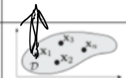
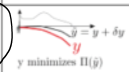
FEM20241016

Wednesday, October 16, 2024 12:52 PM

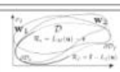
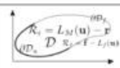
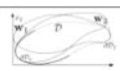
Comparison of different WRMs

Approach	Equation	Figure	Discretization	Discretization method
Balance Law (20)	$\forall \Omega \subset \mathcal{D} : \int_{\partial\Omega}(\mathbf{f}.\mathbf{n})ds - \int_{\Omega} \mathbf{r}\, dv = \mathbf{0}$		Change $\forall \Omega$ to $\{\Omega_1, \Omega_2, \dots, \Omega_n\}$	Similar to subdomain method in WRM
Strong Form (23)	$\forall \mathbf{x} \in \mathcal{D} : \nabla.\mathbf{f} - \mathbf{r} = \mathbf{0}$		Change $\forall \mathbf{x}$ to $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$	Collocation method in WRM. Also FD & FV.
Energy Method (80)	$\forall \tilde{y} \in \mathcal{V} : \Pi(y) \leq \Pi(\tilde{y})$		$\forall\{\tilde{a}_1, \dots, \tilde{a}_n\}$: $\Pi(a_1, \dots, a_n) \leq \Pi(\tilde{a}_1, \dots, \tilde{a}_n) \Rightarrow \frac{\partial \Pi}{\partial a_1} = \dots = \frac{\partial \Pi}{\partial a_n} = 0$	Ritz Energy Method. Also yields Weak Form.

95 / 456

Galerkin

Galerkin

Approach	Equation	Figure	Discretization	Discretization method
Weighted Residual Method (45)	$\forall \mathbf{w} \in \mathcal{W}$: $\int_{\mathcal{D}} \mathbf{w}.\mathcal{R}_i\, dv + \int_{\partial\mathcal{D}_f} \mathbf{w}^f.\mathcal{R}_f\, ds = \mathbf{0}$		Change $\forall \mathbf{w}$ to $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$	Weighted Residual Method (WRM)
Least Square (51)	$R^2 = \int_{\mathcal{D}} \mathcal{R}_i^2\, dv + \int_{\partial\mathcal{D}_f} \mathcal{R}_f^2\, ds = 0$ Always sym. K		Change $R^2 = 0$ to $\forall\{\tilde{a}_1, \dots, \tilde{a}_n\}$: $R^2(a_1, \dots, a_n) \leq R^2(\tilde{a}_1, \dots, \tilde{a}_n) \Rightarrow \frac{\partial R^2}{\partial a_1} = \dots = \frac{\partial R^2}{\partial a_n} = 0$	Least Square method, a WRM for linear L_M (& L_f).
Weak Form (74)	$\forall \mathbf{w} \in \mathcal{W}$ $\int_{\mathcal{D}} L_m^w(\mathbf{w}) L_m(\mathbf{u})\, dv = \int_{\mathcal{D}} \mathbf{w}.\mathbf{r} dv + \int_{\partial\mathcal{D}_f} \mathbf{w}.\bar{\mathbf{f}} ds$		Change $\forall \mathbf{w}$ to $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$	Weak Formulation

96 / 456

$w_i = L_M(\phi_i)$

$w_{f_i} = -L_f(\phi_i)$

Elastostatic

WRS

$V = \{ f \in C^2$ (too strong) $\mid f(x) = \bar{u}(x)$ on $\partial\Omega_u \}$

essential BC

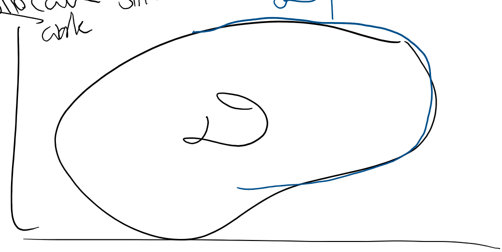
$W = \{ f \in C^0 \mid \text{———} \}$

subdomain & collocation still work

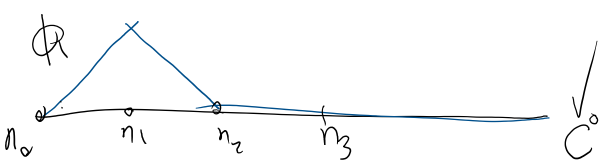
WK

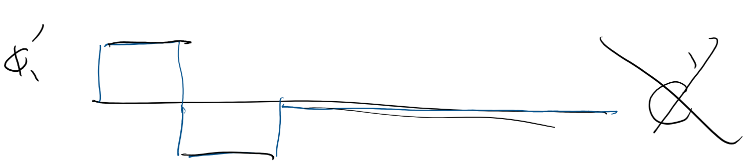
$V = \{ f \in \cancel{C^1}\ C^0 \mid f(x) = \bar{u}(x)$ on $\partial\Omega_u \}$

$W = \{ f \in \cancel{C^1}\ C^0 \mid f(x) = 0$ on $\partial\Omega_u \}$



$f \in C^1(\Omega)$ if f & f' are continuous





PDE order $M = 2m$

diff operator for weak statement

bar $\frac{d}{dx}\left(EA \frac{du}{dx}\right)$ $M = 2$ $m = 1$ $L_m = (\ )$

beam $\frac{d^2}{dx^2}\left(EI \frac{d^2y}{dx^2}\right)$ $M = 4$ $m = 2$ $L_m : (\ )$

in the weak statement we need C^{m-1}

Appendix: Function spaces (optional)

C^k function spaces

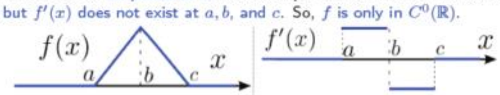
optional

- We define the function spaces

$$C^k(\mathcal{D}) = \{f |\ f \text{ and } \frac{\partial^i f}{\partial \mathbf{x}^i} \text{ exist and are continuus } \forall 0 < i \le k \ \wedge \mathbf{x} \in \mathcal{D}\} \tag{274}$$

$$C^0(\mathcal{D}) = \text{ continuus functions on } \mathcal{D}$$
$$C^1(\mathcal{D}) = \text{ functions with continuus derivative on } \mathcal{D}$$
$$C^\infty(\mathcal{D}) = \text{ infinitely differentiable function on } \mathcal{D}$$

- These conditions are for all points in $\mathcal{D}$. For example the hat function below function f is continuus but $f'(x)$ does not exist at $a, b,$ and c. So, f is only in $C^0(\mathbb{R})$.



We define the corresponding bounded C spaces as,

$$C_b^k(\mathcal{D}) = \{f | f \in C^k(\mathcal{D}), \forall \mathbf{x} : |f(\mathbf{x})| < \infty\} \tag{275}$$

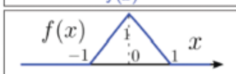
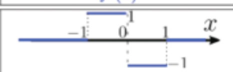
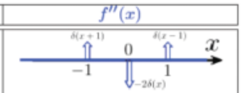
- Clearly,

$$C^{k+1}(\mathcal{D}) \subset C^k(\mathcal{D})$$
$$C_b^k(\mathcal{D}) \subset C^k(\mathcal{D})$$

- For closed sets $\mathcal{D}$ ($\mathcal{D} = \bar{\mathcal{D}}$) $C_b^k(\mathcal{D}) = C^k(\mathcal{D})$. For example $\mathcal{D}_1 = (0,1) = \{x |\ 0 < x < 1\}$ is open while $\mathcal{D}_2 = [0,1] = \{x |\ 0 \le x \le 1\}$ is closed. Function $f(x) = \frac{1}{x}$ is in $C^\infty(\mathcal{D}_1)$ but not $C_b^\infty(\mathcal{D}_1)$.

Comparison of C^k and Sobolev spaces

optional

$f(x)$	$f'(x)$	$f''(x)$
		
$C^0(\mathbb{R})$	$C^1(\mathbb{R})$	$C^2(\mathbb{R})$
Yes	No no derivatives at $\{-1, 0, 1\}$	No not a C^0
$H^0(\mathbb{R}) = L^2(\mathbb{R})$	$H^1(\mathbb{R})$	$H^2(\mathbb{R})$
Yes $\int_{-\infty}^{\infty}(f(x))^2\,dx = \frac{2}{3} < \infty$	Yes $\int_{-\infty}^{\infty}(f(x))^2\,dx = \frac{2}{3} < \infty$	No $\int_{-\infty}^{\infty}(f(x))^2\,dx = \frac{2}{3} < \infty$
	$\int_{-\infty}^{\infty}(f'(x))^2\,dx = 2 < \infty$	$\int_{-\infty}^{\infty}(f'(x))^2\,dx = 2 < \infty$
		$\int_{-\infty}^{\infty}(f''(x))^2\,dx =$ $\int_{-\infty}^{\infty}(\delta(x+1))^2\,dx +$ $\int_{-\infty}^{\infty}(2\delta(x))^2\,dx +$ $\int_{-\infty}^{\infty}(\delta(x-1))^2\,dx$ Not Defined

- So, the hat function $f(x)$ is $C^0(\mathbb{R})$ (continuus) but not $C^1(\mathbb{R})$.
- $f(x)$ is $H^1(\mathbb{R})$ but not $H^2(\mathbb{R})$.
- Sobolev's theorem:

$$\boxed{H^{k+1}(\mathcal{D}) \subset C_b^k(\mathcal{D})} \qquad (280)$$

- As an example $f \in H^1(\mathbb{R}) \quad \Rightarrow f \in C_b^0(\mathbb{R})$

FYI

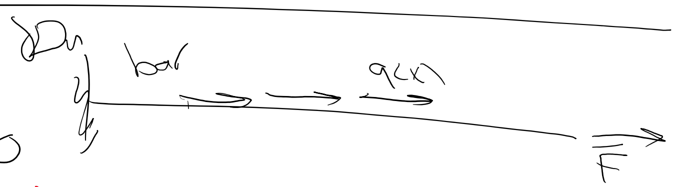
Galerkin Weak Statement Function spaces

optional

1. We first reduce the highest derivative order $M = 2m$ in the strong form (and weighted residual statement) to m in the weak statement.
2. Next, we observe that the functions should only be in $H^m(\mathcal{D})$. We observed that $H^m(\mathcal{D}) \subset C^{m-1}(\mathcal{D})$. In practice, the finite element trial functions that are in $C^{m-1}(\mathcal{D})$ are also $H^m(\mathcal{D})$.

Conventional (continuus) finite element methods:

Strong Form order $M = 2m \quad \Rightarrow$
Trial functions are $\quad C^{m-1}$



$$(EAu')' + q = 0$$

$$V = \{ f \in H^1((0,L)) \mid u(0) = \bar{u} \}$$

$$W = \{ f \in H^1((0,L)) \mid u(0) = 0 \}$$

$$\int_0^L w' EAu'\,dx = \int_0^L w q\,dx + w(L)\bar{F}$$

optional

1D elements

Element types:

1. 1D solid bar element.
2. Truss element.

Concepts:

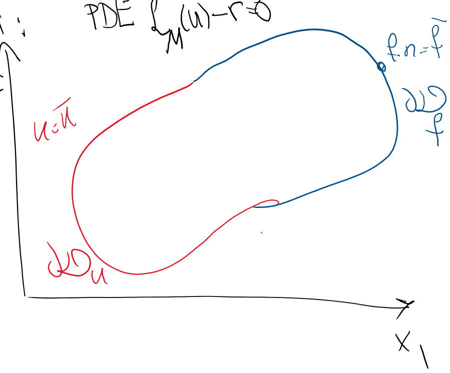
1. Global (weighted residual) vs local (element level) perspectives.
2. Stiffness matrix.
3. Forces: 1.Source term; 2.Natural BC; 3.Essential BC, 4.Nodal.
4. Nodes, elements, shape function, dof.
5. Nodes with more than one dof (truss).
6. Element local coordinate system ξ (bar).
7. Rotation of element local coordinate system (truss).
8. Full stiffness $\mathbf{K}$ (free + prescribed dofs) vs (free only dofs) $\mathbf{K}_{ff}$.
9. High order differential equations (*e.g.*, C^1 beam elements).
10. Multiphysics coupling (beams: axial, bending, & torsional coupling).

225 / 456

general weak statement:

self-adjoint $\Longrightarrow$ K will be symmetric

PDE $\mathcal{L}_M(u) - r = 0$



$$\int_D \mathcal{L}_m(w)\, D\, \mathcal{L}_m(u)\, dV = \underbrace{\int_D w\, r\, dV}_{\text{cont. from source term}} + \underbrace{\int_{\partial D_f} w \bar{f}\, ds}_{\text{cont. from natural BC flux } \bar{F}}$$

$\mathcal{L}_m$: weak statement Diff operator

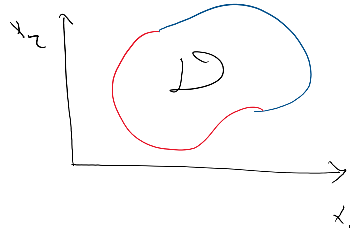
D: material / section property

Weak statement Diff operator	$\mathcal{L}_m$	D	LHS
Bar	$(\ )'$	EA	$\int_0^L w' EA u' \, dx$
Beam	$(\ )''$	EI	$\int_0^L w'' EI y'' \, dx$
2D/3D heat conduction	∇	k	$\int_D \nabla w \, k \, \nabla T \, dV$
2D/3D elasticity	$\frac{\nabla + \nabla^T}{2}$	C	$\int_D \varepsilon(w) \underbrace{C \varepsilon(u)}_{\sigma(u)} dV$

$$\varepsilon(u) = \frac{\nabla u + (\nabla u)^T}{2}$$

Now, we can derive general stiffness and force vector equations for different self-adjoint PDEs:

$$\mathcal{A}(w,u) = \int_{\mathcal{D}} \mathcal{L}_m(w)\, D\, \mathcal{L}_m(u)\, dV \qquad \text{bilinear form}$$



$$(f,g) = \int_{\mathcal{D}} f g \, dV \quad \Big| \text{ eg } (w,r) = \int_{\mathcal{D}} w r \, dV$$

$$\underbrace{(f,g)_N}_{\text{Neumann / Natural BC}} = \int_{\partial \mathcal{D}_f} f g \, ds$$

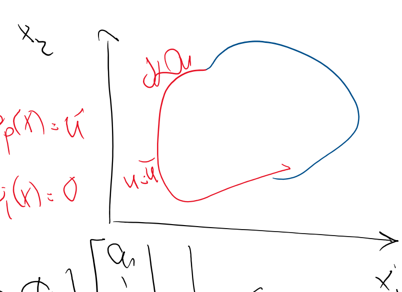
$$\boxed{\begin{aligned} \int_{\mathcal{D}} \mathcal{L}_m(w) D \mathcal{L}_m(u)\, dV &= \int_{\mathcal{D}} w r \, dV + \int_{\partial \mathcal{D}_f} w \bar{F} \, ds \\ \mathcal{A}(w,u) &= (w,r) + (w,\bar{F})_N \end{aligned}} \qquad (1)$$

why $\mathcal{A}$ is called bilinear

always hold $\longleftarrow$ $\mathcal{A}(w_1 + \alpha w_2, u) = \mathcal{A}(w_1, u) + \alpha \mathcal{A}(w_2, u)$

$\mathcal{A}(w, \underbrace{u_1 + \alpha u_2}_{\text{solution}}) = \mathcal{A}(w, u_1) + \alpha \mathcal{A}(w, u_2)$ only holds for linear problem

Now, we want to derive the stiffness matrix and force vectors



$$u^h(x) = \phi_p(x) + \sum_{i=1}^{n} \phi_i(x) a_i \qquad \forall x \in \partial \mathcal{D}_u \quad \phi_p(x) = \bar{u}, \quad \forall i \quad \phi_i(x) = 0$$

$$u^h = \phi_p + \Phi a = \phi_p + [\phi_1 \cdots \phi_n] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \qquad (2)$$

$$\ldots \; \phi_p \; [\phi_1 \; \cdots \; \phi_n] \begin{bmatrix} \vdots \\ a_n \end{bmatrix} \qquad (2)$$

$$w = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_n \end{bmatrix} = \phi^T \quad \text{Galerkin} \qquad (3)$$

Weak statement

for $w_i = \phi_i$ $\quad i = 1 \ldots n \quad$ we have

$$\mathcal{A}(w_i, u^h) = (w_i, r) + (w_i, \bar{F})_N \qquad \text{(eqn 1)}$$

plug eq2 $\quad u^h = \phi_p + \sum_{j=1}^{n} \phi_j a_j$

$$\mathcal{A}\Big(w_i, \underbrace{\phi_p + \sum_{j=1}^{n} a_j \phi_j(x)}\Big) = (w_i, r) + (w_i, \bar{F})_N \qquad / \; w_i = \phi_i$$

$$\mathcal{A}(\phi_i, \phi_p) + \sum_{j=1}^{n} a_j \mathcal{A}(\phi_i, \phi_j) = (\phi_i, r) + (\phi_i, \bar{F})_N$$

$$\left[\mathcal{A}(\phi_i, \phi_1) \;\; \mathcal{A}(\phi_i, \phi_2) \cdots \mathcal{A}(\phi_i, \phi_n)\right] \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_n \end{bmatrix} = -\mathcal{A}(\phi_i, \phi_p) + (\phi_i, r) + (\phi_i, \bar{F})_N$$

$\forall i = 1, \ldots n$

write this for all i (column j)

$$\begin{array}{c} \\ \\ \text{row } i \\ \\ \end{array} \underbrace{\begin{bmatrix} \mathcal{A}(\phi_1,\phi_1) & \cdots & & \mathcal{A}(\phi_1,\phi_n) \\ \mathcal{A}(\phi_2,\phi_1) & \mathcal{A}(\phi_2,\phi_2) & & \\ & & \mathcal{A}(\phi_i,\phi_j) & \\ & & & \mathcal{A}(\phi_n,\phi_n) \end{bmatrix}}_{\text{stiffness matrix}} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = - \underbrace{\begin{bmatrix} \mathcal{A}(\phi_1,\phi_p) \\ \vdots \\ \mathcal{A}(\phi_n,\phi_p) \end{bmatrix}}_{F_D} +$$

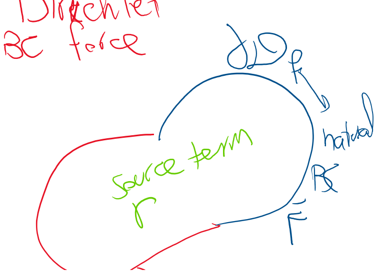
row i: $\quad (F_D)_i = \mathcal{A}(\phi_i, \phi_p) = \int_\Omega \mathcal{L}_m(\phi_i) \, D \, \mathcal{L}_m(\phi_p) \, dv$

F_D Dirichlet BC force

$$\begin{array}{c} \\ i \\ \\ \end{array} \begin{bmatrix} (\phi_1, r) \\ \vdots \\ (\phi_n, r) \end{bmatrix} + \begin{bmatrix} (\phi_1, \bar{F})_N \\ \vdots \\ (\phi_n, \bar{F})_N \end{bmatrix}$$

row i: $\int \phi_i \bar{F} \, ds$

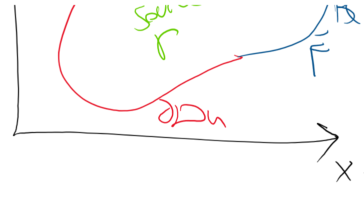
x_2



$L(\phi_n, r)$ $\quad$ $(\phi_n, \bar{F})_N$

F_r: source term force $\quad$ F_N natural BC force

$\int_{\mathcal{D}} \phi_i \, r \, dV$ $\quad$ $\int_{\partial \mathcal{D}_f} \phi_i \bar{F} \, ds$



Weak statement for finite element formulation

optional

- Let the weak statement for a self-adjoint problem be of the form (why the problem is self-adjoint? Change $\mathbf{u}$ and $\mathbf{w}$ on the LHS):

$$\int_{\mathcal{D}} L_m(\mathbf{w}) \mathbf{D} L_m(\mathbf{u}) \, \mathrm{dv} = \int_{\mathcal{D}} \mathbf{w r} \, \mathrm{dv} + \int_{\partial \mathcal{D}_f} \mathbf{w} \bar{\mathbf{F}}.\mathbf{N} \, \mathrm{ds} \tag{336}$$

for example for solid bar we have:

$$\int_0^L \frac{\mathrm{d}w}{\mathrm{d}x} EA \frac{\mathrm{d}u}{\mathrm{d}x} \, \mathrm{d}x = \int_0^L wq \, \mathrm{d}x + (w.\bar{F})_{\partial \mathcal{D}_f}$$

where $\mathcal{D} = [0 \ L]$, $\mathbf{D} = EA$, $L_m = \frac{\mathrm{d}}{\mathrm{d}x}$ and $\partial \mathcal{D}_f$ is either $\{0\}$ or $\{L\}$ (since at least one of these points should be essential BC to prevent rigid body motion for statics, not both points can be in $\partial \mathcal{D}_f$ simultaneously).

- Recalling our general definitions from (288):

$$\mathcal{A}(\mathbf{w}, \mathbf{u}) := \int_{\mathcal{D}} L_m(\mathbf{w}) \mathbf{D} L_m(\mathbf{u}) \, \mathrm{dv} \qquad \text{bilinear form} \tag{337a}$$

$$(\mathbf{w}, \mathbf{r})_r := \int_{\mathcal{D}} \mathbf{w}.\mathbf{r} \, \mathrm{dv} \qquad \text{linear force from source terms} \tag{337b}$$

$$(\mathbf{w}, \bar{\mathbf{F}})_N := \int_{\partial \mathcal{D}_f} \mathbf{w} \bar{\mathbf{F}}.\mathbf{N} \, \mathrm{ds} \qquad \text{linear force from natural BC} \tag{337c}$$

286 / 456

- The weak statement can be rewritten as,

$$\mathcal{A}(\mathbf{w}, \mathbf{u}) = (\mathbf{w}, \mathbf{r})_r + (\mathbf{w}, \bar{\mathbf{F}})_N \tag{338}$$

- Let us discretize this problem to n_{dof} number of dofs, from which n_{f} are free (unknown) and n_{p} are prescribed (known). The corresponding shape vectors are:

$$\mathbf{N} = [\mathbf{N}_1, \ldots, \mathbf{N}_{n_{\mathrm{f}}}] \tag{339a}$$

$$\bar{\mathbf{N}} = [\bar{\mathbf{N}}_{\bar{1}}, \ldots, \bar{\mathbf{N}}_{\bar{n}_{\mathrm{p}}}] \tag{339b}$$

$$Ka = -F_D + F_r + F_N$$

K: $\begin{bmatrix} \mathcal{A}(\phi_1, \phi_1) & \\ & \end{bmatrix}$ $\quad$ $K_{ij} = \mathcal{A}(\phi_i, \phi_j)$

$$K = \begin{bmatrix} & & \\ & & \\ & & \mathcal{A}(\phi_n,\phi_n) \end{bmatrix} \qquad K_{ij} = \mathcal{A}(\phi_i, \phi_j) = \int \mathcal{L}_m(\phi_i)\, D\, \mathcal{L}_m(\phi_j)\, dV$$

$$F_D = \begin{bmatrix} \mathcal{A}(\phi_1, \phi_p) \\ \vdots \\ \mathcal{A}(\phi_n, \phi_p) \end{bmatrix} \quad \text{Dirichlet BC} \qquad F_{D_i} = \int \mathcal{L}_m(\phi_i)\, D\, \mathcal{L}_m(\phi_p)\, dV$$

$$F_r = \begin{bmatrix} (\phi_1, r) \\ \vdots \\ (\phi_n, r) \end{bmatrix} \quad \text{source term} \qquad F_{r_i} = \int_D w\, r\, dV$$

$$F_N = \begin{bmatrix} (\phi_1, \bar{F})_N \\ \vdots \\ (\phi_n, \bar{F})_N \end{bmatrix} \quad \text{Neumann BC} \qquad F_{N_i} = \int_{\partial D_f} w\, \bar{F}\, ds$$

③

K & F or a Galerkin method

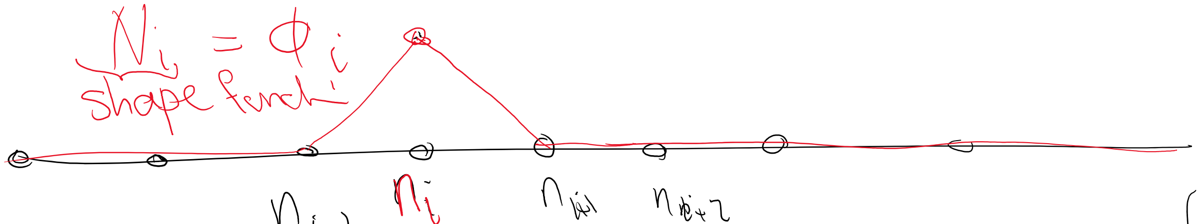
FEM or spectral

$$\phi = \{x, x^2, x^3, \dots\}$$

FEM we have a special Basis funct.

$N_i = \phi_i$

shape funct.



$$N_i(n_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$K_{ij} = \int_{\mathcal{D}} \underbrace{\mathcal{L}_m(\phi_i)}_{B_i} D\, \mathcal{L}_m(\phi_j)\, dV$$

$$N = [N_1, \dots, N_n]$$ vector of shape functions

$$B = [\mathcal{L}_m(N_1) \dots, \mathcal{L}_m(N_n)]$$ B vector (or displacement to strain vector)

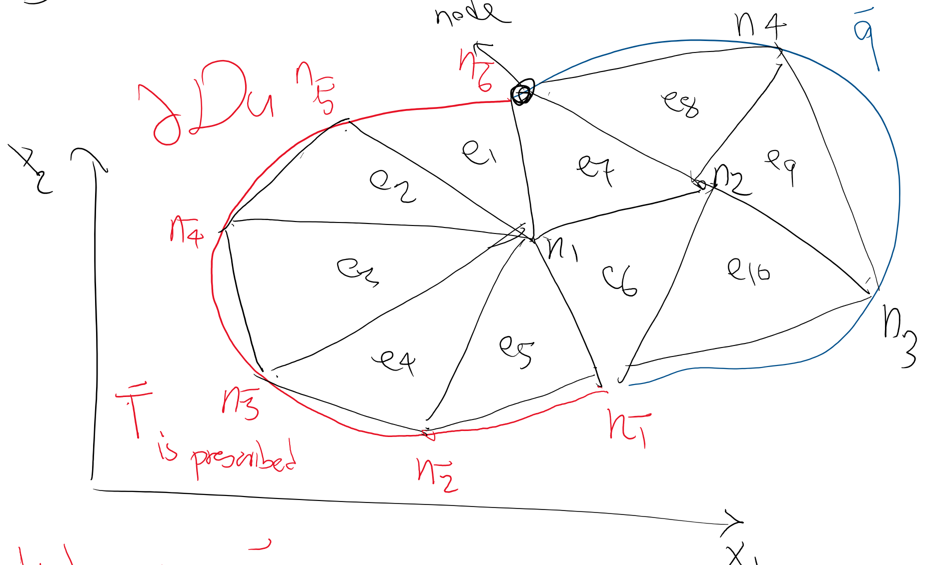
ϕ_p, F_Γ, F_D, F_N in FEM, element

2D heat conduction

$n = 10$ # nodes

$n_e = 10$

$\underbrace{n_p = 6}$ #nodes where primary field (eg Temperature) is prescribed

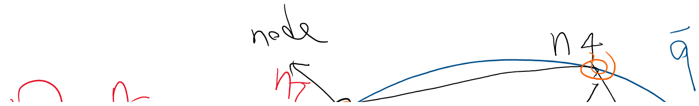


(bar($\bar{\ }$) #s or negative #'s)

$n_F = 4$

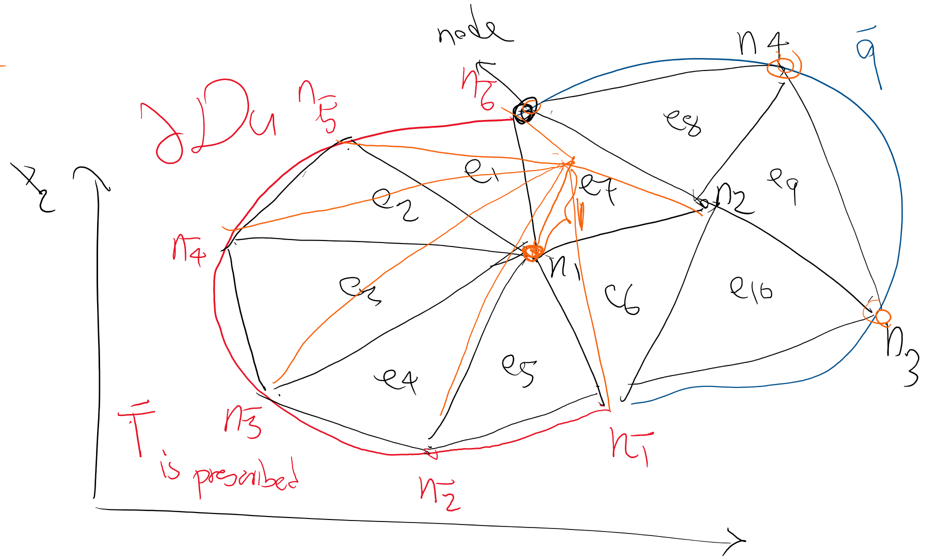
#nodes where primary fields is unknown (regular #'s)

Shape function

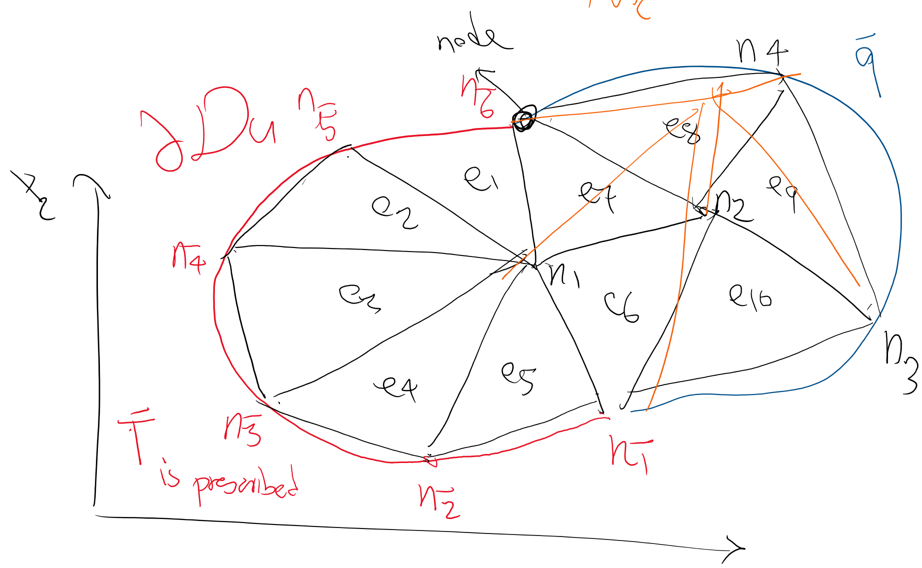


shape function

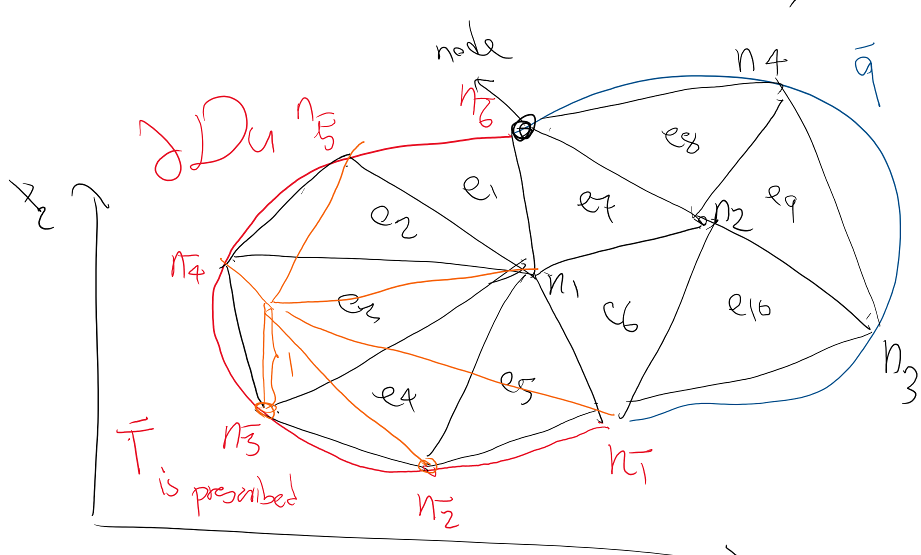
Φ_1



N_2



N_3



$N_{\bar{1}}$ to $N_{\bar{6}}$ will be used to form ϕ_p

[illegible] used to form ϕ_p