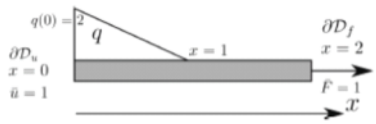
Problem set up



- For this one problem all the quantities are displayed as scalars.
- Consider the 1D bar example shown with the following boundary value problem:

$$\begin{cases} \frac{d\sigma(u(x))}{dx} + q(x) = \frac{d}{dx}\left(EA\frac{du(x)}{dx}\right) + q(x) = 0 & \text{Strong form} \\ u(0) = \bar{u} = 1 & \text{Essential BC on } \partial\mathcal{D}_u = \{0\} \\ F(2) = EA\frac{du(x)}{dx} = \bar{F} = 1 & \text{Natural BC on } \partial\mathcal{D}_f = \{2\} \end{cases} \quad (177)$$

- Material and load properties are,

$$q(x) = \begin{cases} 2 - 2x & x < 1 \\ 0 & 1 \le x \le 2 \end{cases} \qquad E(x) = 1, \quad A(x) = 1, \quad \bar{u} = 1, \quad \bar{F} = 1$$

- According to (177), differential operators and r are:

$$L_M(.) = \frac{d}{dx}\left(EA\frac{d(.)}{dx}\right) \quad (178a)$$

$$L_f(.) = F(.) = EA\frac{d(.)}{dx} \quad (178b)$$

$$r(x) = -q(x) \quad (178c)$$

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Recall, last time we got the stiffness and force vector formulas for 2 unknowns

$$u^h = \phi_p + a_1\phi_1 + a_2\phi_2 \qquad n = 2$$

$$= 1 + a_1 x + a_2 x^2$$

$$a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = ?$$

WRS $\quad K_{2\times2}\, a_{2\times1} = F_{2\times1}$

$$K = \int_0^2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \overbrace{[\,0 \quad 2\,]}^{\phi''} dx - \begin{bmatrix} w_{f_1} \\ w_{f_2} \end{bmatrix}\Bigg|_{x=2} \overbrace{[\,1 \quad 4\,]}^{\phi'|_{x=2}}$$

$$F = \underbrace{-\int_0^1 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} (\overbrace{2-2x}^{q(x)})\, dx}_{F_r \text{ ... term}} \underbrace{- \begin{bmatrix} w_{f_1} \\ w_{f_2} \end{bmatrix}\Bigg|_{x=2}}_{F_N \text{ (natural BC}}$$

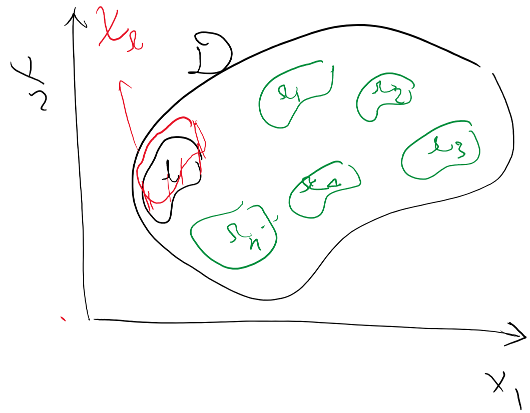
$$\underbrace{\qquad\qquad}_{F_\Gamma \text{ source term}} \qquad \underbrace{\qquad\qquad}_{FN \text{ (Natural BC force)}}$$

Elastostatics

First method: Subdomain

Motivation

$$u = [\phi_1 \cdots \phi_n] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$



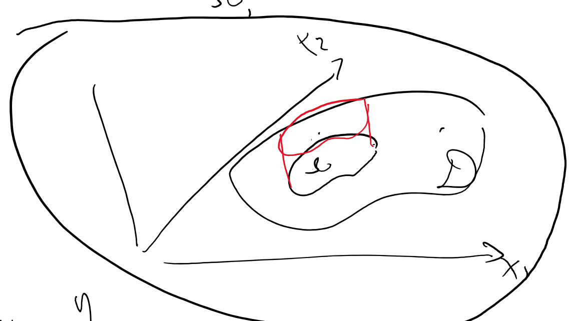
n unknowns:

Balance law $\forall \Omega \quad \int_{\partial\Omega} \sigma \cdot n \, ds + \int_\Omega \rho b \, dV = 0$

to find a_1 to a_n we need n eqns $\Rightarrow$

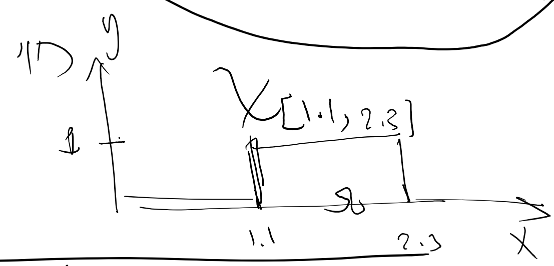
choose n Ω's : $\{\Omega_1, \ldots, \Omega_n\}$

for $i = 1, \ldots, n$ (1) $\int_{\partial\Omega_i} \sigma \cdot n \, ds + \int_{\Omega_i} \rho b \, dV = 0.$ $\int_{\Omega_i} \underbrace{(\nabla\cdot\sigma + \rho b)}_{\text{PDE (strong form)}} dV = 0$



$$\chi_\Omega(x) = \begin{cases} 1 & x \in \Omega \\ 0 & x \notin \Omega \end{cases}$$

(1) $$\int_{\Omega_i} \underbrace{(\nabla\cdot\sigma + \rho b)}_{R_I} dV = 0$$



$$\boxed{\forall i \in \{1,\ldots,n\} \quad \int_D \chi_{\Omega_i} \underbrace{R_I}_{\text{Interior residual}} dV = 0}$$

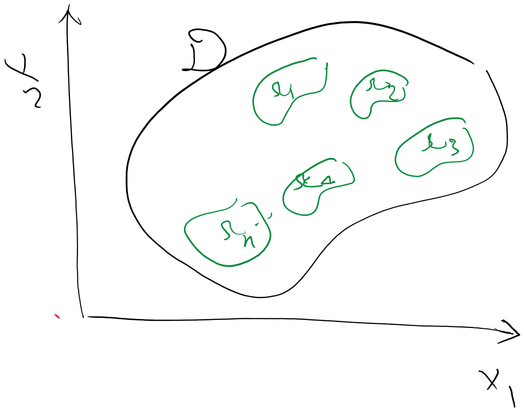
$$\boxed{\text{WRS} \quad \forall i \in \{1,\ldots,n\} \quad \int w_i R_I \, dV = 0}$$

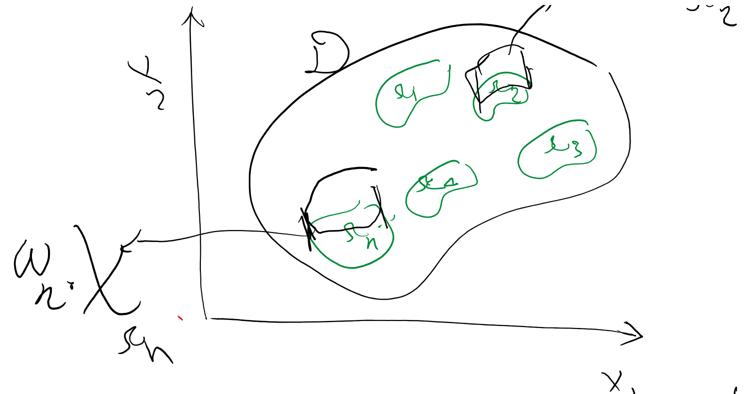
Satisfaction of the balance law is equivalent to a WRS where the weights are characteristic functions of these omegas

Balance law

$w_2 = \chi_{\Omega_2}$

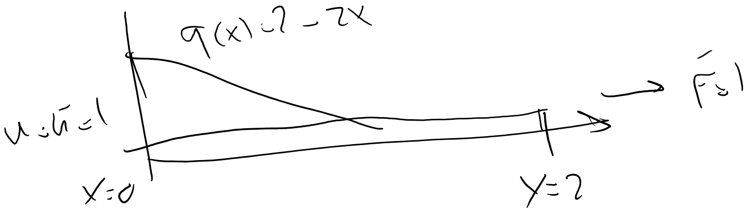
Balance law





see slide 111-2 for more info

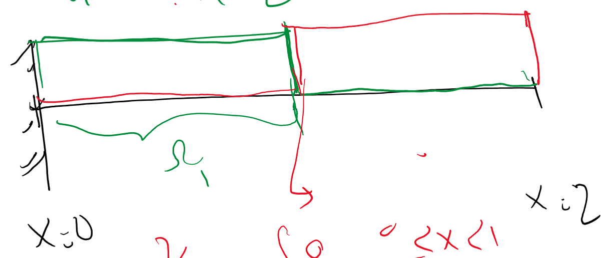
apply this to our 1D problem



$$u = 1 + a_1 x + a_2 x^2$$

$$n = 2$$

$$w_1 = \chi_{\Omega_1} = \begin{cases} 1 & 0 \le x < 1 \\ 0 & 1 < x \le 2 \end{cases}$$



$$w_2 = \chi_{\Omega_2} = \begin{cases} 0 & 0 \le x < 1 \\ 1 & 1 < x < 2 \end{cases}$$

Referring to (*)

$$\underset{2\times 2}{K} = \int_0^2 \begin{bmatrix} w_1 \\ \vdots \\ w_2 \end{bmatrix} [0 \quad 2]\,dx - \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}_{x=2} [1 \quad 4]$$

($w_f = w$ for all except Least square)

First row of K:

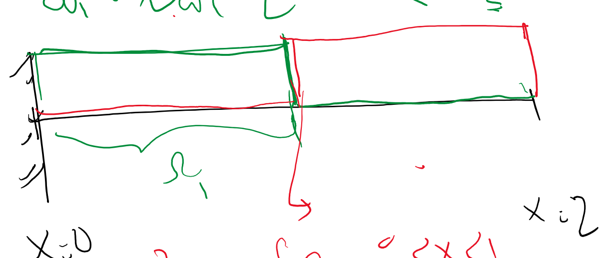
$$K_{1:} = \int_0^2 w_1 [0 \quad 2]\,dx - w_1(x=2)[1 \quad 4]$$

$$= \int_0^1 [0 \quad 2]\,dx - 0\,[1 \quad 4] = [0 \quad 2]$$

$$K_{2:} = \int w_2 [0 \quad 2]\,dx - w_2(x=2)[1 \quad 4] = \int_1^2 [0 \quad 2] - 1[1 \quad 4] = [-1 \quad -2]$$

$$K = \begin{bmatrix} 0 & 2 \\ -1 & -2 \end{bmatrix}$$

$$w_1 = \chi_{\Omega_1} = \begin{cases} 1 & 0 \le x < 1 \\ 0 & 1 < x \le 2 \end{cases}$$



$$\quad \quad \begin{cases} 0 & 0 \le x < 1 \end{cases}$$

Referring to (*)

$$F = -\int_0^1 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} (2 - 2x)\,dx - \begin{bmatrix} w_{f1} \\ w_{f2} \end{bmatrix}_{x=2}$$

$(w_f = w)$

$F_1 = -\int_0^1 \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} (2-2x)dx - \begin{bmatrix} \omega_{F_1} \\ \omega_{F_2} \end{bmatrix}\Big|_{x=2}$ $(\omega_p = \omega)$

$x=0$ $\qquad$ $x=2$

$\omega_2 = \chi_{\omega_2} = \begin{cases} 0 & 0 \le x < 1 \\ 1 & 1 < x < 2 \end{cases}$

$F_1 = \int_0^1 \omega_1(2-2x)dx - \omega_1(x=2)$

$= \int_0^1 (2-2x)dx - 0 = -1$

$F_2 = \int_0^1 \omega_2(2-2x)dx - \omega_2(x=2) = 0 - 1 = -1$ $\qquad F = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$Ka = F \rightarrow \quad a = K^{-1}F = \begin{bmatrix} 2 \\ -1/2 \end{bmatrix}$

S2

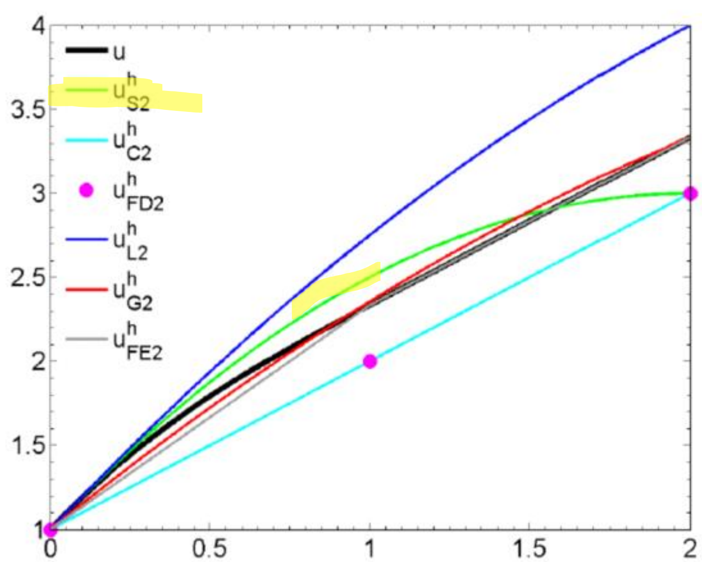
approx

$u^h_{S_2}(x) = \phi_p + \phi a = \phi_p + [\phi_1 \quad \phi_2]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 1 + [x \quad x^2]\begin{bmatrix} 2 \\ -\frac{1}{2} \end{bmatrix} = 1 + 2x - \frac{x^2}{2}$

subdomain

n (# of unknowns)

Bar example, $n = 2$, Comparison of solutions



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$u = 1 + 2x - \frac{x^2}{2}$

$x=0$ $\Sigma F = 0$ $\quad x=1$

$F = 1$

$x=1$ $\quad \Sigma F = 0$ $\quad x=2$

each segment satisfies the balance law

$R_I = u'' + q$

Method 2: Collocation method

Imagine all we know are the DE and the BCs

$u = \bar{u} = 1$

$q(x) = \begin{cases} 2-2x & 0 < x < 1 \end{cases}$

$F = u' = \bar{F}$

$= 1$

$R \quad = \quad u'' + q \quad\quad 0 < x < 2$

$u = \bar{u} = 1$ $\quad q(x) = \begin{cases} 2-2x & 0<x<1 \\ 0 & \end{cases}$ $\quad \bar{F} = u' = \bar{F} = 1$

$$\boxed{\begin{cases} R_I = u'' + q & 0 < x < 2 \\ \text{Essential BC} \quad u(0) = \bar{u} = 1 & x = 0 \\ \text{Natural BC} \quad R_f = \bar{F} - F = 1 - u' & x = 2 \end{cases}}$$

$u^h = 1 + a_1 x + a_2 x^2$

$\phi_0 + \sum a_i \phi_i \rightarrow$ satisfies Essential BC

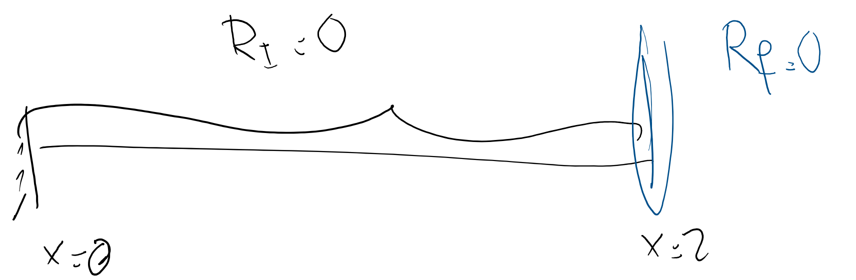
$R_I = u^{h''} + q = 2a_2 + q(x) = \begin{cases} 2 - 2x + 2a_2 & 0 < x \le 1 \\ 2a_2 & 1 \le x < 2 \end{cases}$

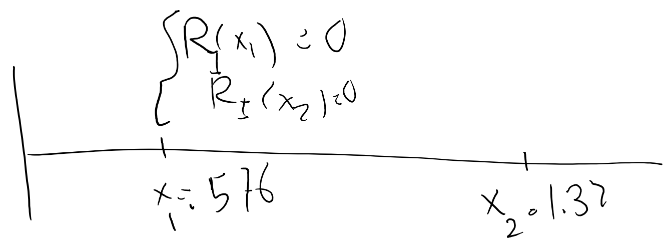
$(0 < x < 2)$

$R_f = 1 - u^{h'} = 1 - a_1 - 2a_2 x \Big|_{x=2} = 1 - a_1 - 4a_2$

$(x = 2)$

need to choose 2 eqns out of these $(n = 2)$





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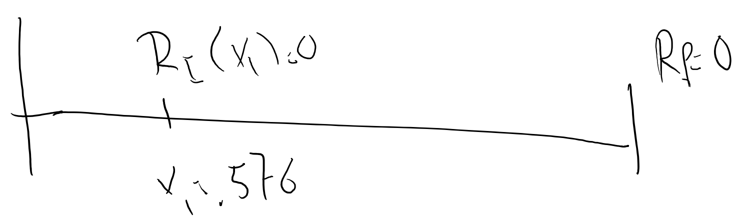


My choice: 1 pt for R_I

My choice: 1 pt for R_I

\+ R_f

$R_I(x=1) = 0 \qquad 2 - 2x + 2a_2 = 0 \quad @\, x=1$

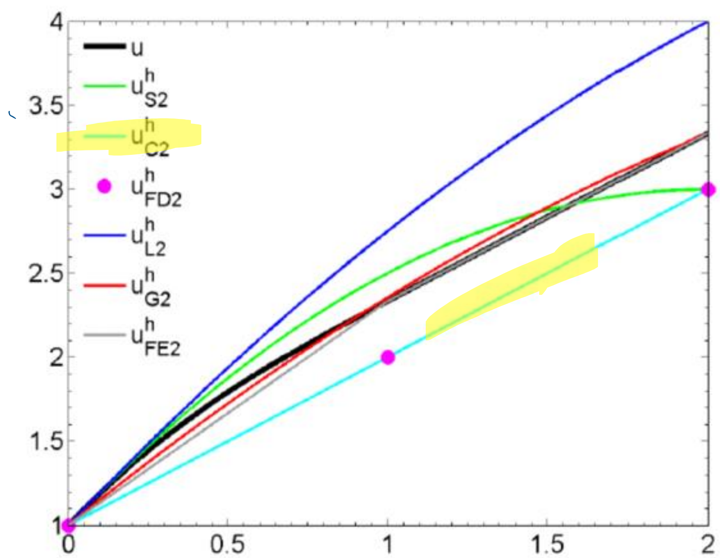
$R_f = 0 \qquad 1 - a_1 - 4a_2 = 0$

$$\underbrace{\begin{bmatrix} 0 & 2 \\ -1 & -4 \end{bmatrix}}_{K} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_{F} \Longrightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} +1 \\ 0 \end{bmatrix}$$

$$\boxed{u^h_{C2}(x) = \phi_p + a_1 x + a_2 x^2 = 1 + x}$$

collocation

Bar example, $n = 2$, Comparison of solutions



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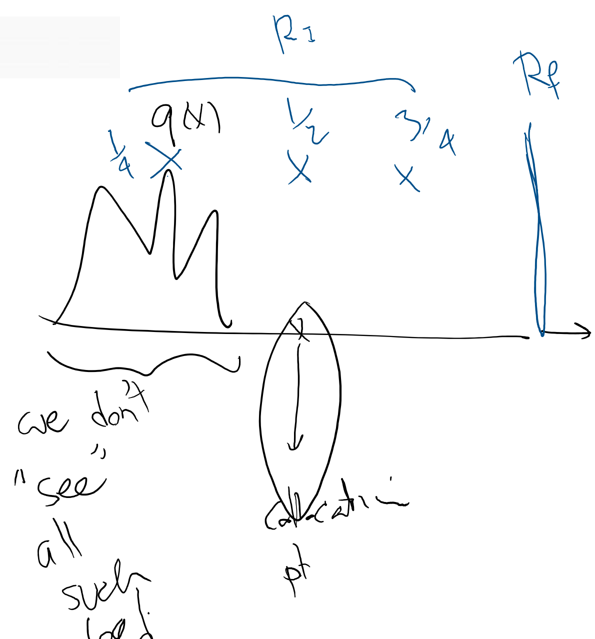


WRS eg

Subdomain

Balance law

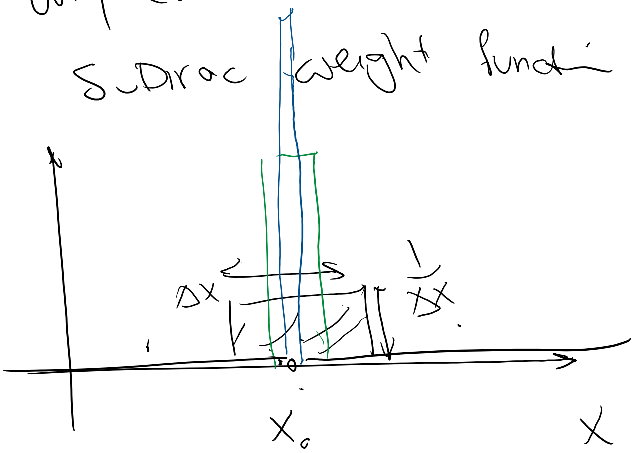
Not doing an integration is its strength (very cheap and versatile) but also its weakness as it's not going to "see" all

Not doing an integration is its strength (very cheap and versatile) but also its weakness as it's not going to "see" all source term, etc.

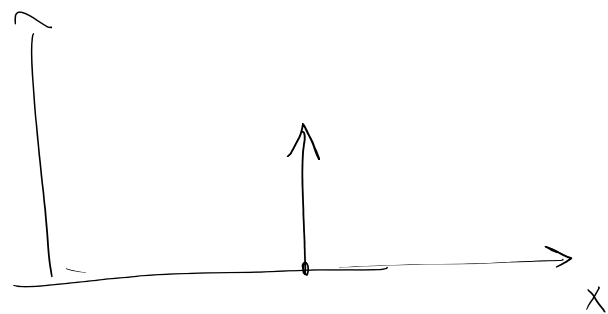
Collocation method — δ Dirac functions

Diff Eqn + BC → WRS → NK

Why collocation is a WRS with δ-Dirac weight function



$\Delta x \to 0$



$$f(x) = \delta(x - x_0) = \begin{cases} \infty & x = x_0 \\ 0 & x \neq x_0 \end{cases}$$

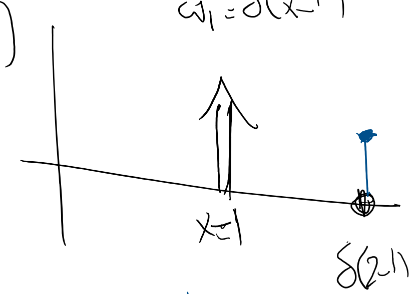
$$\boxed{\int_{-\infty}^{+\infty} \underline{\delta(x-x_0)}\, g(x)\,dx = g(x_0)}$$

What about $\omega_1 = \delta(x-1)$

$$K = \int_0^2 \begin{bmatrix}\omega_1 \\ \omega_2\end{bmatrix} [0 \quad 2]\,dx - \begin{bmatrix}\omega_1 \\ \omega_2\end{bmatrix}(x=2)\,[1 \quad 4]$$

$$F = -\int_0^1 \begin{bmatrix}\omega_1 \\ \omega_2\end{bmatrix}(2 - 2x)\,dx - \begin{bmatrix}\omega_1 \\ \omega_2\end{bmatrix}(x=2)$$

$\omega_1 = \delta(x-1)$



$$\omega_2 = \begin{cases} 0 & x < 2 \\ 1 & x = 2 \end{cases}$$

$$K_{1\cdot} = \int^2 \delta(x-1)[0 \quad 2]\,dx - \delta(2-1)[1 \quad 4]$$

$$K_{1:} = \int_0^2 \underbrace{\delta(x-1)}_{\omega_1} \underbrace{[0\ \ 2]}_{g} dx - \underbrace{\delta(2-1)}_{\omega_1(x=2)} [1\ \ 4]$$

$$\begin{cases} \dots & x<2 \\ 1 & x=2 \end{cases}$$

$$[0\ \ 2]\big|_{x=1} - 0\,[1\ \ 4] = [0\ \ 2]$$

$$K_{2:} = \underbrace{\int_0^2 \omega_2 [0\ \ 2]\, dx}_{} - \omega_2(x=2)[1\ \ 4]$$

$$= \quad 0 \quad - [1\ \ 4] = [-1\ \ -4]$$

$$K = \begin{bmatrix} 0 & 2 \\ -1 & -4 \end{bmatrix}$$

you can show $F = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

matches direct satisfaction of DE

HW collocation

$$u^h = a_0 + a_1 x + \dots + a_4 x^4$$

$$R_i = u^{h\,''} + q(x)$$

choose $x = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ for this

3 eqns

$$R_f = 1 - u^{h\,'}$$

+ 1 eqn here

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4 eqs, 4 unknowns $\Rightarrow a = K^{-1} F$