Least Square Method

We want to obtain the discrete solution corresponding to the continuum least square method (51):

$$\text{Find } \mathbf{u} \in \mathcal{V} = \{\mathbf{v} \mid \mathbf{v} \in C^M(\mathcal{D}, L_u(\mathbf{u}) = \bar{\mathbf{u}}\} \text{ such that}$$
$$\int_{\mathcal{D}} \mathcal{R}_i^2(\mathbf{u})\,\mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathcal{R}_f^2(\mathbf{u})\,\mathrm{ds} = \int_{\mathcal{D}} (L_M(\mathbf{u}) - \mathbf{r})^2\,\mathrm{dv} + \int_{\partial\mathcal{D}_f} (\bar{\mathbf{f}} - L_f(\mathbf{u}))^2\,\mathrm{ds} = 0$$

By changing $\mathbf{u}$ to $\mathbf{u}^h \in \mathcal{V}^h$ and minimizing R^2 with respect to solution coefficients a instead of continuum condition $R^2 = 0$, we have

var: $(EA()')'$ $\quad EA)'$

Find $\mathbf{u}^h \in \mathcal{V}^h$ such that (154)

$\forall \tilde{\mathbf{u}}^h \in \mathcal{V}^h: \quad R^2(\mathbf{u}^h) \le R^2(\tilde{\mathbf{u}}^h)$ where

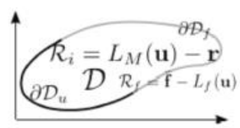
$$R^2(\tilde{\mathbf{u}}^h) = \int_{\mathcal{D}} \mathcal{R}_i^2(\tilde{\mathbf{u}}^h)\,\mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathcal{R}_f^2(\tilde{\mathbf{u}}^h)\,\mathrm{ds} = \int_{\mathcal{D}} (L_M(\tilde{\mathbf{u}}^h) - \mathbf{r})^2\,\mathrm{dv} + \int_{\partial\mathcal{D}_f} (\bar{\mathbf{f}} - L_f(\tilde{\mathbf{u}}^h))^2\,\mathrm{ds}$$

Noting that $\mathcal{V}^h$ is an n-dimensional space (122) and $\mathbf{u}^h = \sum_{j=1}^n a_j\phi_j + \phi_p$ (117a) the minimum condition can be expressed as,

Find $[\mathrm{a}] \in \mathbb{R}^n$ such that (155)

$\forall [\tilde{\mathrm{a}}] \in \mathbb{R}^n: \quad R^2([\mathrm{a}]) \le R^2([\tilde{\mathrm{a}}])$ where

$R^2([\tilde{\mathrm{a}}]) = R^2(\tilde{\mathbf{u}}^h)$ for $\tilde{\mathbf{u}}^h = [\phi]^{\mathrm{T}}[\tilde{\mathrm{a}}] + \phi_p$



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- The minimum condition for the solution $[\mathrm{a}](\mathbf{u}^h = [\phi]^{\mathrm{T}}[\mathrm{a}] + \phi_p)$ in (155) can be expressed as,

$$[\mathrm{a}] \text{ is minimizer } \Rightarrow \frac{\partial R^2}{\partial a_i} = 0 \Rightarrow$$
$$\int_{\mathcal{D}} 2\frac{\partial L_M(\mathbf{u}^h)}{\partial a_i}(L_M(\mathbf{u}^h) - \mathbf{r})\,\mathrm{dv} + \int_{\partial\mathcal{D}_f} (-2)\frac{\partial L_f(\tilde{\mathbf{u}}^h)}{\partial a_i}(\bar{\mathbf{f}} - L_f(\tilde{\mathbf{u}}^h))\,\mathrm{ds} = 0 \quad (156)$$

Linear

- Noting the linearity of L_M and L_f and $[\mathrm{a}](\mathbf{u}^h = [\phi]^{\mathrm{T}}[\mathrm{a}] + \phi_p)$ we observe,

$$L_M(\mathbf{u}^h) = a_i L_M(\phi_i) + \phi_p \Rightarrow \frac{\partial L_M(\mathbf{u}^h)}{\partial a_i} = L_M(\phi_i) \quad (157a)$$
$$L_f(\mathbf{u}^h) = a_i L_f(\phi_i) + \phi_p \Rightarrow \frac{\partial L_f(\mathbf{u}^h)}{\partial a_i} = L_f(\phi_i) \quad (157b)$$

$(f+g)'' = f'' + g''$

$\nabla\cdot((\rho v)\otimes v)$

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$\nabla\cdot(\rho(v_1+v_2)\otimes(v_1+v_2))$

$\neq \nabla\cdot(\rho v_1 \otimes v_1)$

$+ \nabla\cdot(\rho v_2 \otimes v_2)$

- Equations (156) and (157) yield,

$$\forall \phi_i \in \mathcal{V}^h: \int_{\mathcal{D}} L_M(\phi_i)\underbrace{(L_M(\mathbf{u}^h) - \mathbf{r})}_{\mathcal{R}_i}\,\mathrm{dv} + \int_{\partial\mathcal{D}_f} (-L_f(\phi_i))\underbrace{(\bar{\mathbf{f}} - L_f(\tilde{\mathbf{u}}^h))}_{\mathcal{R}_f}\,\mathrm{ds} = 0 \quad (158)$$

- In comparison to (126) for the general statement of weighted residual methods we observe,

$$\forall \mathbf{w} \in \mathcal{W}^h: \int_{\mathcal{D}} \mathbf{w}.\mathcal{R}_i(\mathbf{u}^h)\,\mathrm{dv} + \int_{\partial\mathcal{D}_f} \mathbf{w}^f.\mathcal{R}_f(\mathbf{u}^h)\,\mathrm{ds} = 0$$

- Discrete Least Square problem for linear differential operators L_M and L_f is equivalent to a discrete weighted residual statement with the weight functions:

R^2 method

Weight functions corresponding to Least Square Method

$$\mathbf{w} = L_M(\phi) \quad (159a)$$
$$\mathbf{w}^f = (-L_f(\phi)) \quad (159b)$$

$$\mathbf{w} = L_M(\phi) \qquad (159a)$$
$$\mathbf{w}^f = (-L_f(\phi)) \qquad (159b)$$

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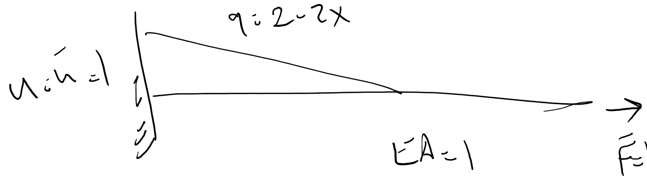
Bar problem:



$$R_i(u) = \mathcal{L}_M(u) - r$$
$$= (EAu')' + q$$

$\mathcal{L}_M = (EA(\,)')'$ $\quad r = -q$

$EA = 1 \implies \boxed{\mathcal{L}_M = (\,)''}$ (1a)

$$R_f = \bar{F} - F = \bar{F} - \mathcal{L}_f(u) = \bar{F} - EA(u)'$$

$\left.\begin{matrix}\mathcal{L}_f = EA(\,)' \\ EA = 1\end{matrix}\right\} \rightarrow \boxed{\mathcal{L}_f = (\,)'}$ (1b)

$$\mathbf{w} = L_M(\phi)^t \qquad (159a)$$
$$\mathbf{w}^f = (-L_f(\phi))^t \qquad (159b)$$

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$$u^h = \phi_p + \phi a = 1 + [x \quad x^2]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

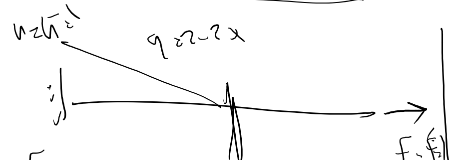
$$W = L_M(\phi)^t = (\phi)^{t\,''} = \left([x \quad x^2]''\right)^t = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\omega_f = (-L_f(\phi))^t = -\begin{bmatrix} x \\ x^2 \end{bmatrix}' = \begin{bmatrix} -1 \\ -2x \end{bmatrix}$$

$$\boxed{\omega = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad \omega_f = \begin{bmatrix} -1 \\ -2x \end{bmatrix}} \quad (2)$$

for R^2 $n = 2$

$$WRS: \quad \int_0^2 \omega R_{Inside}\, dx + \omega_f R_f \Big|_{x=2}$$



$$R_{Inside} = (\underbrace{EA}_{1} u^h)' + q = u^{h\,''} + q = [x \quad x^2]''\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + q$$

& eqn

$$R_{inside} = (\underbrace{EA}_{1} u^{h\prime})' + q = u^{h\prime\prime} + q = [x \;\; x^2]'' \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + q$$

$$R_f = \bar{F} - F = 1 - \underbrace{EA}_{1} u^{h\prime} = 1 - [x \;\; x^2]' \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 1 - [1 \;\; 2x] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

& eqn 2

$$\int_0^2 \begin{bmatrix} 0 \\ 2 \end{bmatrix} \left([0 \;\; 2] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + q \right) dx + \underbrace{\begin{bmatrix} -1 \\ -2x \end{bmatrix}}_{w_f} \left(1 - [1 \;\; 2x] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right) \Bigg|_{x=2} = 0$$

$$\left(\int_0^2 \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} dx \right) \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \int_0^1 \begin{bmatrix} 0 \\ 2 \end{bmatrix} \overbrace{(2-2x)}^{q(x)} dx + \begin{bmatrix} -1 \\ -4 \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ 4 & 16 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ 4 & 24 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1/4 \end{bmatrix}$$

$$\boxed{u^h_{L2} = 1 + 2x - \frac{1}{4} x^2}$$

In R2 method the stiffness matrix is always symmetric (for linear problems)

$$K = \int_0^2 \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}'' [\phi_1 \;\; \phi_2]'' dx + \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}' [\phi_1 \;\; \phi_2]' \Bigg|_{x=2}$$

in general

$$K = \int_D \begin{bmatrix} L_M(\phi_1) \\ \vdots \\ L_M(\phi_n) \end{bmatrix} \left[L_M(\phi_1) \cdots L_M(\phi_n) \right] dV + \int_{\partial\Omega_f} \begin{bmatrix} L_f(\phi_1) \\ \vdots \\ L_f(\phi_n) \end{bmatrix} \left[L_f(\phi_1) \cdots L_f(\phi_n) \right] ds$$

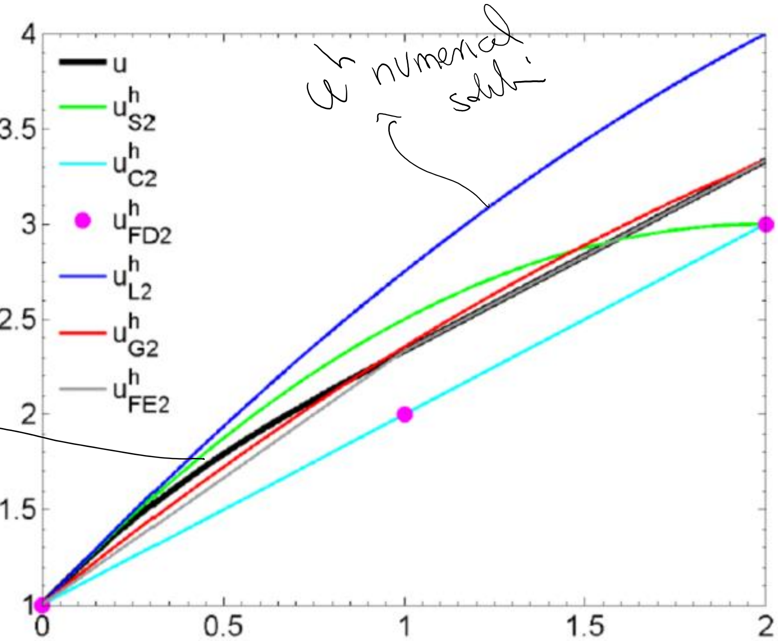
ω

$-\omega_f$

Bar example, $n = 2$, Comparison of solutions

u^h numerical solution

exact
u (u_{exact})



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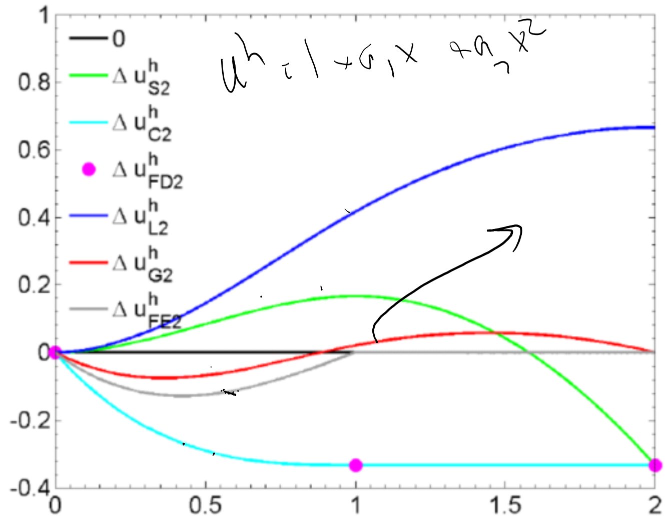
Bar example, $n = 2$, Comparison of solutions



$u^h = 1 + a_1 x + a_2 x^2$

error $= u^h - u$

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Scalar error measures

$$\|u^h - u\|_2 = \sqrt{\int_0^2 (u^h - u)^2 \, dx}$$

$$\left(\|f\|_2 = \sqrt{\int_\Omega f^2 \, dV} \right)$$

Another reference error measure for the this problem

$$V(\tilde{u}) = \int_0^2 \tilde{u}' \underbrace{EA}_{1} \tilde{u}' \, dx$$

internal energy

$$V(\tilde{u}) = \int_0^2 (\tilde{u}')^2 \, dx$$

energy-based error $= V(u^h - u) = \int_0^2 (u^{h\prime} - u')^2 \, dx$

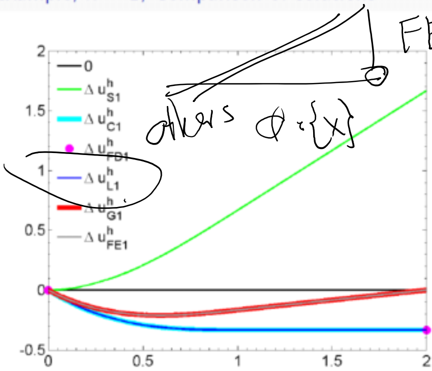
Galerkin $\equiv$ Ritz $\Longrightarrow$ Guaranteed to give the lowest energy error.

We often care about the energy the most

Galerkin is a very reasonable choice

Errors for n = 1 to n = 4

Bar example, $n = 1$, Comparison of solutions



FEM

others $\Phi = \{x\}$

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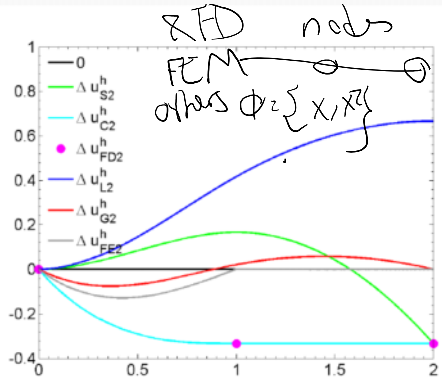
Bar example, $n = 2$, Comparison of solutions



& FD nodes

FEM

others $\Phi = \{x, x^2\}$

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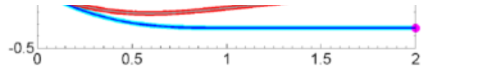
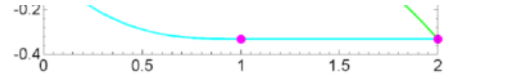
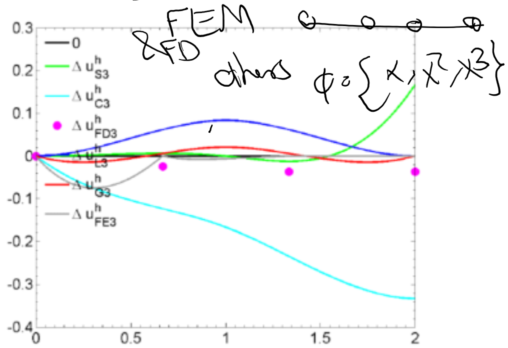




Bar example, $n = 3$, Comparison of solutions

FEM & FD

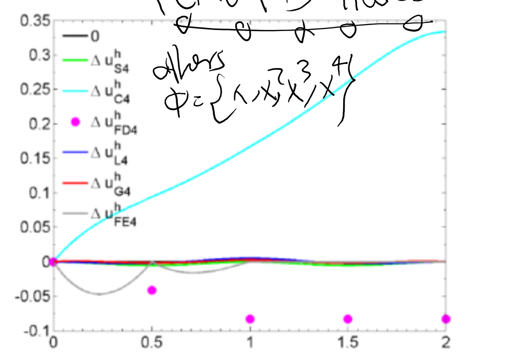
others $\phi = \{x, x^2, x^3\}$



Bar example, $n = 4$, Comparison of solutions

FEM & FD nodes

others $\Phi = \{x, x^2, x^3, x^4\}$

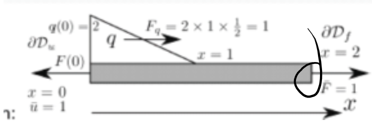


In 1D FEM solution is exact at FEM nodes!

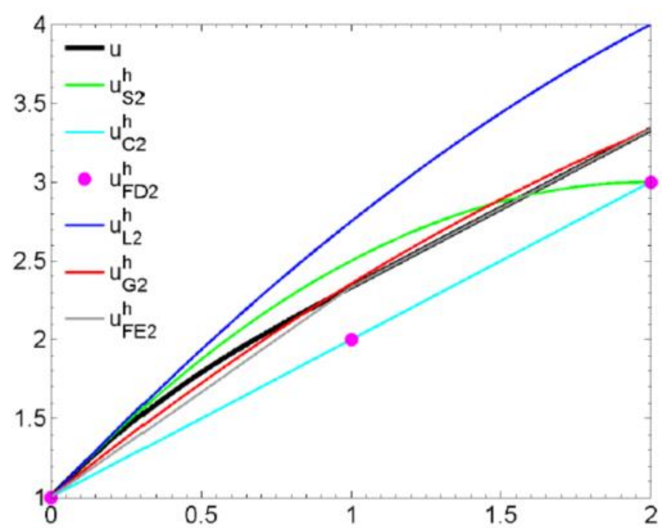
Exact solution

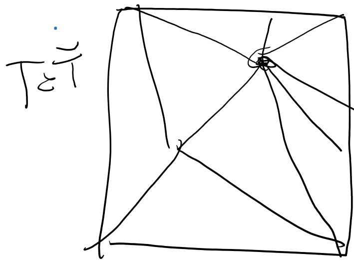
- The exact solution can be summarized as,

$$u(x) = \begin{cases} \frac{x^3}{3} - x^2 + 2x + 1 & 0 \le x \le 1 \\ x + \frac{4}{3} & 1 < x \le 2 \end{cases} \tag{179}$$



Bar example, $n = 2$, Comparison of solutions





we're not getting the exact nodal solutions with FEM in 2D/3D

$$u^h = \phi_p + a_1 \phi_1 + a_2 \phi_2 + a_3 \phi_3$$

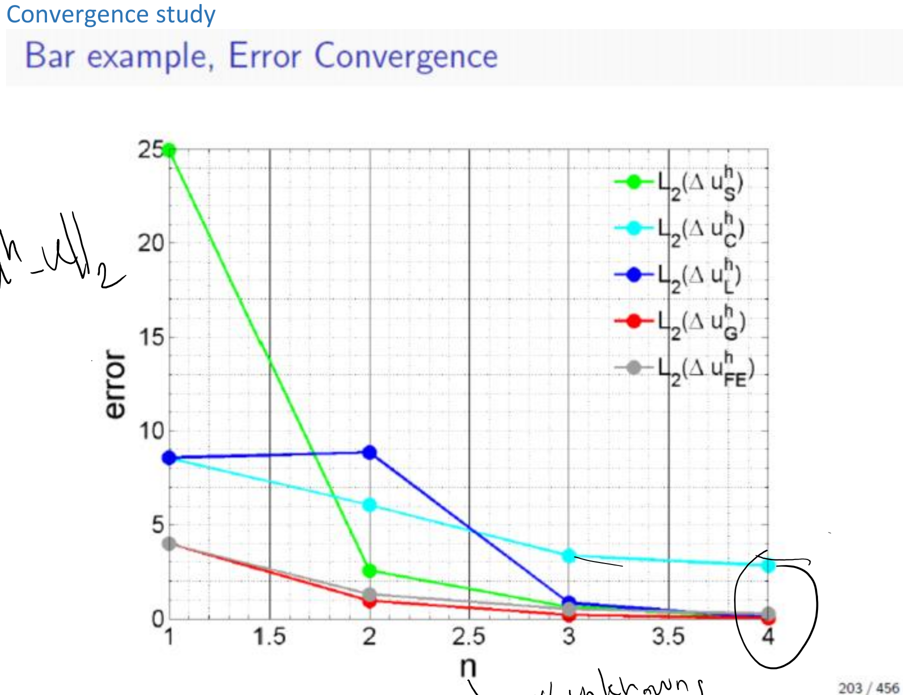
within this solution set LS minimizes $R^2 = \int_{\Omega} R_i^2 \, dx + \int_{\partial \Omega_q} R_q^2 \, ds$

Not $\|u - u^h\|_2$: so the actual error

$\|u - u^h\|_2$ can be larger than other methods

Convergence study

$\|u^h - u\|_2$

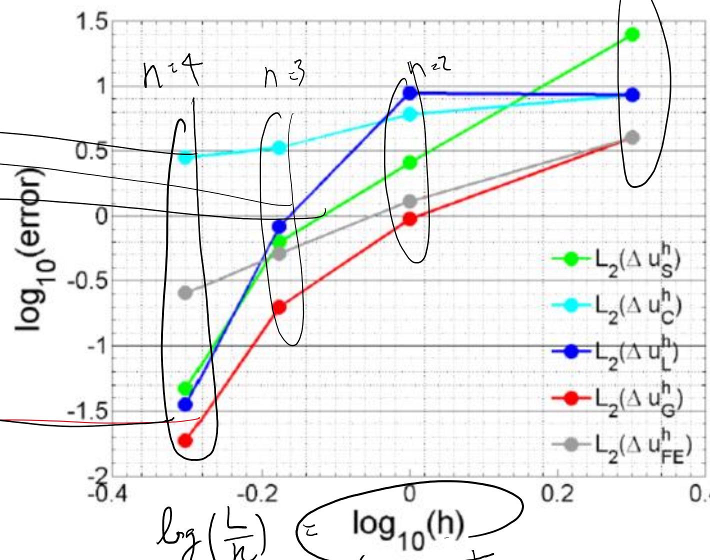


→ # unknowns

Bar example, Error Convergence

$\{x, x^2, x^3, x^4\}$

$n=4$ $n=3$ $n=2$ $n=1$



spectral Galerkin

average element size

FEM

h

L

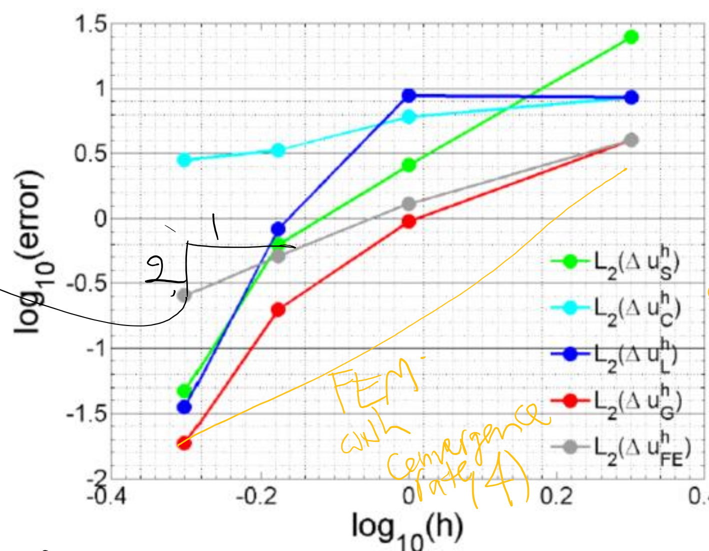
$hn = L$

unknowns for this problem

$\underbrace{\log\left(\frac{L}{n}\right)}_{-\log n} = \underbrace{\log_{10}(h)}_{\text{FEM \& FD}}$

Bar example, Error Convergence

$p=1$ element



2

1

FEM with convergence rate (4)

$p=3$ element

for FEM if the mesh is "fine enough" (we have reached asymptotic convergence rate")

$$\log(\text{error}) = A + 2\log h$$

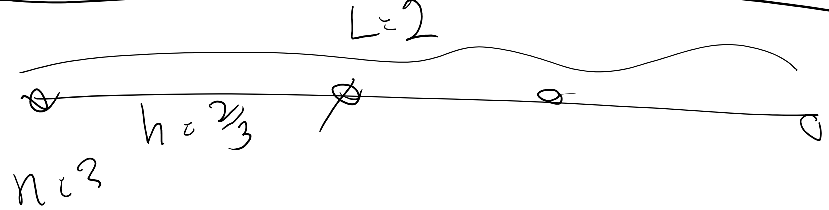
$0, \log(\text{error})$ $A, 2\log(h)$

$$e^{\log(\text{error})} = e^{A + 2\log(h)} \Rightarrow$$

$$\text{error} = \underbrace{(e^{A})}_{C}\, e^{2\log h}$$

$$\boxed{\text{error} = C\,h^2 = C\left(\frac{L}{n}\right)^2}$$

asymptotic eqn for FEM error



FOR FEM

$$\text{error} = C\,h^{ap+b}$$

some numbers (a, b)

element order (p)

$h \to 0$

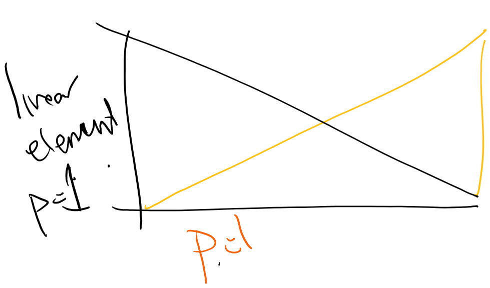
for L^2 error

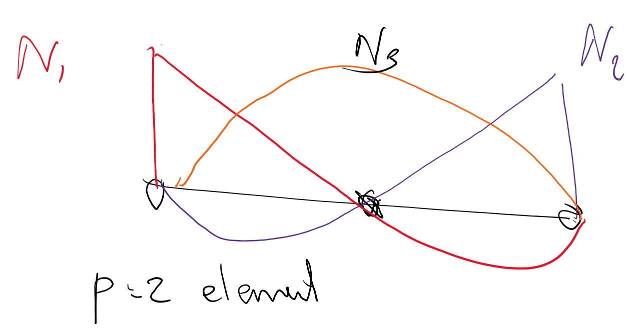
$$\text{error} = C\,h^{p+1}$$

$a = 1$

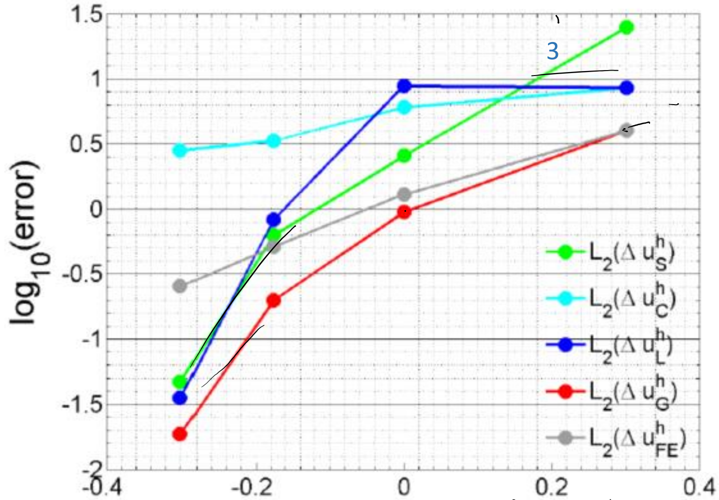
$b = 1$

$h \to 0$





Bar example, Error Convergence



$\underbrace{\log_{10}(h)}_{\text{FEM, FD}} = \underbrace{\log\left(\frac{L}{n}\right)}_{\text{others}}$

FEM | others (spectral)

$n=1$ | $\Phi = \{x\}$

$n=2$ ϕ_1 ϕ_2 linear | $\Phi = \{x, x^2\}$ $p=2$

$h=3$ still piecewise linear | $\Phi = \{x, x^2, x^3\}$ $p=3$

$n=4$ 〃 | $\Phi = \{x, x^2, x^3, x^4\}$ $p=4$

exponential decay of error to zero

Why then we don't use spectral methods?

Bar example, $n = 4$

	S	C	FD
$\mathbf{K}$	$\begin{bmatrix} 0 & 1 & \frac{3}{4} & \frac{1}{2} \\ 0 & 1 & \frac{9}{4} & \frac{7}{2} \\ 0 & 1 & \frac{15}{4} & \frac{19}{2} \\ -1 & -3 & -\frac{27}{4} & -\frac{27}{2} \end{bmatrix}$ X	$\begin{bmatrix} 0 & 2 & 3 & 3 \\ 0 & 2 & 6 & 12 \\ 0 & 2 & 9 & 27 \\ -1 & -4 & -12 & -32 \end{bmatrix}$ X	$\begin{bmatrix} -8 & 4 & 0 & 0 \\ 4 & -8 & 4 & 0 \\ 0 & 4 & -8 & 4 \\ 0 & 0 & 2 & -2 \end{bmatrix}$ X
$\mathbf{F}^T$	$\begin{bmatrix} -\frac{3}{4} & -\frac{1}{4} & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -5 & 0 & 0 & -1 \end{bmatrix}$
$\mathbf{a}^T$	$\begin{bmatrix} 2 & -\frac{13}{12} & \frac{1}{2} & -\frac{1}{12} \end{bmatrix}$	$\begin{bmatrix} \frac{7}{3} & -\frac{3}{2} & \frac{5}{6} & -\frac{1}{6} \end{bmatrix}$	$u_i = \begin{bmatrix} \frac{7}{4} & \frac{9}{4} & \frac{11}{4} & \frac{13}{4} \end{bmatrix}$
	LS	G, Spectral	FE G, FE
$\mathbf{K}$	✓ $\begin{bmatrix} 1 & 4 & 12 & 32 \\ 4 & 24 & 72 & 192 \\ 12 & 72 & 240 & 672 \\ 32 & 192 & 672 & \frac{9728}{5} \end{bmatrix}$ Always symmetric	✓ $\begin{bmatrix} 2 & 4 & 8 & 16 \\ 4 & \frac{32}{3} & 24 & \frac{256}{5} \\ 8 & 24 & \frac{288}{5} & 128 \\ 16 & \frac{256}{5} & 128 & \frac{2048}{7} \end{bmatrix}$	✓ $\begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix}$
$\mathbf{F}^T$	$\begin{bmatrix} 1 & 2 & 10 & 30 \end{bmatrix}$	$\begin{bmatrix} \frac{7}{3} & \frac{25}{6} & \frac{81}{10} & \frac{241}{15} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} & \frac{1}{12} & 0 & 1 \end{bmatrix}$
$\mathbf{a}^T$	$\begin{bmatrix} 2 & -\frac{17}{16} & \frac{23}{48} & -\frac{5}{64} \end{bmatrix}$	$\begin{bmatrix} \frac{97}{48} & -\frac{9}{8} & \frac{17}{32} & -\frac{35}{384} \end{bmatrix}$	$\begin{bmatrix} \frac{19}{24} & \frac{4}{3} & \frac{11}{6} & \frac{7}{3} \end{bmatrix}$

which ones are symmetric

only sym K when the problem/weak statement is self-adjoint

- System Matrix $\mathbf{K}$ is nonsymmetric for Subdomain, Collocation and Finite Difference methods.
- System Matrix $\mathbf{K}$ is always symmetric for Least Square method.
- For this self adjoint problem $\mathbf{K}$ is symmetric for Galerkin methods ($\mathbf{w} = [x^i]$ and FE hat functions).
- Finite Element trial functions are local leading to sparse structure of $\mathbf{K}$ matrix.
- Spectral trial functions are continuous and span the entire domain. The matrix $\mathbf{K}$ is dense.
- Spectral methods have better convergence properties than FE methods, while their use is most often is limited to simple geometries.

Bar example, $n = 4$

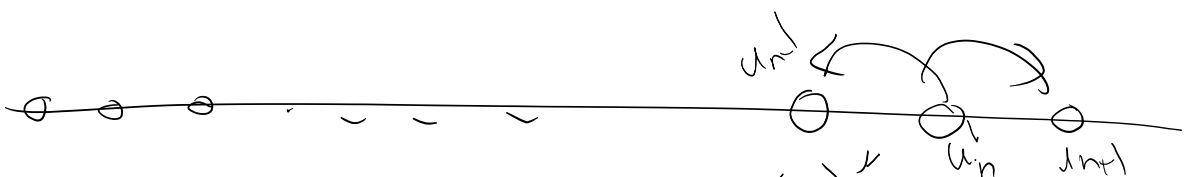
	S	C	FD
	$\begin{bmatrix} 0 & 1 & \frac{3}{4} & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} 0 & 2 & 3 & 3 \end{bmatrix}$	$\begin{bmatrix} -8 & 4 & 0 & 0 \end{bmatrix}$

dense

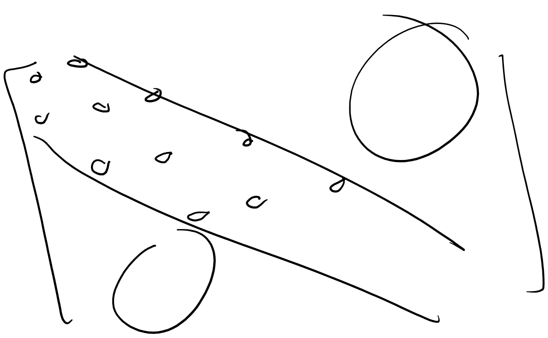
	S	C	FD
$\mathbf{K}$	$\begin{bmatrix} 0 & 1 & \frac{3}{4} & \frac{1}{2} \\ 0 & 1 & \frac{9}{4} & \frac{7}{2} \\ 0 & 1 & \frac{15}{4} & \frac{19}{2} \\ -1 & -3 & -\frac{27}{4} & -\frac{27}{2} \end{bmatrix}$	$\begin{bmatrix} 0 & 2 & 3 & 3 \\ 0 & 2 & 6 & 12 \\ 0 & 2 & 9 & 27 \\ -1 & -4 & -12 & -32 \end{bmatrix}$	$\begin{bmatrix} -8 & 4 & 0 & 0 \\ 4 & -8 & 4 & 0 \\ 0 & 4 & -8 & 4 \\ 0 & 0 & 2 & -2 \end{bmatrix}$
$\mathbf{F}^T$	$\begin{bmatrix} -\frac{3}{4} & -\frac{1}{4} & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -5 & 0 & 0 & -1 \end{bmatrix}$
$\mathbf{a}^T$	$\begin{bmatrix} 2 & -\frac{13}{12} & \frac{1}{2} & -\frac{1}{12} \end{bmatrix}$	$\begin{bmatrix} \frac{7}{3} & -\frac{3}{2} & \frac{5}{6} & -\frac{1}{6} \end{bmatrix}$	$u_i = \begin{bmatrix} \frac{7}{4} & \frac{9}{4} & \frac{11}{4} & \frac{13}{4} \end{bmatrix}$
	LS	G	FE
$\mathbf{K}$	$\begin{bmatrix} 1 & 4 & 12 & 32 \\ 4 & 24 & 72 & 192 \\ 12 & 72 & 240 & 672 \\ 32 & 192 & 672 & \frac{9728}{5} \end{bmatrix}$	$\begin{bmatrix} 2 & 4 & 8 & 16 \\ 4 & \frac{32}{3} & 24 & \frac{256}{5} \\ 8 & 24 & \frac{288}{5} & 128 \\ 16 & \frac{256}{5} & 128 & \frac{2048}{7} \end{bmatrix}$	$\begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix}$
$\mathbf{F}^T$	$\begin{bmatrix} 1 & 2 & 10 & 30 \end{bmatrix}$	$\begin{bmatrix} \frac{7}{3} & \frac{25}{6} & \frac{81}{10} & \frac{241}{15} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} & \frac{1}{12} & 0 & 1 \end{bmatrix}$
$\mathbf{a}^T$	$\begin{bmatrix} 2 & -\frac{17}{16} & \frac{23}{48} & -\frac{5}{64} \end{bmatrix}$	$\begin{bmatrix} \frac{97}{48} & -\frac{9}{8} & \frac{17}{32} & -\frac{35}{384} \end{bmatrix}$	$\begin{bmatrix} \frac{19}{24} & \frac{4}{3} & \frac{11}{6} & \frac{7}{3} \end{bmatrix}$

sparse matrix

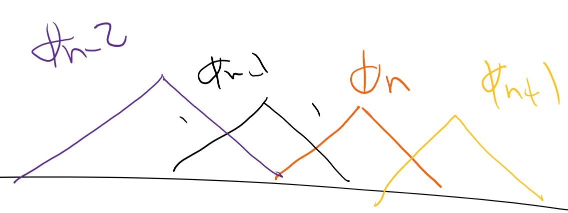
- System Matrix $\mathbf{K}$ is nonsymmetric for Subdomain, Collocation and Finite Difference methods.
- System Matrix $\mathbf{K}$ is always symmetric for Least Square method.
- For this self adjoint problem $\mathbf{K}$ is symmetric for Galerkin methods ($\mathbf{w} = [x^i]$ and FE hat functions).
- Finite Element trial functions are local leading to sparse structure of $\mathbf{K}$ matrix.
- Spectral trial functions are continuus and span the entire domain. The matrix $\mathbf{K}$ is dense.
- Spectral methods have better convergence properties than FE methods, while their use is most often is limited to simple geometries.



$$(\ )'' \simeq \frac{2u_n - u_{n+1} - u_{n-1}}{n^2}$$

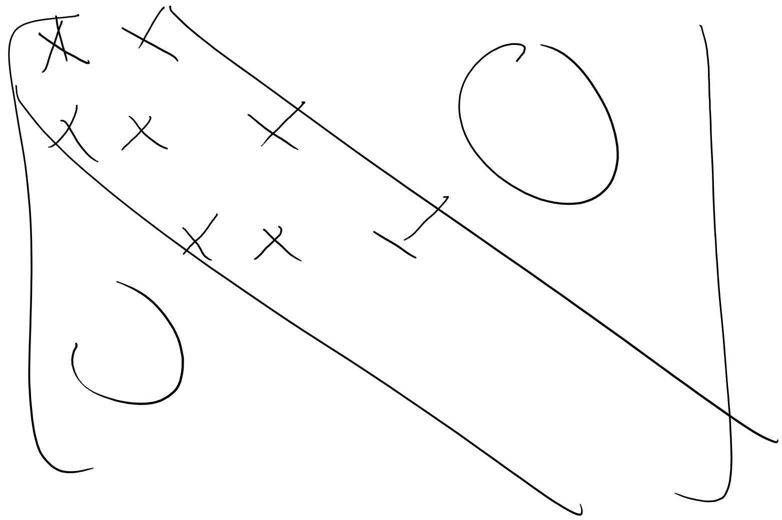


FEM

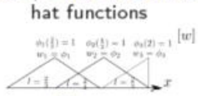
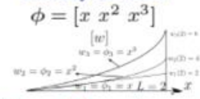
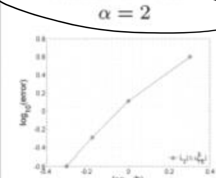
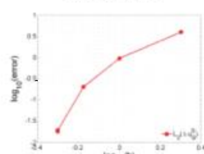


$$K_{n-2,n} = 0$$

like FD



Observations: FE versus spectral methods

Feature	Finite Element	Spectral Methods
Trial Functions Example	Local / Finite Regularity hat functions 	Globally Smooth $\phi = [x\ x^2\ x^3]$ 
Matrix $\mathbf{K}$ Example	Sparse $\begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix}$	Full (diagonal for orthogonal ϕ) $\begin{bmatrix} 2 & 4 & 8 & 16 \\ 4 & \frac{32}{3} & 24 & \frac{256}{5} \\ 8 & 24 & \frac{288}{5} & 128 \\ 16 & \frac{256}{5} & 128 & \frac{2048}{7} \end{bmatrix}$
order of accuracy of u^h (p)	fixed ($e.g.$, $p = 1$)	$\nearrow$ vs. n ($e.g.$, $p = n$)
Convergence Example	Linear: $e = Ch^{\alpha}$ $\alpha = 2$ 	higher than linear exponential 
Geometry	Very general geometries	simple ($e.g.$, rectangular) in practice to get diagonal $\mathbf{K}$

Diagonal matrix for spectral methods

- The global nature of trial functions ϕ in spectral method results in full $\mathbf{K}$ matrices that are expensive to solve.
- To circumvent this problem we employ trial functions that make $\mathbf{K}$ diagonal.
- In weak statement $K_{ij} := \mathcal{A}(\phi_i, \phi_j) = \int_{\mathcal{D}} L_m^w(\phi_i) L_m(\phi_j)\,\mathrm{dv}$.
- If the problem is self-adjoint $\mathcal{A}(.,.)$ is an inner product and we can construct an orthogonal trial function basis ϕ_i for example using Gram Schmidt method.
- Given the particular form of $\mathcal{A}$ (from L_m^w and L_m) and domain of integration $\mathcal{D}$ ([0 1], [−1 1], semi-infinite, infinite, *etc.*) we employ various trigonometric and orthogonal polynomial spaces. Some examples are:
  - $\phi_k(x) = e^{ikx}$ Fourier spectral method.
  - $\phi_k(x) = T_k(x)$ Chebyshev spectral method.
  - $\phi_k(x) = L_k(x)$ or $P_k(x)$ Legendre spectral method.
  - $\phi_k(x) = \mathcal{L}_k(x)$ Laguerre spectral method.
  - $\phi_k(x) = H_k(x)$ Hermite spectral method.

  where $T_k(x)$, $L_k(x)(P_k(x))$, $\mathcal{L}_k(x)$, and $H_k(x)$ are the Chebyshev, Legendre, Laguerre, and Hermite polynomials of degree k, respectively.
- The orthogonal property of these functions is for simple geometries. That is why spectral methods are more popular for simple geometries where we can take advantage of their exponential convergence property while keeping computational costs low by using orthogonal trial functions.

$$\int \phi_k(x)\,\phi_j(x)\,dx = \begin{cases} \neq 0 & k = j \\ = 0 & k \neq j \end{cases}$$

Use a trick

$$K = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

$I a = F$

$a = F$