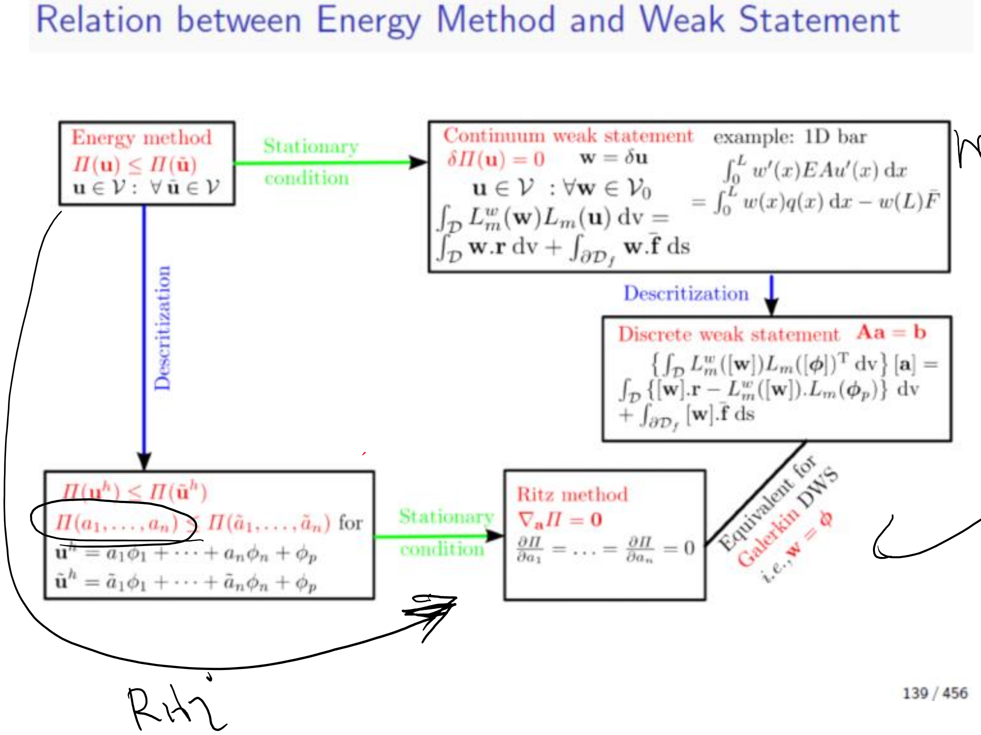
# FEM20241009

Wednesday, October 9, 2024 12:59 PM

## Why the weak statement + Galerkin gives the same solution as the Ritz method?



Relation between Energy Method and Weak Statement

Energy method
$\Pi(\mathbf{u}) \le \Pi(\tilde{\mathbf{u}})$
$\mathbf{u} \in \mathcal{V}: \ \forall \tilde{\mathbf{u}} \in \mathcal{V}$

Stationary condition

Continuum weak statement
$\delta\Pi(\mathbf{u}) = 0 \qquad \mathbf{w} = \delta\mathbf{u}$
$\mathbf{u} \in \mathcal{V} \ : \forall \mathbf{w} \in \mathcal{V}_0$
$\int_{\mathcal{D}} L_m^w(\mathbf{w}) L_m(\mathbf{u}) \, \mathrm{dv} = \int_{\mathcal{D}} \mathbf{w}.\mathbf{r} \, \mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathbf{w}.\bar{\mathbf{f}} \, \mathrm{ds}$

example: 1D bar
$\int_0^L w'(x) E A u'(x) \, \mathrm{d}x = \int_0^L w(x) q(x) \, \mathrm{d}x - w(L)\bar{F}$

Descritization

Discrete weak statement $\mathbf{A}\mathbf{a} = \mathbf{b}$
$\left\{ \int_{\mathcal{D}} L_m^w([\mathbf{w}]) L_m([\boldsymbol{\phi}])^{\mathrm{T}} \, \mathrm{dv} \right\} [\mathbf{a}] = \int_{\mathcal{D}} \{ [\mathbf{w}].\mathbf{r} - L_m^w([\mathbf{w}]).L_m(\boldsymbol{\phi}_p) \} \, \mathrm{dv} + \int_{\partial\mathcal{D}_f} [\mathbf{w}].\bar{\mathbf{f}} \, \mathrm{ds}$

Descritization

$\Pi(\mathbf{u}^h) \le \Pi(\tilde{\mathbf{u}}^h)$
$\Pi(a_1, \ldots, a_n) \le \Pi(\tilde{a}_1, \ldots, \tilde{a}_n)$ for
$\mathbf{u}^h = a_1\phi_1 + \cdots + a_n\phi_n + \phi_p$
$\tilde{\mathbf{u}}^h = \tilde{a}_1\phi_1 + \cdots + \tilde{a}_n\phi_n + \phi_p$

Stationary condition

Ritz method
$\nabla_{\mathbf{a}}\Pi = \mathbf{0}$
$\frac{\partial\Pi}{\partial a_1} = \ldots = \frac{\partial\Pi}{\partial a_n} = 0$

Equivalent for Galerkin DWS
i.e., $\mathbf{w} = \boldsymbol{\phi}$

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Ritz

weak statement (continuum)

Discrete weak

$\phi_1 = x$

$\phi_2 = x^2$

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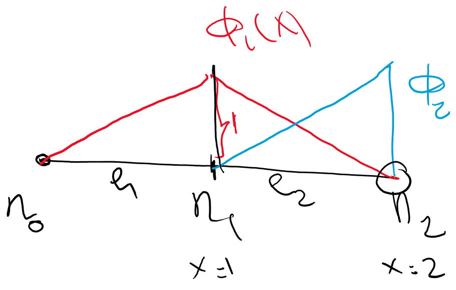
Let's look at another Galerkin method where the basis functions are different

$w \; \phi$

### Finite Element Method (FEM)

$\phi_1$ & $\phi_2$ are piecewise linear basis functions

(in FEM they're also called <u>shape functions</u>)



$$\phi_1(x) = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 < x \le 2 \end{cases}$$

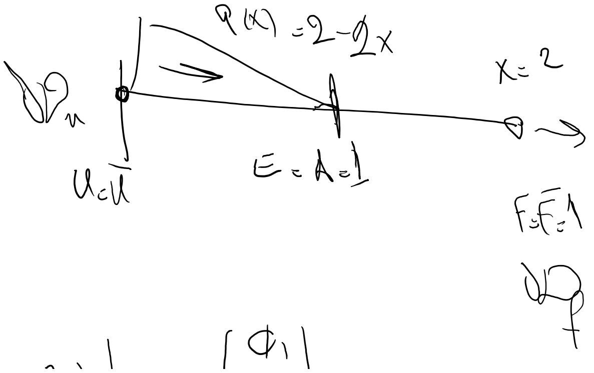
$$\phi_2(x) = \begin{cases} 0 & 0 \le x \le 1 \\ x-1 & 1 \le x \le 2 \end{cases}$$

We derived this last time

weak statement

$$\left( \int_0^L \phi^{t'} EA \phi'' dx \right) a = \int_0^L \phi^t q dx + \phi^t(x=L)\bar{F}$$

$(\text{for } \phi_p = 1)$

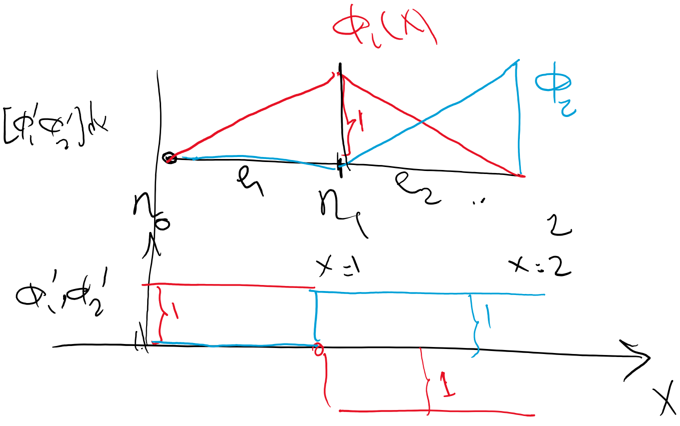


$n=2$ & our problem

$$\underbrace{\left[\int_0^2 \begin{bmatrix}\phi_1'\\ \phi_2'\end{bmatrix}[\phi_1' \ \ \phi_2']\,dx\right]}_{K_{2\times 2}}\begin{bmatrix}a_1\\ a_2\end{bmatrix} = \int_0^1 \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}(2-2x)\,dx + \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}(x=2)$$

$u^h = \phi_p + a_1\phi_1 + a_2\phi_2$ / we're still using $\phi_p = 1$

$$K = \int_0^2 \begin{bmatrix}\phi_1'\\ \phi_2'\end{bmatrix}[\phi_1' \ \ \phi_2']\,dx = \int_{e_1} \begin{bmatrix}\phi_1'\\ \phi_2'\end{bmatrix}[\phi_1' \ \ \phi_2']\,dx + \int_{e_2} \begin{bmatrix}\phi_1'\\ \phi_2'\end{bmatrix}[\phi_1' \ \phi_2']\,dx$$



$$= \underbrace{\int_0^1}_{\int_{e_1}} \begin{bmatrix}1\\ 0\end{bmatrix}[1 \ \ 0]\,dx + \int_1^2 \begin{bmatrix}-1\\ 1\end{bmatrix}[-1 \ \ 1]\,dx$$

$$= \int_0^1 \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}dx + \int_1^2 \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}dx = \underbrace{\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}}_{K}$$

$$\boxed{\phi_i(n_j) = \delta_{ij} = \begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}}$$

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$$F = \int_0^1 \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}(2-2x)\,dx + \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}\underbrace{(x=2)}_{n_2} = \int_0^1 \begin{bmatrix}x\\ 0\end{bmatrix}(2-2x)\,dx + \begin{bmatrix}0\\ 1\end{bmatrix}$$

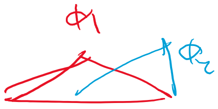
$$= \begin{bmatrix}1/3\\ 0\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix} \Longrightarrow \quad F = \begin{bmatrix}1/3\\ 1\end{bmatrix}$$

$$a = K^{-1}F = \begin{bmatrix}4/3\\ 7/3\end{bmatrix}$$

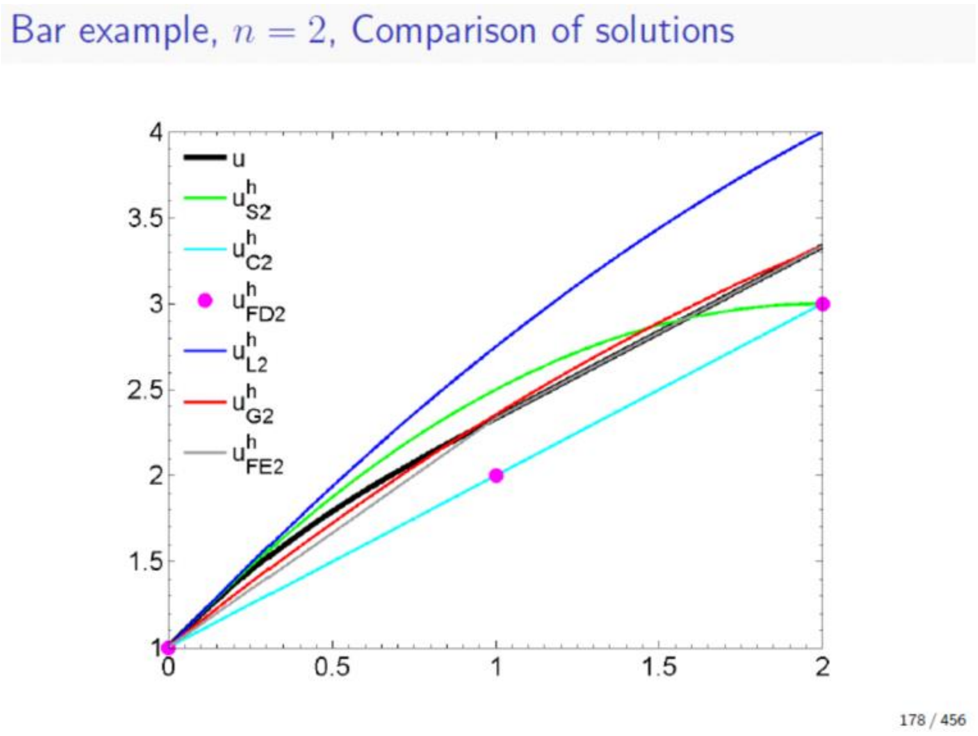
FEM solution $n=2$

$$u^h_{FE,2}(x) = \phi_p(x) + a_1\phi_1(x) + a_2\phi_2(x)$$

$$= 1 + \frac{4}{3}\phi_1(x) + \frac{7}{3}\phi_2(x)$$



$$\cancel{= 1 + \frac{4}{3}x + \frac{7}{3}x^2}$$



## Bar example, $n = 2$, Comparison of solutions

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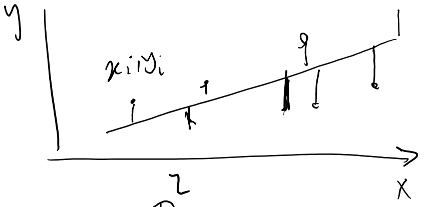
Least square (LS) method

The LS method is used in many contents for example in linear regression

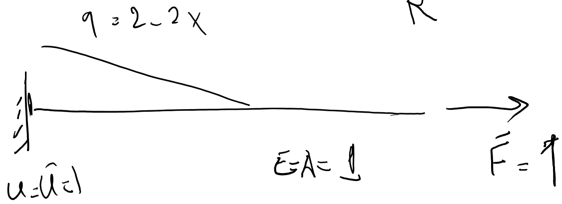


$R^2$

R2 in our context:

① $R_i = u'' + q$      $0 \leq x \leq 2$

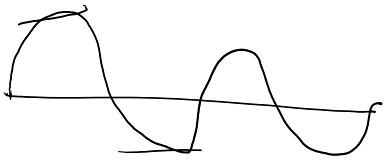
② $R_f = \bar{F} - F = 1 - u'$



& use $\phi_p = 1$ / $\phi_1 = x$, $\phi_2 = x^2$

Let's discretize the problem  $u^h = \phi_p + a_1\phi_1 + a_2\phi_2$

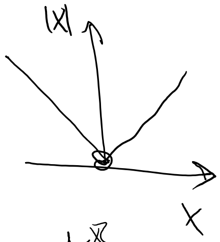
③ $u^h = 1 + a_1 x + a_2 x^2 \Rightarrow u^{h\prime} = a_1 + 2a_2 x$, $u^{h\prime\prime} = 2a_2$



plug ③ into ①&②

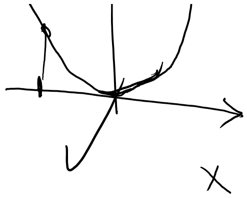
$$\begin{cases} R_i = 2a_2 + q(x) \\ R_f = 1 - a_1 - 2a_2 x \end{cases}$$

$$R^2 = \int_0^2 (2a_2 + q(x))^2 dx + (1 - a_1 - 2a_2 x)^2_{x=2}$$



$$R^2 = \int_0^2 R_i^2 \, dx + R_f^2 (x=2)$$

$$R = \boxed{\int_0 \underline{R_i}\, dx} + \boxed{R_f\,(x=2)}$$



for all x
$x^2 \geq 0$
$\& \, c_1$

$$R^2 = 0 \Longleftrightarrow \text{Exact soln.}$$

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to get the numerical soln for a given n
we minimize $R^2$
"Least square"

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here

$$R^2 = \int_0^2 \left(2a_2 + q(x)\right)^2 dx + \left(1 - a_1 - 2a_2 x\right)^2_{x=2}$$

$$= \int_0^1 (2a_2 + \overbrace{2 - 2x}^{q(x)\ (0<x<1)})^2\, dx + \int_1^2 (2a_2 + 0)^2\, dx + \left(1 - a_1 - 4a_2\right)^2$$

$$R^2(a_1, a_2) = \underline{1} + a_1^2 + \boxed{24}\, a_2^2 - 2a_1 - 8a_2 + 8a_1a_2$$

minimize $R^2$

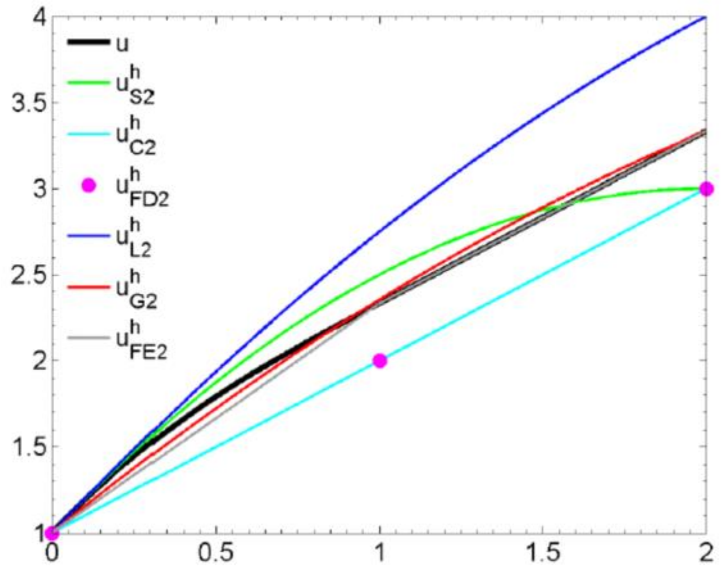
$$\nabla R^2 = \begin{pmatrix} \partial R^2/\partial a_1 \\ \partial R^2/\partial a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{bmatrix} 2a_1 + 8a_2 - 2 \\ 8a_1 + 4a_2 - 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow Ka = F \quad K = \begin{bmatrix} 2 & 8 \\ 8 & 48 \end{bmatrix}, \ F = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \Rightarrow a = K^{-1}F = \begin{bmatrix} 2 \\ -1/4 \end{bmatrix}$$

$$\boxed{u^h = 1 + 2x - \frac{1}{4}x^2}$$

$$\boxed{u^h_{L2} = 1 + 2x - \frac{1}{4}x^2}$$

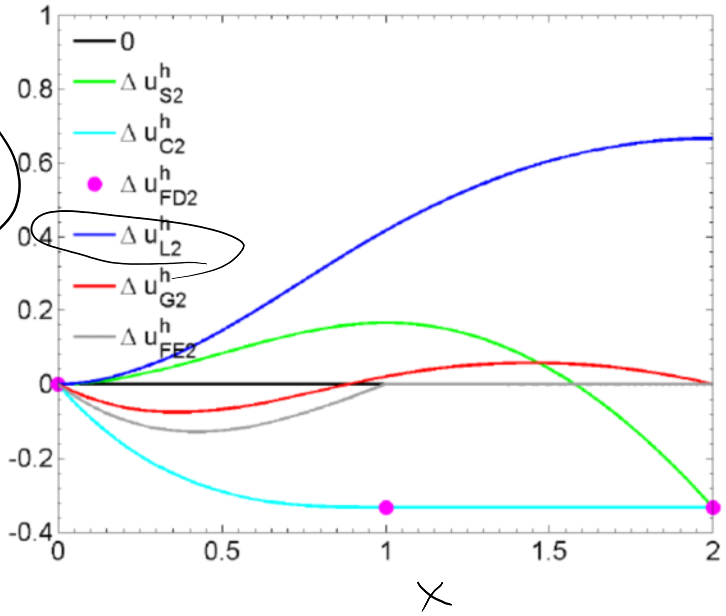
## Bar example, $n = 2$, Comparison of solutions



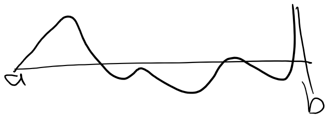
$u^h$

## Bar example, $n = 2$, Comparison of solutions

$u^h - u_{exact}$



$L^2$ norm

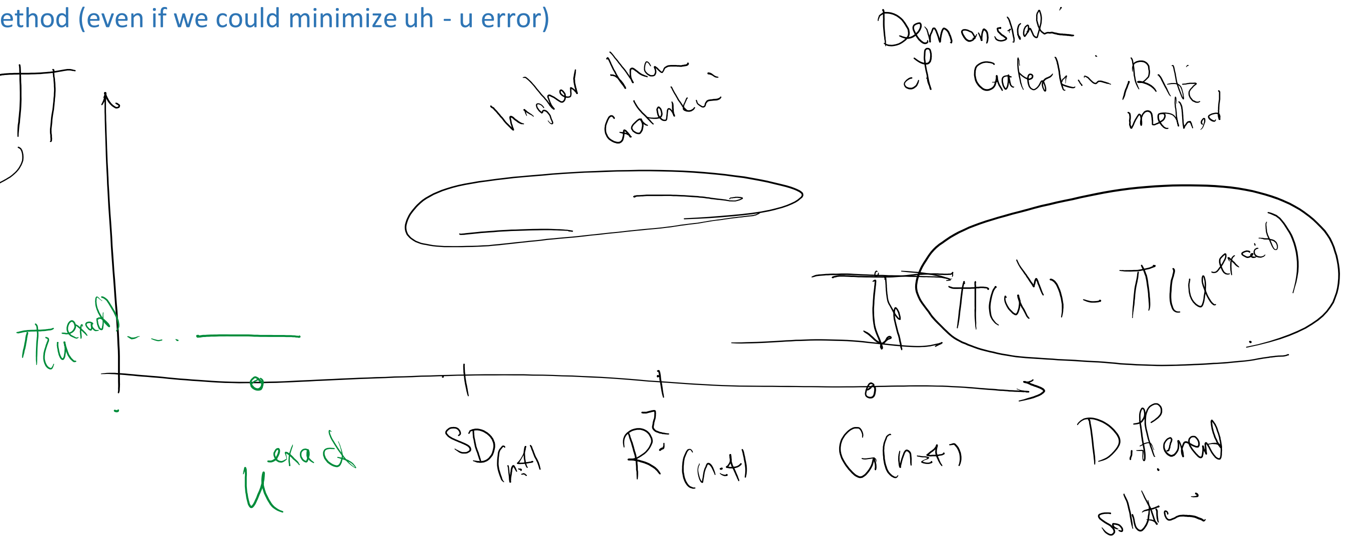


$$L^2_{[a,b]}(f) := \|f\| = \int_a^b (f(x))^2 dx$$

| | | | |
|---|---|---|---|
| $\|u^h - u\|$ | minimum | | |
| $\Pi(u^h)$ | " | Ritz ≡ Galerkin | ✓ |
| error in PDE ~~& BCs~~ natural | " | $R^2$ | |

In fact, in many cases, we still would have gone with minimizing the enregy method (even if we could minimize uh - u error)

FYI $\Pi$ potential energy



$u^h = \phi_p + a_1\phi_1 + \dots + a_n\phi_n$ (e.g $1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$
this kind of solution

$$\text{energy of error} = \Pi(\underline{u^h - u^{exact}})$$

$$= \int_0^l \Delta\varepsilon : \Delta\sigma dV = \int (\varepsilon^h - \varepsilon^{exact}) : EA (\varepsilon^h - \varepsilon^{exact}) dv$$

Galerkin method: — minimizes the $\Pi$ energy (pot.)

— Error of energy is also minimum

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Doing Least square another way.

$$R^2 = \int_0^L R_i^2\,dx + R_f\,(x=L)$$

bar problem

$$u^h = \phi_p + \Sigma a_i \phi_i$$

bar problem

$$R_i = \mathcal{L}_M(u^h) - r \qquad R_i = (EAu^{h\prime})' + q \;:\; \mathcal{L}_M = (EA(\,)')' \qquad r = -q$$

$$R_f = \bar{F} - \mathcal{L}_f(u^h) \qquad R_f = \bar{F} - EAu^{h\prime} \qquad \mathcal{L}_f = EA(\,)'$$

$$R^2(a_1 \ldots a_n) = \int_0^L \left[\mathcal{L}_M(u^h) - r\right]^2 dx + \left(\bar{F} - \mathcal{L}_f(u^h)\right)^2 \Big|_{x=L}$$

$$u^h = \phi_p + \sum_{i=1}^n a_i \phi_i$$

$$\nabla R^2 = 0 \Rightarrow \frac{\partial \nabla^2 R}{\partial a_i} = 0 \;: \quad R^2 = \int_0^L R_I^2\,dx + R_f^2$$

$$\boxed{\frac{\partial R^2}{\partial a_i} = \int_0^L 2 \overset{w}{\frac{\partial R_I}{\partial a_i}} R_I\,dx + 2 R_f \overset{w_f}{\frac{\partial R_f}{\partial a_i}} = 0}$$

$$\frac{\partial R_I}{\partial a_i} = \frac{\partial\, \mathcal{L}_M(u^h) - r}{\partial a_i} = \frac{\partial\, \mathcal{L}_M(a_1\phi_1 + \cdots + a_n\phi_n + \phi_p)}{\partial a_i} = \frac{\partial \sum_{j=1}^n a_j \mathcal{L}_M(\phi_j)}{\partial a_i}$$

$\phi$

$\overline{\partial a_i} \; - \qquad \partial a_i \qquad \partial a_i \qquad \partial a_i$

$$= \frac{\partial \left(a_1 L_M(\phi_1) + a_2 L_M(\phi_2) \cdots + a_n L_M(\phi_n)\right)}{\partial a_i}$$

$$\boxed{\frac{\partial R_I}{\partial a_i} = L_M(\phi_i)} \qquad \boxed{\frac{\partial R_f}{\partial a_i} = -L_f(\phi_i)}$$

$W_i = L_M(\phi_i)$ $\qquad$ $(W_f)_i = -L_f(\phi_i)$