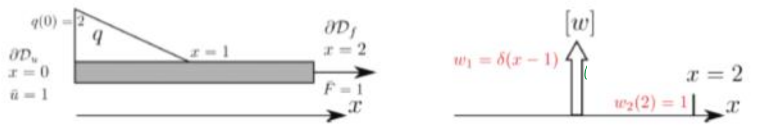
FEM20241002

Wednesday, October 2, 2024 12:53 PM

Bar example, $n = 2$, Collocation method



- Equations (216) and (217) yield,

$$\mathbf{K} = \begin{bmatrix} 0 & 2 \\ -1 & -4 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \tag{218}$$

- From $\mathbf{Ka} = \mathbf{F}$ (125) and (218) we get,

$$\mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{219}$$

- From $u^h = a_j\phi_j + \phi_p$ (117a), $[\phi] = \{x, x^2\}$ (*cf.* (196b)), and $\phi_p = 1$ (182) we have

$$\boxed{u^h_{C2} = 1 + x} \tag{220}$$

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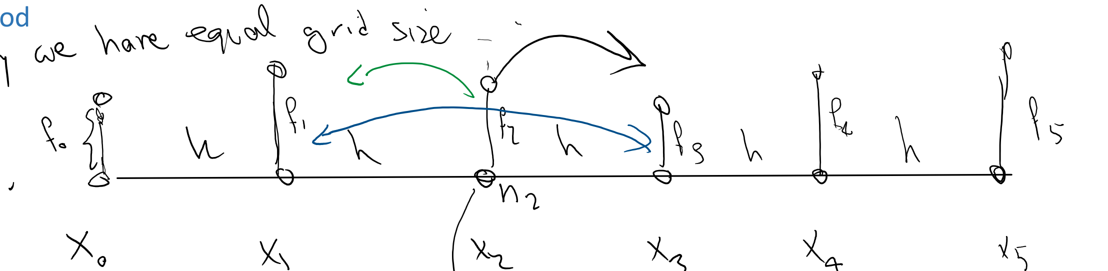
Finite Difference method

commonly we have equal grid size



derivative at n_2

f' errors:

$O(h)$

$$f'(x_2) \approx \frac{f(x_2+h) - f(x_2)}{h} = \frac{f_3 - f_2}{h}$$ forward difference

$O(h)$

$$f'(x_2) \approx \frac{f(x_2) - f(x_2-h)}{h} = \frac{f_2 - f_1}{h}$$ backward difference

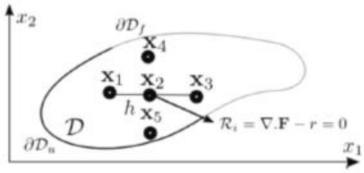
more accurate $O(h^2)$

$$f'(x_2) \approx \frac{f(x_2+h) - f(x_2-h)}{2h} = \frac{f_3 - f_1}{2h}$$ central diff

f''

$$f''(x_2) = f_2'' \approx \frac{f_1 + f_3 - 2f_2}{h^2} = \frac{f_1 - 2f_2 + f_3}{h^2}$$

Collocation method versus Finite Difference

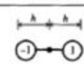
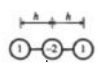
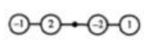
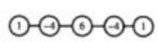
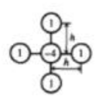
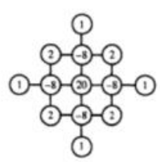


- Both Collocation and Finite Difference methods directly work with the strong form and boundary conditions.
- Collocation method is a particular class of weighted residual method where the solution is interpolated as $\mathbf{u}^h = a_j\phi_j + \phi_p$.
- Finite Difference does not interpolate the solution with trial function. Rather, it uses discrete values of the function on often regular grids to approximate differential operators.
- Differential operators in Finite Difference method are approximate, where as in collocation method the solution $\mathbf{u}^h$ exactly satisfies the strong form at $\mathbf{x}_i$.
- As an example, let us assume the differential operator L_M in $\mathcal{R}_i$ includes a Laplacian operator $\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$. The finite difference approximation of Laplacian on a uniform grid with size h would be,

$$\Delta u(\mathbf{x}_2) = \frac{1}{h^2}\left(u(\mathbf{x}_1) + u(\mathbf{x}_3) + u(\mathbf{x}_4) + u(\mathbf{x}_5) - 4u(\mathbf{x}_2)\right) \tag{150}$$

Finite Difference Stencils

TABLE 3.1 *Finite difference approximations for various differentiations*

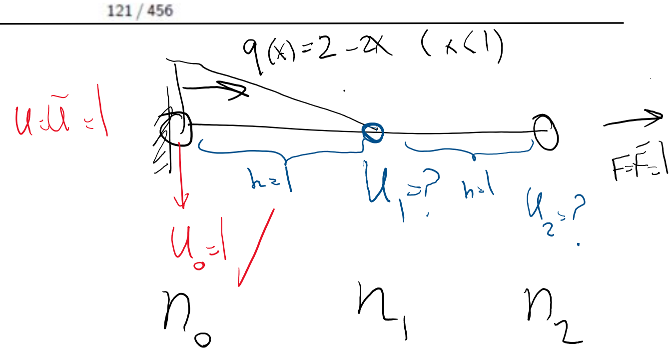
Differentiation	Finite difference approximation	Molecules
$\left.\frac{dw}{dx}\right\|_i$	$\frac{w_{i+1} - w_{i-1}}{2h}$	
$\left.\frac{d^2w}{dx^2}\right\|_i$	$\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2}$	
$\left.\frac{d^3w}{dx^3}\right\|_i$	$\frac{w_{i+2} - 2w_{i+1} + 2w_{i-1} - w_{i-2}}{2h^3}$	
$\left.\frac{d^4w}{dx^4}\right\|_i$	$\frac{w_{i+2} - 4w_{i+1} + 6w_i - 4w_{i-1} + w_{i-2}}{h^4}$	
$\nabla^2 w\|_{i,j}$	$\frac{-4w_{i,j} + w_{i+1,j} + w_{i,j+1} + w_{i-1,j} + w_{i,j-1}}{h^2}$	
$\nabla^4 w\|_{i,j}$	$[20w_{i,j} - 8(w_{i+1,j} + w_{i-1,j} + w_{i,j+1} + w_{i,j-1}) + 2(w_{i+1,j+1} + w_{i-1,j+1} + w_{i-1,j-1} + w_{i+1,j-1}) + w_{i+2,j} + w_{i-2,j} + w_{i,j+2}$ $w_{i,j-2}]/h^4$	

$\nabla^2 \omega$

$\nabla^4 \omega$

Source: Bathe's book, section 3.3.5.

How do we use Finite Difference (FD) for our 1D problem



$n = 2$

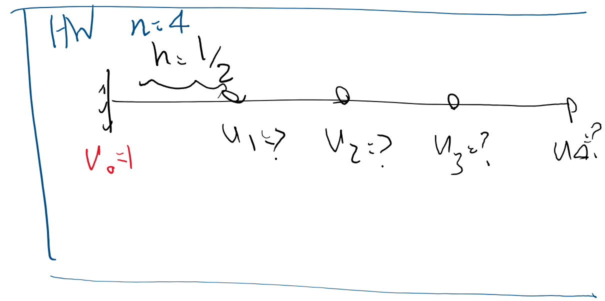
unknowns are $\{u_1, u_2\}$

$u_1 \rightarrow x = 1$, $u_2 \rightarrow x = 2$

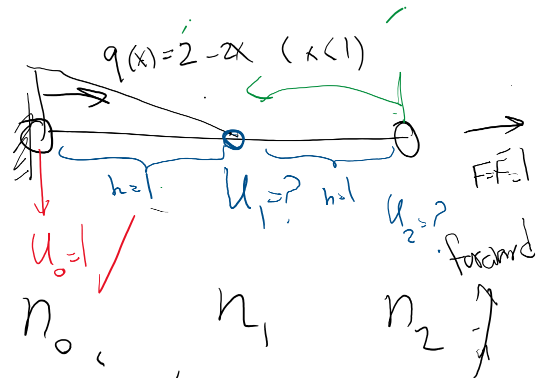
$R_i(x) = u^{h''} + q(x), \quad 0 < x < 2$

HW $n = 4$, $h = 1/2$

$$\begin{cases} R_i(x) = u'' + q(x) & 0 < x < 2 \\ R_f = \bar{F} - F = 1 - u^{h\prime} & x = 2 \end{cases}$$



$$\begin{cases} @\, n_1 : R_i(x_1) = 0 & u_1'' + q(x_1) = 0 \quad \text{(eq1)} \\ @\, n_2 : R_f = 0 & 1 - u_2' = 0 \quad \text{(eq2)} \end{cases}$$



n_0, n_1, n_2

my choice

$$u_1'' = \frac{u_2 + u_0 - 2u_1}{h^2}$$

(eq1) $$\frac{u_2 + u_0 - 2u_1}{\underset{1}{h^2}} + \underset{0}{q(x=1)} = 0$$

(eq2) $$1 - \left(\frac{u_2 - u_1}{\underset{h=1}{h}}\right) = 0$$

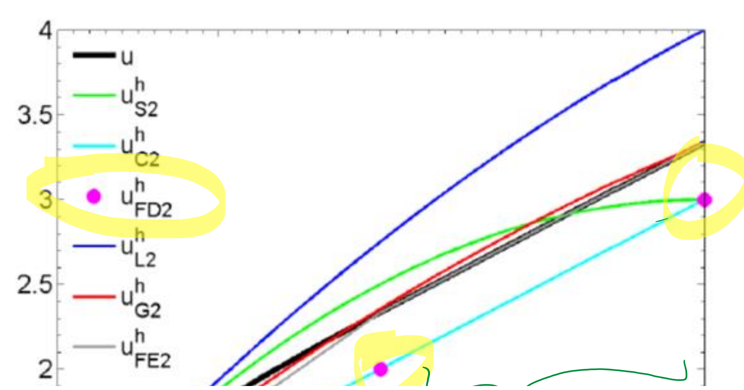
$$u_2' = \frac{u_2 - u_1}{h} \quad \text{BD}$$

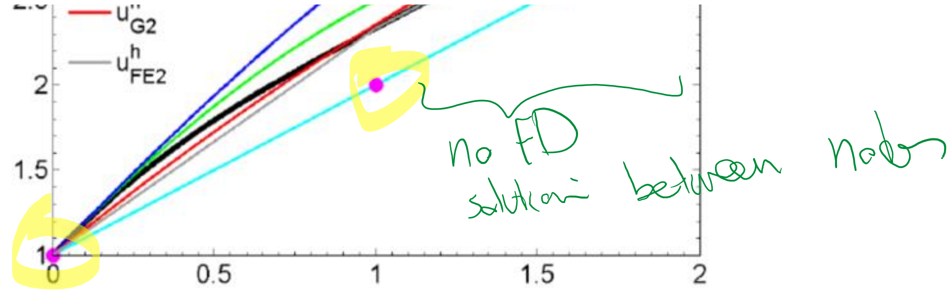
$$\begin{cases} \overset{u_0}{1} + u_2 - 2u_1 = 0 \\ u_1 - u_2 = -1 \end{cases}$$

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}}_{K} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ -1 \end{bmatrix}}_{F}$$

$$\rightarrow \boxed{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{FD} \quad n=2}$$

Bar example, $n = 2$, Comparison of solutions





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Similarities of FD and collocation:

- They satisfy the equations at the nodes (interior residual at interior nodes, natural BC residual at natural boundary nodes)
- They don't involve any integrations.
- They are fast (especially the FD).
- Both are not that accurate …

What's the difference?

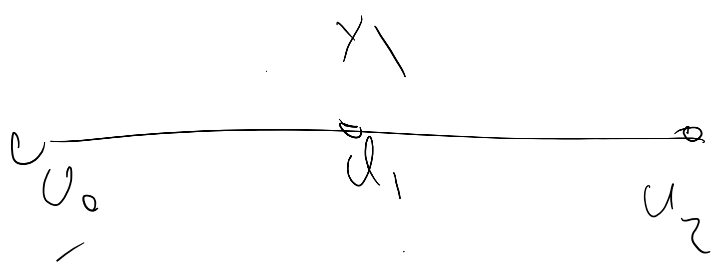
$\underbrace{\text{Collocation}}$ $u^h(x) = \phi_p(x) + a_1\phi_1(x) + a_2\phi_2(x)$

eg here $1 + a_1 x + a_2 x^2$

we find a function

$u^{h''} = 2a_2$ exact

FD we only have nodal values



$$u^{h''}(x_1) \approx \frac{u_2 + u_0 - 2u_1}{h^2}$$

not exact (unlike Collocation!)

$h \to 0$ good $(\ )'$, $(\ )''$ mathemally $\to 0$

But (bad) we deal with cancelation error due to finite precision calculation,

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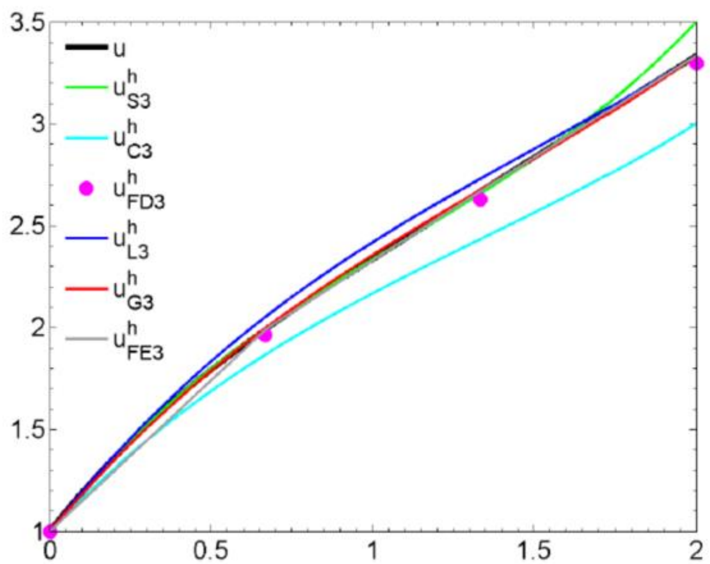
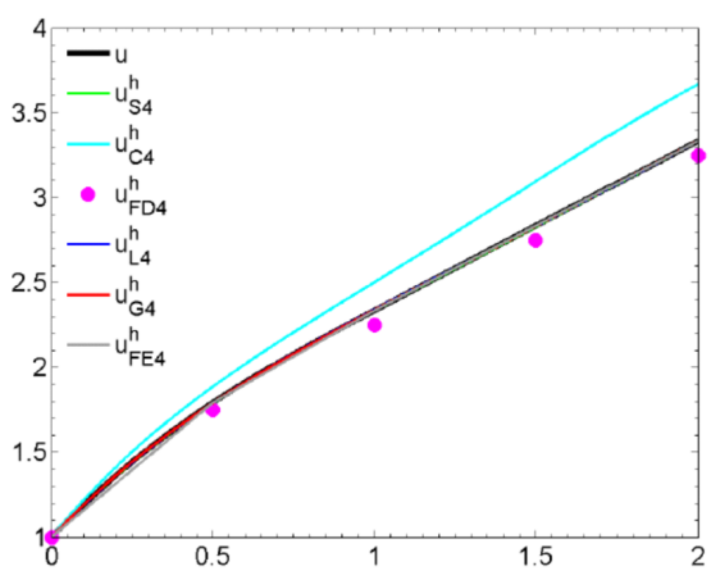
Bar example, $n = 3$, Comparison of solutions



Bar example, $n = 4$, Comparison of solutions



Galerkin method

$$\boxed{W = \phi}$$

④ Spectral Galerkin

basis function $\{x, x^2, x^3, x^4, \ldots\}$

WRS: $\int_0^2 \omega R_i \, dx + \omega_f R_f = 0$

$\omega_f = \omega$ (for all other than Least square)

From last week

$$K = \int_0^2 \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} [0 \quad 2] dx - \begin{bmatrix} \omega_{f_1} \\ \omega_{f_2} \end{bmatrix} (x=2) [1 \quad 4]$$

$$u^h = \phi_p + a_1 \phi_1 + a_2 \phi_2 = 1 + a_1 x + a_2 x^2$$

Galerkin $\omega_1 = \omega_{f_1} = \phi_1 = x$

$\omega_2 = \omega_{f_2} = \phi_2 = x^2$

$$K = \int_0^2 \begin{bmatrix} x \\ x^2 \end{bmatrix} [0 \quad 2] dx - \begin{bmatrix} x \\ x^2 \end{bmatrix}_{x=2} [1 \quad 4] \Rightarrow$$

$$K = \int_0 [x^2][\ldots]\,dx \quad [x^2]\Big|_{x=2} \qquad \Rightarrow$$

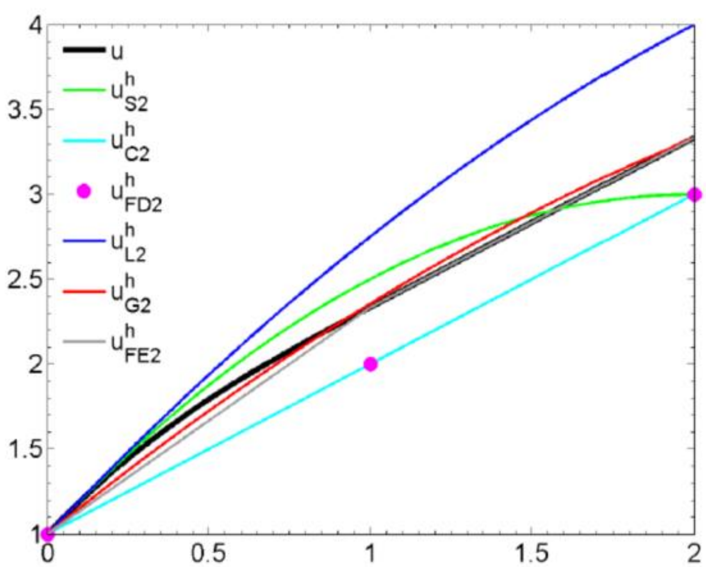
$$F = -\int_0^1 \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}(2-2x)\,dx - \begin{bmatrix} \omega\phi_1 \\ \omega\phi_2 \end{bmatrix}_{x=2} = \int_0^1 \begin{bmatrix} x \\ x^2 \end{bmatrix}(2-2x)\,dx - \begin{bmatrix} x \\ x^2 \end{bmatrix}_{x=2}$$

$$K = \begin{bmatrix} -2 & -4 \\ -4 & -\frac{32}{3} \end{bmatrix} \quad F = \begin{bmatrix} -7/3 \\ -25/6 \end{bmatrix}, \quad Ka = F \Rightarrow a = \begin{bmatrix} 37/24 \\ -3/16 \end{bmatrix}$$

$$u^h_{G2} = \phi_p + a_1\phi_1 + a_2\phi_2 = 1 + \frac{37}{24}x - \frac{3}{16}x^2$$

Galerkin (G), $n=2$

Bar example, $n = 2$, Comparison of solutions



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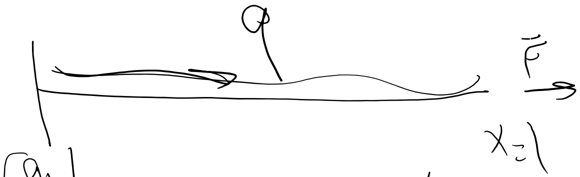
Galerkin method using the weak statement (WK)

Galerkin method using the weak statement

$$\int_0^L \omega' EA u' dx = \int_0^L \omega q \, dx + \omega(L)\bar{F} \quad (Wk)$$



$n=2$

$$\omega = \begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix} \qquad u = \phi_p + [\phi_1 \ \phi_2]\begin{bmatrix}a_1\\ a_2\end{bmatrix} \qquad \text{plug them in } (Wk)$$

$$\int_0^L \begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}' EA\left([\phi_1 \ \phi_2]'\begin{bmatrix}a_1\\ a_2\end{bmatrix} + \phi_p'\right) = \int_0^L \begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix} q \, dx + \begin{bmatrix}\omega_1(L)\\ \omega_2(L)\end{bmatrix}\bar{F}$$

$$Ka = F_r + F_N - F_D$$

$$K = \int_0^L \begin{bmatrix}\omega_1'\\ \omega_2'\end{bmatrix} EA \, [\phi_1' \ \phi_2'] \, dx, \quad F_r = \underbrace{\int_0^L \begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix} q \, dx}_{\text{source term}} \quad \underbrace{F_N}_{\text{Neumann BC}} = \begin{bmatrix}\omega_1(L)\\ \omega_2(L)\end{bmatrix}\bar{F}$$

$$F_D = \int_0^L \begin{bmatrix}\omega_1'\\ \omega_2'\end{bmatrix} EA \phi_p' \, dx$$

Dirichlet BC

$$u^h = \phi_p + \sum_i a_i \phi_i$$

ϕ_p satisfies E.BC ($\phi_p(0) = \bar{u}$ here)

ϕ_i : homog. E.BC ($\phi_p(0) = 0$ here)

ϕ_i's are suitable weights

weight function for a weak statement $\leftarrow$ ω_i homog E.BC ($\omega_i(0) = 0$ here)

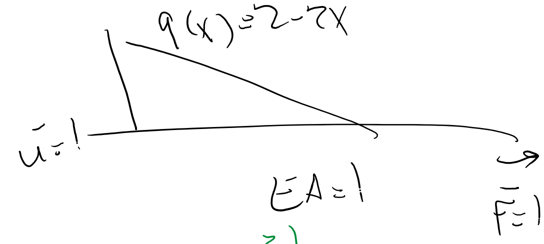
Galerkin method can also be used in the weak statement (in fact, for FEM, that's the only option and WRS does not work)

$$Ka = F_r + F_N - F_D$$

$$K = \int_0^L \begin{bmatrix} \omega_1' \\ \omega_2' \end{bmatrix} EA \,[\phi_1' \;\; \phi_2']\,dx \;,\quad F_r = \int_0^L \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} q\,dx \qquad F_N = \begin{bmatrix} \omega_1(L) \\ \omega_2(L) \end{bmatrix} \bar{F}$$

source term — Neumann BC

$$F_D = \int_0^L \begin{bmatrix} \omega_1' \\ \omega_2' \end{bmatrix} EA\,\phi_p'\,dx$$



$$\Phi = [\phi_1 \;\; \phi_2] = [x \;\; x^2]$$

$$\omega = \Phi = [x \;\; x^2]$$

$$K = \int_0^2 \begin{bmatrix} x \\ x^2 \end{bmatrix}' [x \;\; x^2]'\,dx = \int_0^2 \begin{bmatrix} 1 \\ 2x \end{bmatrix} [1 \;\; 2x]\,dx = \int_0^2 \begin{bmatrix} 1 & 2x \\ 2x & 4x^2 \end{bmatrix} dx$$

$$\Longrightarrow \qquad K = \begin{bmatrix} 2 & 4 \\ 4 & 32/3 \end{bmatrix}$$

$$F_D = 0 \quad \phi_p' = (1)' = 0 \qquad F_r = \int_0^2 \begin{bmatrix} x \\ x^2 \end{bmatrix} (2 - 2x)\,dx \;,\quad F_N = \begin{bmatrix} x \\ x^2 \end{bmatrix}_{x=2} \times 1$$

$$F = F_r + F_N - F_D = \begin{bmatrix} 7/3 \\ 25/6 \end{bmatrix}$$

$$a = K^{-1} F = \begin{bmatrix} 37/24 \\ -3/16 \end{bmatrix} \qquad \boxed{u^h_{(2)} = 1 + \frac{37}{24}x - \frac{3}{16}x^2}$$

Using the weak statement

As expected, it matches the solution from WRS for Galerkin method

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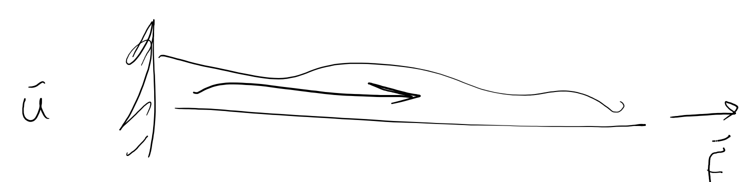
Ritz method:
The idea is that here we
1. Discretize the solution
2. Minimize the energy

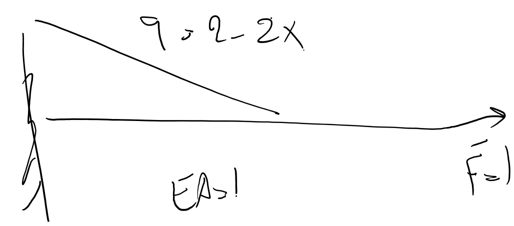


$$\Pi(u) = V(u) - W(u) = \int_0^L \frac{1}{2} EA (u')^2 dx - \left(\int_0^L q u\,dx + u(L)\bar{F} \right)$$

$$\Pi(u) = \underbrace{V(u)}_{\text{internal energy}} - \underbrace{W(u)}_{\text{external work}} = \underbrace{\int_0^L \frac{1}{2} EA u'^2 \, dx}_{V} - \underbrace{\left(\int_0^L uq \, dx + u(L)\bar{F}\right)}_{W}$$

$$\boxed{\Pi(u) = \int_0^2 \frac{1}{2} u'^2 \, dx - \int_0^1 (2-2x)u \, dx - u(2)}$$

$u=1$ ⊛



$$u^h = \phi_p + a_1\phi_1 + a_2\phi_2 = 1 + a_1 x + a_2 x^2$$

for $\phi_p = 1$, $\phi_1 = x$, $\phi_2 = x^2$

plug in ⊛

$$u^{h\prime} = a_1 + 2a_2 x$$

$$\Pi(u^h) = \int_0^2 \frac{1}{2}(a_1 + 2a_2 x)^2 dx - \int_0^1 (2-2x)(1 + a_1 x + a_2 x^2) - \left[1 + a_1(2) + a_2(2)^2\right]$$

. . .

$$\Pi(a_1, a_2) = \left(a_1^2 + 4a_1a_2 + \frac{16}{3}a_2^2\right) - \left(\frac{7}{3}a_1 + \frac{25}{6}a_2 + 2\right)$$

We want to minimize the energy

$$\nabla\Pi = \begin{bmatrix} \frac{\partial \Pi}{\partial a_1} \\ \frac{\partial \Pi}{\partial a_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2a_1 + 4a_2 - \frac{7}{3} \\ 4a_1 + \frac{32}{3}a_2 - \frac{25}{6} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow \quad Ka = F \qquad K = \begin{bmatrix} 2 & 4 \\ 4 & 32/3 \end{bmatrix} \qquad F = \begin{bmatrix} 7/3 \\ 25/6 \end{bmatrix}$$

$Ka = F$ $\quad K = \begin{bmatrix} 4 & 32/3 \end{bmatrix}$ $\quad F = \begin{bmatrix} 25/6 \end{bmatrix}$

$$a = K^{-1}F = \begin{bmatrix} 37/24 \\ -3/16 \end{bmatrix} \qquad u^h = 1 + \frac{37}{24}x - \frac{3}{16}x^2$$