dx + \begin{bmatrix} w_{f_1} \\ \vdots \\ w_{f_n} \end{bmatrix}(x=L)\, L_f(\phi_p)(x=L)$$

$$- \bar{F} \begin{bmatrix} w_{f_1} \\ \vdots \\ w_{f_n} \end{bmatrix}(x=L)$$

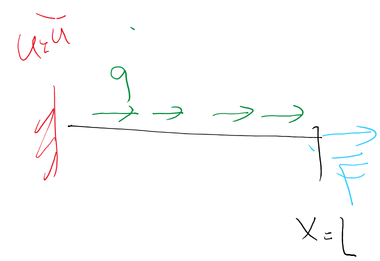
$$K_{n\times n}\, a_{n\times 1} = F_{n\times 1} = F_r + F_D + F_N$$

$$K = \left(\int_0^L \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_n \end{bmatrix} [\mathcal{L}_M(\phi_1) \cdots \mathcal{L}_M(\phi_n)]\, dx - \begin{bmatrix} \omega_{f_1}(L) \\ \vdots \\ \omega_{f_n}(L) \end{bmatrix} [L_f(\phi_1) \cdots L_f(\phi_n)](x=L) \right)$$



$$F_r = -\int \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_n \end{bmatrix} q\, dx$$

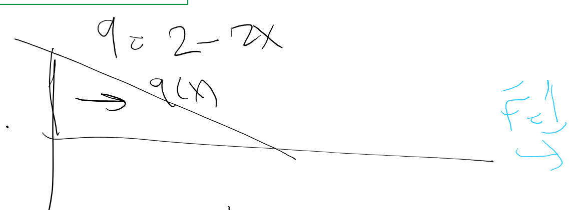
$$\phi_p(x=0) = \bar{u}$$



$$F_D = -\int_0^L \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_n \end{bmatrix} L_M(\phi_p)\, dx + \begin{bmatrix} \omega_{f_1} \\ \vdots \\ \omega_{f_n} \end{bmatrix}(x=L)\, L_f(\phi_p)(x=L)$$

$$F_N = \bar{F} \begin{bmatrix} \omega_{f_1}(L) \\ \vdots \\ \omega_{f_n}(L) \end{bmatrix}$$

Our problem



$$n = 2 \qquad \phi = [\phi_1 \ \ \phi_2] = [x \ \ x^2]$$

$$\phi_p = \frac{1}{2}$$

$$F_D = \int_0^2 \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \Big(\underbrace{EA}_{1}\, \cancel{(\phi_p')'}\Big)\, dx + \begin{bmatrix} \omega_{f_1} \\ \omega_{f_2} \end{bmatrix}(x=2) \Big(\underbrace{EA}_{1}\, (\underbrace{\cancel{1}}_{\phi_p})'\Big) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$K = \int_0^2 \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} [L_M(\phi_1) \ \ L_M(\phi_2)]\, dx - \begin{bmatrix} \omega_{f_1}(2) \\ \omega_{f_2}(2) \end{bmatrix} [L_f(\phi_1) \ \ L_f(\phi_2)](x=2)$$

$$L_M = (\underbrace{EA}_{1}(\,)')' \qquad L_M = (\,)'' \qquad L_P = \underbrace{EA}_{1}(\,)' = (\,)'$$

$$\phi_1 = x \qquad \phi_2 = x^2$$

$$K = \int_0^2 \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \left(x'' \quad {x^2}'' \right) dx - \begin{bmatrix} \omega_{f_1}(2) \\ \omega_{f_2}(2) \end{bmatrix} \Big[\underset{1}{\underset{\downarrow}{x'}} \quad \underbrace{{x^2}'}_{2x} \Big] \Big|_{x=2}$$

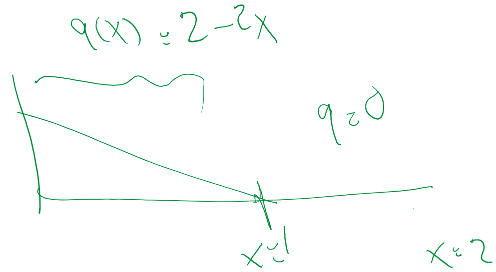
$$= \int_0^2 \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \begin{bmatrix} 0 & 2 \end{bmatrix} dx - \begin{bmatrix} \omega_{f_1}(2) \\ \omega_{f_2}(2) \end{bmatrix} \begin{bmatrix} 1 & 2x \end{bmatrix} \Big|_{x=2}$$

$$\boxed{K = \int_0^2 \begin{bmatrix} 0 & 2\omega_1 \\ 0 & 2\omega_2 \end{bmatrix} dx - \begin{bmatrix} \omega_{f_1}(2) \\ \omega_{f_2}(2) \end{bmatrix} \begin{bmatrix} 1 & 4 \end{bmatrix}}$$

$$F_N = -\underset{1}{\underset{\downarrow}{F}} \begin{bmatrix} \omega_{f_1} \\ \omega_{f_2} \end{bmatrix} (x=2)$$

$$\boxed{F_N = \begin{bmatrix} \omega_{f_1}(2) \\ \omega_{f_2}(2) \end{bmatrix}}$$

$$F_r = \int_0^2 \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} q(x)\, dx$$



$$\int_0^1 \begin{bmatrix} \omega_1(x) \\ \end{bmatrix} (2 - 2x)\, dx$$

$$\boxed{F_r = \int_0^1 \begin{bmatrix} \omega_1(x) \\ \omega_2(x) \end{bmatrix} (2-2x)\, dx}$$