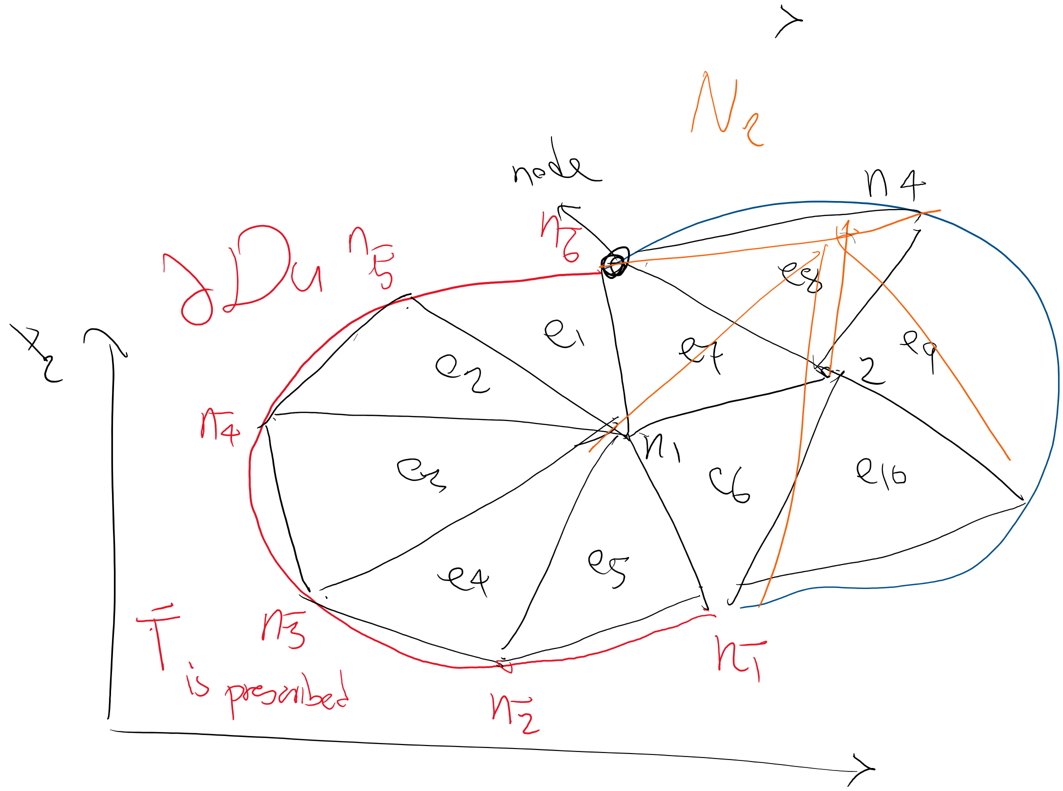
FEM20241021

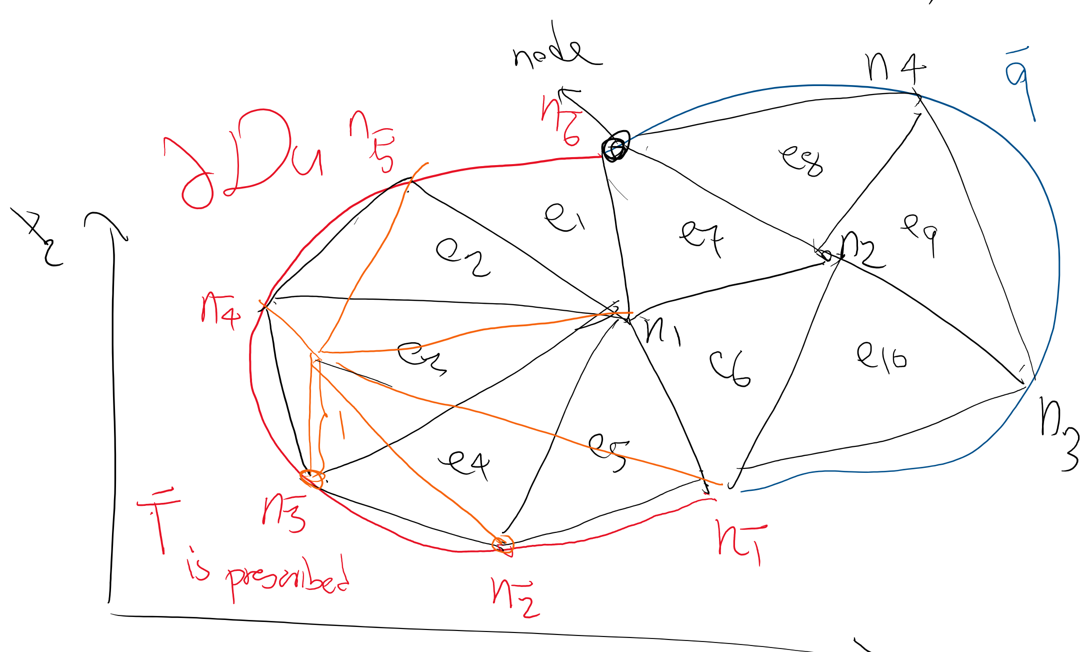
Monday, October 21, 2024 3:52 PM

$n_p = 6$

$n_f = 6$

$n = n_p + n_f = 12$





N_1 to N_6 will be used to form ϕ_p

Recall

basis function ϕ_i $\longrightarrow$ FEM shape function N_i

Recall $\quad$ basis function $\phi_i \longrightarrow$ shape function N_i

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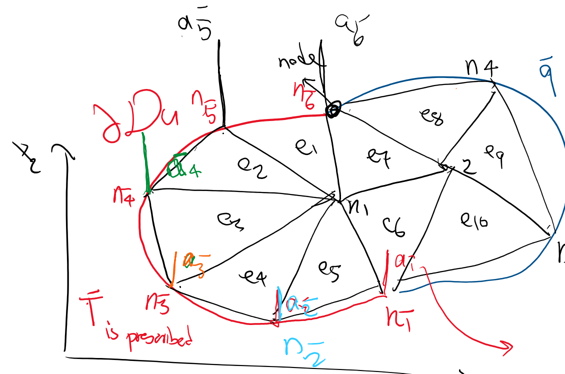
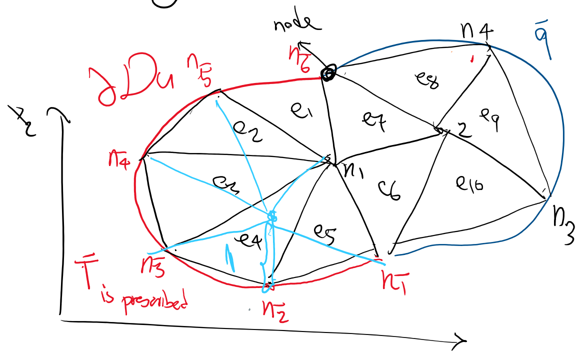
Dirichlet BC (phi_p) using shape functions:

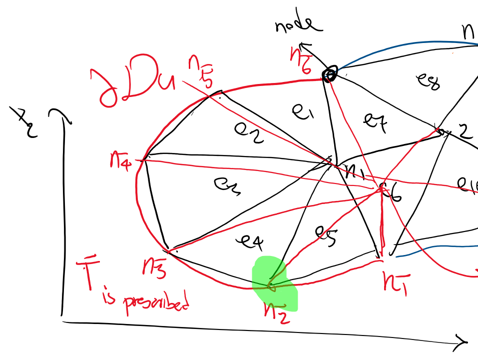


temperature at $n_{\bar 1}$

$$\phi_p(x) = a_{\bar 1} N_{\bar 1}(x) + a_{\bar 2} N_{\bar 2}(x) + \cdots + a_{\bar 6} N_{\bar 6}(x)$$

$$\phi_p(n_{\bar 2}) = a_{\bar 1} \cancel{N_{\bar 1}(n_{\bar 2})}^{0} + a_{\bar 2} N_{\bar 2}(n_{\bar 2})^{1} + a_{\bar 3} \cancel{N_{\bar 3}(n_{\bar 2})}_{0} + \cdots + a_{\bar 6} \cancel{N_{\bar 6}(n_{\bar 2})}_{0}$$





$N_{\bar 1}(x)$

$$\phi_p(n_{\bar 2}) = a_{\bar 2}$$

Similarly: for $i = 1, \ldots, 6$ $\quad \phi_p(n_{\bar i}) = a_{\bar i}$

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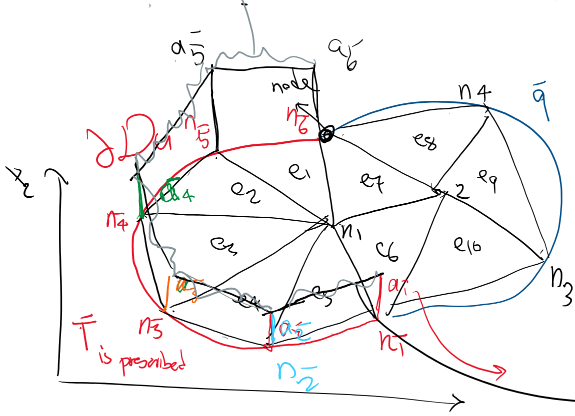
exact $\bar T$ on $\partial\Omega_u$



we needed

$\forall x \in \partial\Omega_u \quad \phi_p(x) = \bar T(x)$

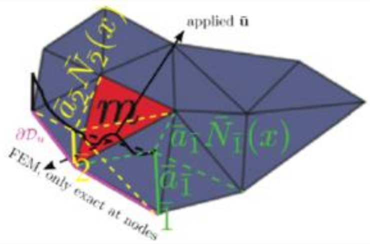
But we only satisfy this condition @ prescribed nodes!

FEM

$\phi_p(x)$

$\Phi_p(x)$

B. Essential Boundary Conditions



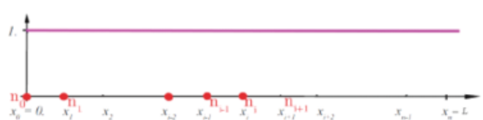
$n_{\bar{p}}$	:= number of (prescribed) dof on $\partial\mathcal{D}_u$	
$\bar{(.)}$	= decoration for prescribed dofs	
$\{\bar{1},\ldots,\bar{n_p}\}$	= global dofs on $\partial\mathcal{D}_u$	(308)
$\bar{\mathbf{a}} = [\bar{a}_{\bar{1}},\ldots,\bar{a}_{\bar{n_p}}]^T$	= vector of prescribed values for these dofs	
$\bar{\mathbf{N}} = [\bar{N}_{\bar{1}},\ldots,\bar{N}_{\bar{n_p}}]$	= (row) vector of shape functions for these dofs	
$\bar{\mathbf{B}} = [\bar{B}_{\bar{1}},\ldots,\bar{B}_{\bar{n_p}}]$	= (row) vector of "displacement to strain" for these dofs	

- ϕ_p is formed by,

$$\boxed{\phi_p = \bar{\mathbf{N}}\bar{\mathbf{a}} = \sum_{i=1}^{n_{\bar{p}}} \bar{a}_{\bar{i}}\bar{N}_{\bar{i}} \;\Rightarrow\; \frac{d\phi_p}{dx} = \bar{\mathbf{B}}\bar{\mathbf{a}} = \sum_{i=1}^{n_{\bar{p}}} \bar{a}_{\bar{i}}\bar{B}_{\bar{i}}} \qquad (309)$$

- While this construction guarantees satisfaction of essential boundary conditions at nodes $\bar{1},\ldots,\bar{n_p}$, as shown in the figure the intermediate values may not match $\bar{u}$.

B. Essential Boundary Conditions

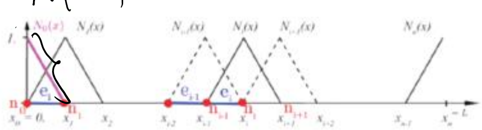


- Previously to satisfy the essential conditions for ϕ_p we used,

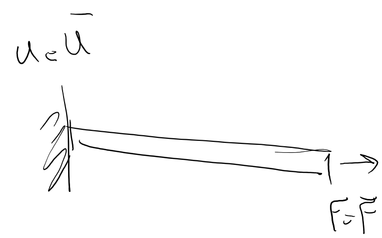
$$\phi_p = \bar{u}1 \quad\Rightarrow\quad \phi_p(0) = \bar{u}$$

- However, Finite Element Shape functions provide a natural way to construct ϕ_p by using (302): $\qquad N_i(x_j) = \delta_{ij}$

$N_{\bar{I}}(x)$

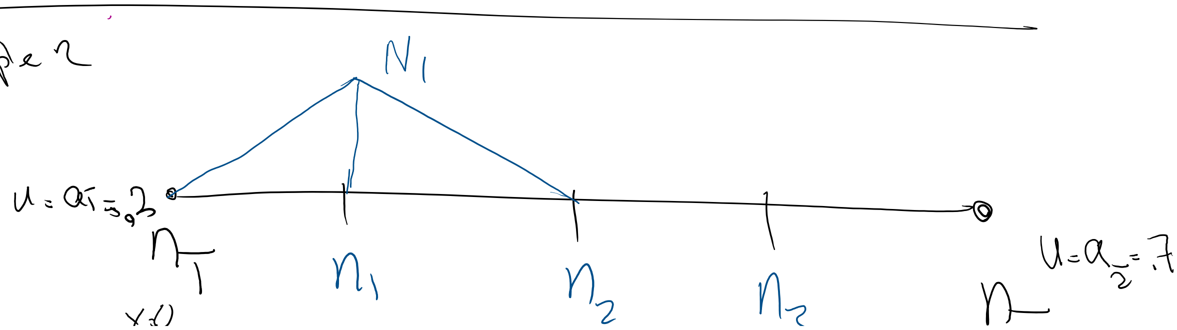


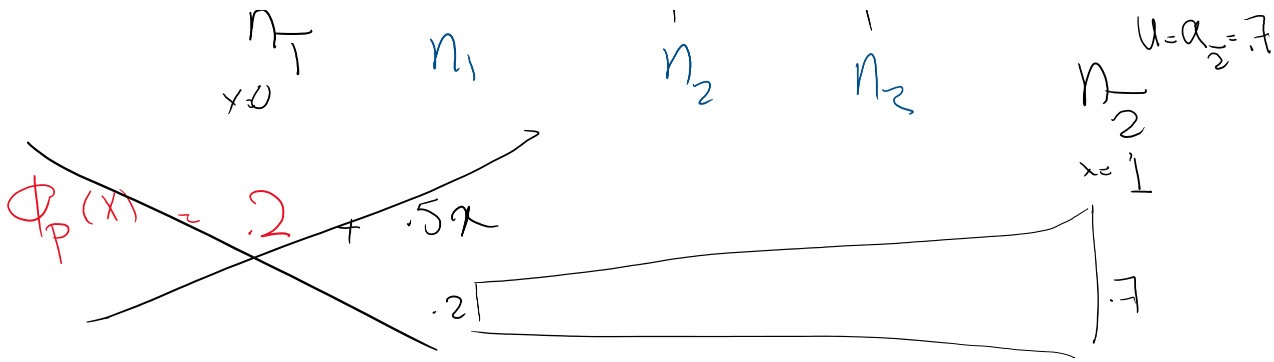
- We let, $\quad \phi_p = \bar{u}N_0 \quad \Rightarrow \phi_p(0) = \phi_p(x_0) = \bar{u}N_0(x_0) = \bar{u}\delta_{11} = \bar{u} \qquad (307)$
- It is clear that for all the trial functions (*i.e.*, shape functions corresponding to unknowns - $I \in \{N_1,\ldots,N_{n_f}\}$, $N_I(0) = N_I(x_0) = \delta_{I0} = 0$. That is, trial functions satisfy homogeneous essential boundary condition.

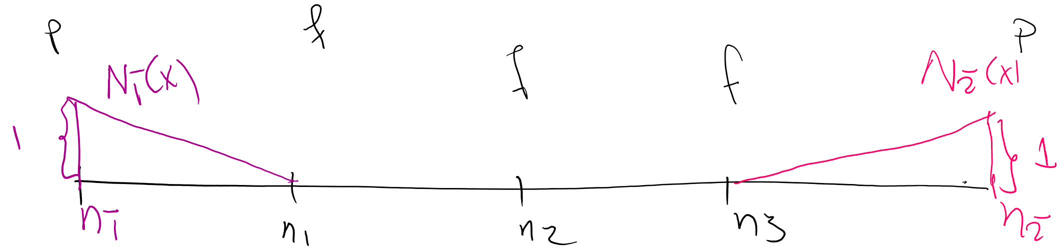


$$\Phi_p = \underbrace{a_{\bar{1}}}_{\bar{u}}\, N_{\bar{1}}(x) = \bar{u}\, N_{\bar{1}}(x)$$

Example 2







$$\Phi_p(x) = a_{\bar 1} N_{\bar 1}(x) + a_{\bar 2} N_{\bar 2}(x)$$
$$= 0.2\, N_{\bar 1}(x) + .7\, N_2(x)$$

K_f

Same example

$$u^h(x) = \Phi_p(x) + \sum_{i=1}^{n_f} a_i N_i$$
$$\Phi_p(x) = \sum_{i=1}^{n_p} a_{\bar i} N_{\bar i}(x)$$

$\Rightarrow$ # free dof

$n_f = 3$

$n_p = 2$

$$u^h = \underbrace{\sum_{i=1}^{n_p} a_{\bar i} N_{\bar i}(x)}_{\Phi_p} + \sum_{i=1}^{n_f} a_i N_i(x)$$

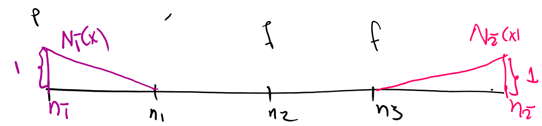
$$= \underbrace{\begin{bmatrix} N_{\bar 1} & N_{\bar 2} & \cdots & N_{\bar n_p} \end{bmatrix}}_{N_p} \underbrace{\begin{bmatrix} a_{\bar 1} \\ \vdots \\ a_{\bar n_p} \end{bmatrix}}_{a_p} + \underbrace{\begin{bmatrix} N_1(x) & \cdots & N_{n_f}(x) \end{bmatrix}}_{N_f} \underbrace{\begin{bmatrix} a_1 \\ \vdots \\ a_{n_f} \end{bmatrix}}_{a_f}$$

$$= N_p a_p + N_f a_f$$

$$u^h = \underbrace{N_p}_{\Phi_p} a_p + N_f a_f$$

$$N_p = [N_{\bar 1} \cdots N_{\bar n_p}] \qquad a_p = \begin{bmatrix} a_{\bar 1} \\ \vdots \\ a_{\bar n_p} \end{bmatrix}$$

$$N_f = [N_1 \cdots N_{n_f}] \qquad a_f = \begin{bmatrix} a_1 \\ \vdots \\ a_{n_f} \end{bmatrix}$$



Ex
$n_p = 2$
$n_f = 3$

(1)

$$\varepsilon = u^{h\prime} = \left(\sum_{i=1}^{n_p} N_{\bar i}(x)\, a_{\bar i}\right)' + \left(\sum_{i=1}^{n_f} N_i(x)\, a_i\right)'$$

$$= \sum_{i=1}^{n_p} N_{\bar i}'(x)\, a_i + \sum_{i=1}^{n_f} N_i'(x)\, a_i$$

$$\mathcal{L}_m(u) = B_p a_p + B_f a_f$$

$$B_p = \mathcal{L}_m(N_p) = [\mathcal{L}_m(N_{\bar 1}), \ldots, \mathcal{L}_m(N_{\bar n_p})]$$

(2)

(for a bar $\mathcal{L}_m(\,) = (\,)'$

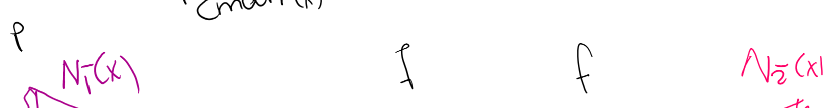
$B_p = [N_{\bar i}', \ldots N_{\bar n_p}']$)

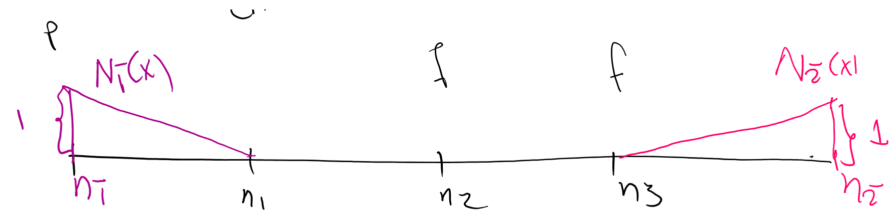
$$B_f = \mathcal{L}_m(N_f) = [\mathcal{L}_m(N_1), \ldots, \mathcal{L}_m(N_{n_f})]$$

(for a bar problem

$B_f = [N_1', \ldots, N_{n_f}']$)

for bar $\underbrace{\varepsilon(x)}_{\text{strain}} = \underbrace{B_p(x)}_{\substack{\text{Displacement} \\ \text{to strain} \\ \text{operator} \\ \text{(matrix)}}} \underbrace{a_p}_{\substack{\text{nodal} \\ \text{disp}}} + \underbrace{B_f(x)}_{\substack{\text{displacement} \\ \text{to strain} \\ \text{operator}}} \underbrace{a_f}_{\substack{\text{nodal} \\ \text{disp}}}$

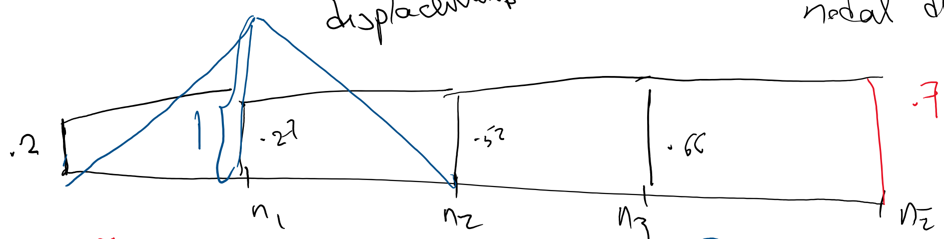




$a_p = \begin{bmatrix} .2 \\ .7 \end{bmatrix}$ $\quad a_f = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ $\quad$ FEM solution $\quad a_f = \begin{bmatrix} .27 \\ .52 \\ 0.66 \end{bmatrix}$

$$\underbrace{\varepsilon(x)}_{\text{strain}} = [B_{\bar{1}}(x) \;\; B_{\bar{2}}(x)] \underbrace{\begin{bmatrix} .2 \\ .7 \end{bmatrix}}_{a_p} + [B_1(x) \;\; B_2(x) \;\; B_3(x)] \underbrace{\begin{bmatrix} .27 \\ .52 \\ .66 \end{bmatrix}}_{a_f}$$

nodal displacements $\qquad$ nodal displacement



$$u^h = [N_{\bar{1}} \;\; N_{\bar{2}}] \begin{bmatrix} .2 \\ .7 \end{bmatrix} + [N_1(x) \;\; N_2(x) \;\; N_3(x)] \begin{bmatrix} .27 \\ .52 \\ .66 \end{bmatrix}$$

$$u^h(n_2) = 0.52 = a_2.$$

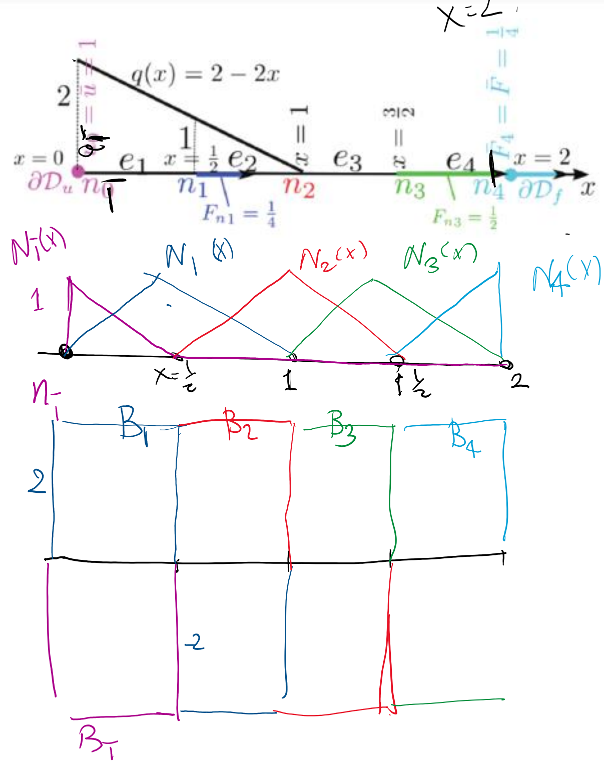
$$u^h(n_{\bar{1}}) = 0.2 = a_{\bar{1}}$$

$$\boxed{\begin{gathered} a_i = \text{Solution @ node } i \\ N_i(n_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \end{gathered}}$$

Bar Example: Overview

$n_p = 1$

$n_f = 4$



step1 : shape functions & B's

$p = \bar{a} = [a_1] = [1]$

$$a_f = a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = ?$$

$$\boxed{\begin{array}{l} B's \ \mathcal{L}_m = (\ )' \\ B = N' \end{array}}$$

in general

$$K = K_{ff} = \int_D \begin{bmatrix} \mathcal{L}_m(\phi_1) \\ \vdots \\ \mathcal{L}_m(\phi_n) \end{bmatrix} D \left[\mathcal{L}_m(\phi_1), \dots \mathcal{L}_m(\phi_n)\right] dV$$

FEM $\quad \phi \to N$

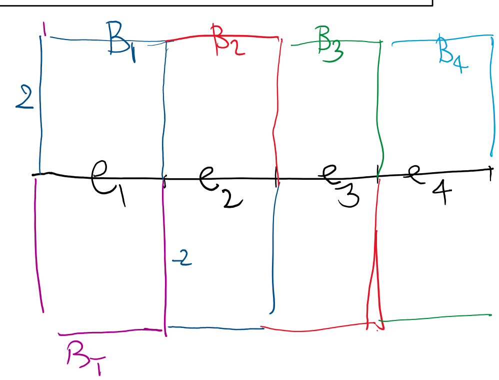
$\mathcal{L}_m(\phi) \to B$

$n \to n_f$

3

$$\boxed{\begin{aligned} K = K_{ff} &= \int_D \begin{bmatrix} B_1 \\ \vdots \\ B_{nf} \end{bmatrix} D \left[B_1 \dots B_{nf}\right] dV \\ &= \int_D B_f^t D B_f \, dV \end{aligned}}$$

$$K = \int_0^2 \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} D \left[B_1 \ B_2 \ B_3 \ B_4\right] dx$$



$$K_{11} = \int_0^2 B_1 B_1 \, dx = \int B_1 B_1 \, dx + \int B_1 B_1 \, dx$$

B_T

$$K_{11} = \int_{\mathcal{D}} B_1 B_1\, dx = \int_{e_1} B_1 B_1 dx + \int_{e_2} B_1 B_1 dx + \int_{e_3} B_1 B_1\, dx + \int_{e_4} B_1 B_1\, dx = \int_{x=0}^{.5} (2)(2)\, dx + \int_{.5}^{1} (-2)(-2)\, dx + 0 + 0 =$$

$$.5 \times 2 \times 2 + .5 \times (-2) \times (-2) = \underline{4}$$

$$K_{12} = \int_0^2 B_1 B_2\, dx = \int_{e_2} B_1 B_2\, dx = \int_{.5}^{1} (-2)(2)\, dx = .5 \times (-2) \times (2) = \underline{-2}$$

$$K = \begin{bmatrix} 4 & -2 & & & \\ -2 & 4 & -2 & & \\ & -2 & 4 & -2 & \\ & & -2 & 2 & \end{bmatrix}$$

(4)

stiffness for the test problem

Recall from last time

$$F_D = \mathcal{A}(\phi^t, \phi_p) = \mathcal{A}\left(\begin{bmatrix} \phi_1 \\ \vdots \\ \phi_n \end{bmatrix}, \phi_p \right)$$

N_{np}

general

$$F_D = \int_{\mathcal{D}} \mathcal{L}_m \begin{bmatrix} N_1 \\ \vdots \\ N_{np} \end{bmatrix} D\, \mathcal{L}_m(\phi_p)\, dV$$

FEM

$$\phi_p = N_p\, a_p = [N_1, \ldots, N_{\bar{n}_p}] \begin{bmatrix} a_1 \\ \vdots \\ a_{\bar{n}_p} \end{bmatrix}$$

$$F_D = \int_D \underbrace{\mathcal{L}_m \begin{bmatrix} N_1 \\ \vdots \\ N_{n_f} \end{bmatrix}}_{B_f^t} D \underbrace{\mathcal{L}_m \left(\begin{bmatrix} N_{\bar{1}} & \cdots & N_{n_{\bar{p}}} \end{bmatrix} \right)}_{B_p} \begin{bmatrix} a_{\bar{1}} \\ \vdots \\ a_{\bar{n}_p} \end{bmatrix}$$

$$F_D = \left(\int_D B_f^t D B_p \, dV \right) a_p$$

$D \leftarrow$ domain

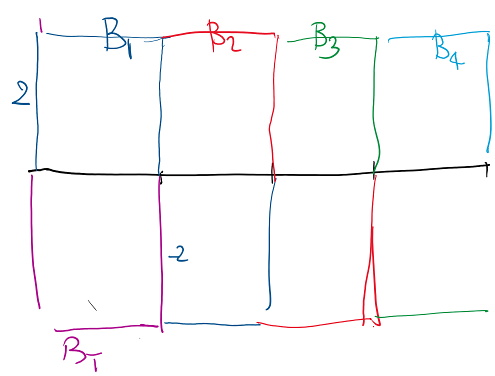
$$F_D = K_{fp} \, a_p \qquad (5)$$

Recall

$$K = K_{ff} = \int_D \underline{B_f^t D B_f \, dV}$$

$$F_D = K_{fp} \, a_p$$

$$K_{fp} = \int_0^2 \underbrace{\begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix}}_{B_f} \underbrace{D}_{EA=1} \begin{bmatrix} B_{\bar{1}} \end{bmatrix} dx$$



$$K_{fp} = \begin{bmatrix} \int_{e_1} B_1 B_{\bar{1}} \, dx \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} .5 \times (-2)(2) \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$a_p = a_T = [1]$

$$\boxed{F_D = K_{fp} a_p = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix} [1] = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$ 6.