From last time

$$K = K_{ff} = \int_{\Omega} B_f^t \, D \, B_f \, dV$$

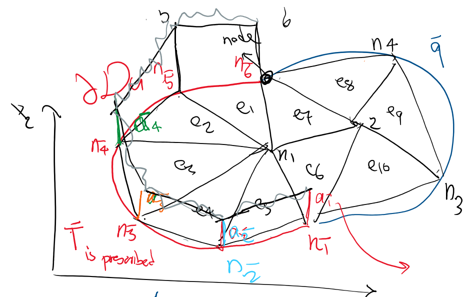
$\Omega \rightarrow$ domain; $D \rightarrow$ section/material property

$$F_D = K_{fp} a_p \qquad K_{fp} = \int B_f^t D B_p \, dV$$

Other examples of FD

Ex 1

$$\underset{4\times1}{F_D} = \underset{4\times6}{(K_{fp})} \underset{6\times1}{a_p}, \quad a_p = \begin{bmatrix} a_{\bar{1}} \\ a_{\bar{2}} \\ \vdots \\ a_{\bar{6}} \end{bmatrix}$$

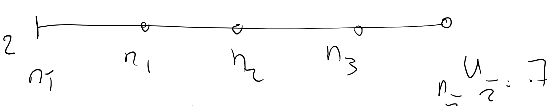


$n_f = 4$

$n_p = 6$

Ex 2

$$\underset{3\times1}{(F_D)} = \underset{3\times2}{(K_{fp})} \underset{2\times1}{(a_p)} = K_{fp} \begin{bmatrix} .2 \\ .7 \end{bmatrix}$$



$$K_{fp} = \int_{\Omega} B_f^t D B_p \, dV = \int_0^L \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} EA \begin{bmatrix} B_{\bar{1}} & B_{\bar{2}} \end{bmatrix} dx$$

$n_f = 3$

$n_p = 2$

Source term force:

$$F_r = \int_{\Omega} N_f^t \, r \, dV$$

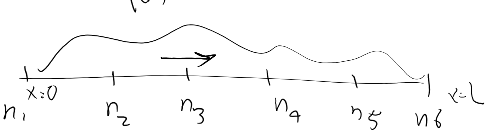
$r \rightarrow$ source term

e.g 1D bar source

$$F_r = \int_0^L \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_{n_f} \end{bmatrix} q(x) \, dx$$

source term $\leftarrow q(x)$

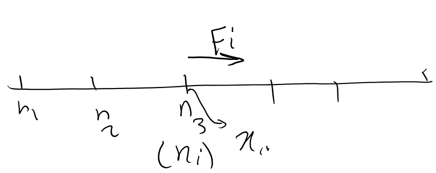
a) Distributed source



$n_f = 6$

b) point force

point force F_i is exerted @ node n_i



L q(x)

@ node n_i

$n_1 \quad n_2 \quad n_3 \quad (n_i) \; x_i$

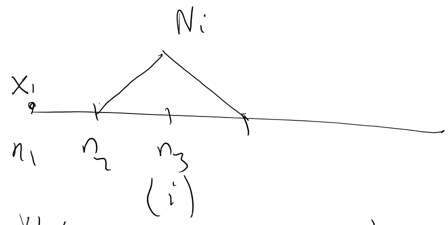
$$F_r = \int_0^L \begin{bmatrix} N_1 \\ \vdots \\ N_j(x) \\ \vdots \\ N_{n_f}(x) \end{bmatrix} \overbrace{\left(F_i\, \delta(x - x_i)\right)}^{q(x)} dx$$

$$q(x) = F_i \underbrace{\delta(x - x_i)}\quad \left(\begin{matrix}\text{no}\\ \text{summation}\\ \text{on } i\end{matrix}\right)$$

$$= \begin{bmatrix} F_i \int_0^L N_1(x)\, \delta(x - x_i)\, dx \\ \vdots \\ F_i \int_0^L N_j(x)\, \delta(x - x_i)\, dx \\ \vdots \\ F_i \int N_{n_f}(x)\, \delta(x - x_i)\, dx \end{bmatrix}$$

$$\boxed{\int_0^L f(x)\, \delta(x - x^*)\, dx = f(x^*)}$$

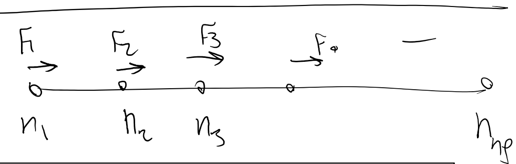
$$= \begin{bmatrix} F_i\, N_1(x_i) \\ \vdots \\ F_i\, N_i(x_i) \\ \vdots \\ F_i\, N_{n_f}(x_i) \end{bmatrix}$$



$N_i(x_1) = \dots = 0$ except

$N_i(x_i) = 1$

$$= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ F_i \\ 0 \\ 0 \\ 0 \end{bmatrix} \leftarrow \text{row } i$$



the corresponding F_r is

this will be added to the RHS

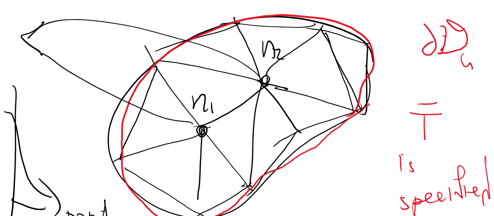
$$Ka = \underbrace{F_r + F_N - F_D}_{\text{Come from elements}} + F_n \qquad F_n = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_{nf} \end{bmatrix}$$

Nodal force vector

(1)

2D heat conduction

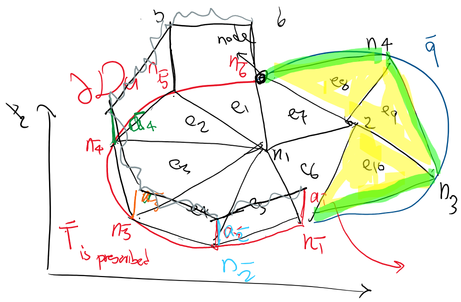
$$F_n = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$ point heat sources like laser



F_N : Neumann BC

$$F_N = \int_{\partial D} N^t \bar{F} \, ds$$

$\bar{F}$: prescribed flux on Neumann BC



Example on the right

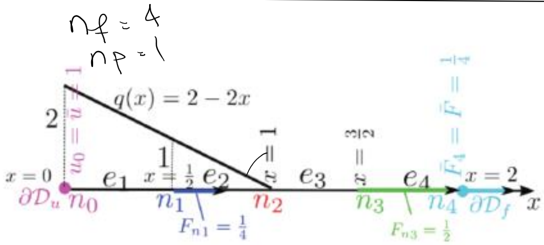
$$F_N = \int_{\partial D} \begin{bmatrix} N_1 \\ \vdots \\ N_4 \end{bmatrix} \bar{q} \, ds = \sum_{e} \int \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \bar{q} \, ds \qquad n_f = 4$$

e: elements → here only e_8, e_9, e_{10} involved

When the elements are 1D (bars, beams, trusses, frames) we don't form FN as shown below.

$n_f = 4$

$n_p = 1$



$$Ka = F_n + F_N + F_r - F_D$$

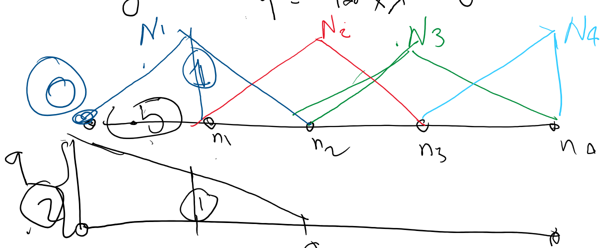
last time

$$K = \begin{bmatrix} 4 & -2 & & \\ -2 & 4 & -2 & \\ & -2 & 4 & -2 \\ & & -2 & 2 \end{bmatrix}$$

$$F_D = (K_{fp})_{4\times1} (a_p)_{1\times1} = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$F_r = \int_0^2 \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} q \, dx = \int_0^1 \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \underbrace{(2-2x)}_{q(x)} dx = \begin{bmatrix} \int_0^1 N_1 q \, dx \\ \int N_2 q \, dx \\ 0 \\ 0 \end{bmatrix}$$

$q = 0$ for $x > 1$

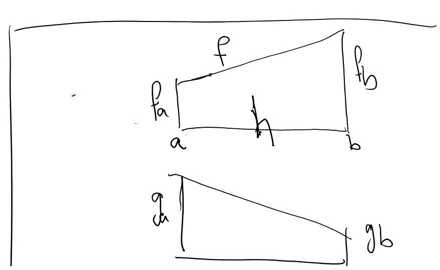


Example

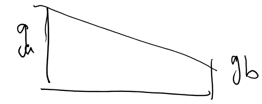
$$\int_0^2 N_1 q \, dx = \int_{e_1} N_1 q \, dx + \int_{e_2} N_1 q \, dx$$

$$\int_{e_1} N_1 q \, dx = \frac{.5}{6}(2\times0\times2 + 2\times1\times1$$

$$+ 0\times1 + 1\times0)$$



$+0\times1+1\times0)$



f&g liner

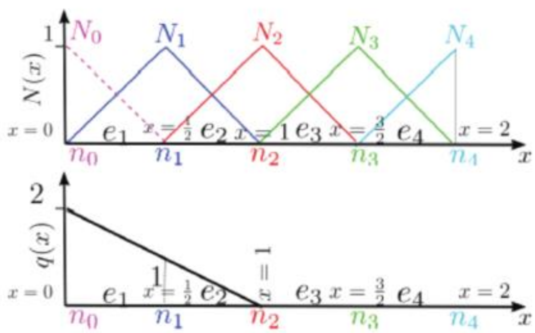
$\int_a^b f(x)\,g(x)\,dx$

$\frac{h}{6}\left(2f_a g_a + 2 f_b g_b + f_a g_b + f_b g_a\right)$

So, Fr is calculated as follows:

Bar Example: Step 2.1: Source term force

done above

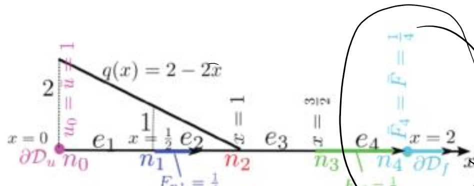


From (312a),

$$\mathbf{F_r} = \int_0^L \begin{bmatrix} N_1 \\ \vdots \\ N_{n_f} \end{bmatrix} q\,dx = \begin{bmatrix} \int_0^2 N_1(x)q(x)\,dx \\ \int_0^2 N_2(x)q(x)\,dx \\ \int_0^2 N_3(x)q(x)\,dx \\ \int_0^2 N_4(x)q(x)\,dx \end{bmatrix} = \begin{bmatrix} \int_{e_1} N_1(x)q(x)\,dx + \int_{e_2} N_1(x)q(x)\,dx \\ \int_{e_2} N_2(x)q(x)\,dx \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1/2}{6}\left((2).(0).(2)+(2).(1).(1)+(0).(1)+(1).(2)\right)+\frac{1/2}{6}\left((2).(1).(1)+(2).(0).(0)+(1).(0)+(0).(1)\right) \\ \frac{1/2}{6}\left((2).(0).(1)+(2).(1).(0)+(0).(0)+(1).(1)\right) \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{F_r} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{12} \\ 0 \\ 0 \end{bmatrix} \tag{317}$$

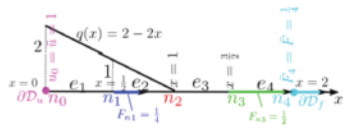


$$F_n = \begin{bmatrix} 1/4 \\ 0 \\ 1/2 \\ 1/4 \end{bmatrix}$$

in 1D natural BC is already included in Fn $\Rightarrow$ we shouldn't form FN

$$F_N = \int_{\partial\Omega_f} N^t \bar F\,ds = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix}_{x=2} \left(\frac{1}{4}\right) = \begin{bmatrix} N_1(x=2) \\ \vdots \\ N_4(x=2) \end{bmatrix} \left(\frac{1}{4}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{4} \end{bmatrix}$$

Bar Example: FEM Solution



- From (311) we have

$$\mathbf{F} = \mathbf{F}_r + \cancel{\mathbf{F}_N} + \mathbf{F}_n - \mathbf{F}_D$$

- Obtaining the individual values from (317), (318), (319), and (320) we obtain,

$$\mathbf{F} = \mathbf{F}_r + \mathbf{F}_N + \mathbf{F}_n - \mathbf{F}_D = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{12} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{4} \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ 0 \\ \frac{1}{2} \\ \cancel{0}\ \frac{1}{4} \end{bmatrix} - \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{11}{4} \\ \frac{1}{12} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}$$

- Recalling the value for the stiffness matrix (316) and $\mathbf{K}\mathbf{a} = \mathbf{F}$ we obtain,

$$\mathbf{K} = \begin{bmatrix} 4 & -2 & 0 & 0 \\ & 4 & -2 & 0 \\ \text{sym.} & & 4 & -2 \\ & & & 2 \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} \frac{11}{4} \\ \frac{1}{12} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix} \quad \Rightarrow \quad \boxed{\mathbf{a} = \begin{bmatrix} \frac{43}{24} \\ \frac{53}{24} \\ \frac{31}{12} \\ \frac{65}{24} \end{bmatrix}} \qquad (321)$$

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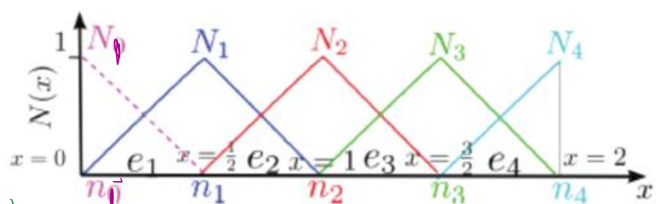
$$u^h(x) = \phi_p(x) + \sum_{i=1}^{nf} a_i N_i(x) = a_{\bar{i}} N_{\bar{i}}(x) + \sum_{i=1}^{4} a_i N_i$$

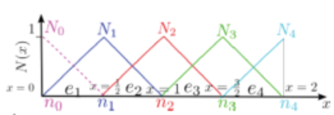
$=$



$$\underbrace{1}_{a_{\bar{i}}} N_{\bar{i}}(x) + \overbrace{\frac{43}{24}}^{a_1} N_1(x) + \frac{53}{24} N_2 + \frac{31}{12} N_3 + \frac{65}{24} N_4$$

$$\boxed{u^h(n_2) = \frac{53}{24} = a_2} \qquad \boxed{a_i = u^h(n_i)}$$

Bar Example: FEM Solution



- The FEM solution is,

$$u^h = a_i \phi_i + \phi_p = a_i N_i + \phi_p$$

- ϕ_p is given by (309),

$$\phi_p = \sum_{i=1}^{n_p} \bar{u}_{\bar{i}} N_{\bar{i}} = \bar{u}_0 N_0 = \bar{u} N_0 = 1 N_0 \quad \Rightarrow$$

$$\boxed{u^h = \bar{u} N_0 + \{a_1 N_1 + a_2 N_2 + a_3 N_3 + a_4 N_4\} = 1.N_0 + \frac{43}{24}.N_1 + \frac{53}{24}.N_2 + \frac{31}{12}.N_3 + \frac{65}{24}.N_4} \qquad (322)$$

- What is $u^h(\frac{1}{2})$?

$$u^h(\frac{1}{2}) = u^h(n_1) = \bar{u} N_0(n_1) + a_1.N_1(n_1) + a_2.N_2(n_1) + a_3.N_3(n_1) + a_4.N_4(n_1)$$

$$= \bar{u}.\delta_{01} + a_1.\delta_{11} + a_2.\delta_{21} + a_3.\delta_{31} + a_4.\delta_{41} = +a_1 = \frac{43}{24}$$

- In general,

$$\boxed{u^h(n_I) = a_I} \qquad (323)$$

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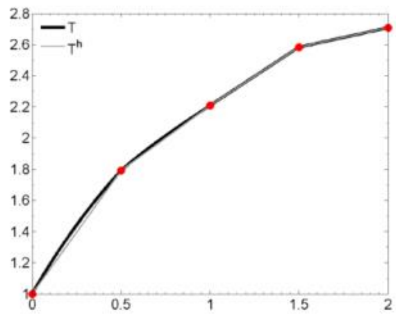
Bar Example: solution values

u



- u^h and u match at all nodes n_0, n_1, n_2, n_3, and n_4. This holds for 1D solid elements with uniform AE and **does not hold** in general.

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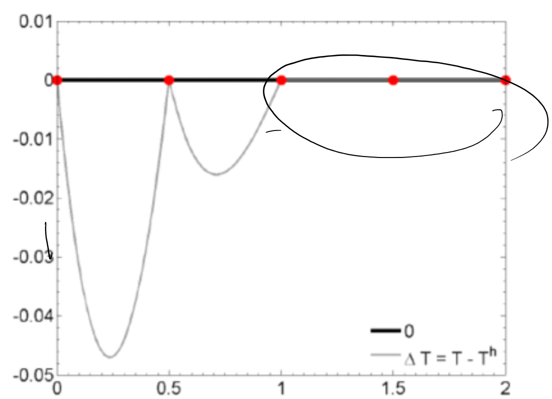
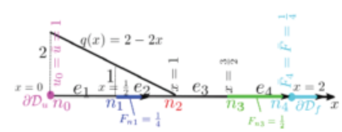
Bar Example: error in solution values





- As mentioned before, the solution error at all nodes n_0, n_1, n_2, n_3, and n_4 is zero. This does not hold in general for FEM method.

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- The Exact solution u and its derivative $\frac{\mathrm{d}u}{\mathrm{d}x}$ are

$$u = \begin{cases} \frac{x^3}{3} - x^2 + 2x + 1 & 0 \le x < \frac{1}{2} \\ \frac{x^3}{3} - x^2 + \frac{7}{4}x + \frac{9}{8} & \frac{1}{2} \le x < 1 \\ \frac{3}{4}x + \frac{35}{24} & 1 \le x < \frac{3}{2} \\ \frac{1}{4}x + \frac{53}{24} & \frac{3}{2} \le x \le 2 \end{cases} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \begin{cases} x^2 - 2x + 2 & 0 \le x < \frac{1}{2} \\ x^2 - 2x + \frac{7}{4} & \frac{1}{2} \le x < 1 \\ \frac{3}{4} & 1 \le x < \frac{3}{2} \\ \frac{1}{4} & \frac{3}{2} \le x \le 2 \end{cases} \qquad (324)$$

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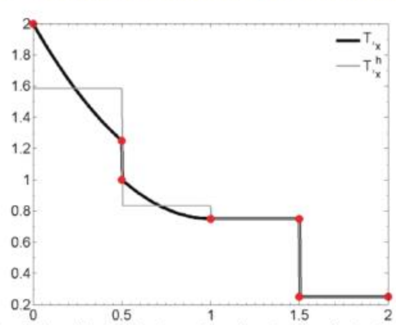
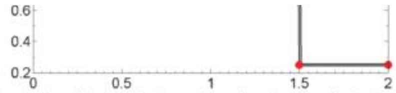
Bar Example: solution derivatives ($\propto$ axial force)

$\varepsilon = u'$



element order p
(for use $p = 1$)
$\|u^h - u_{exact}\| \approx C h^{p+1}$
guaranteed

- The errors in solution derivative is larger than those in the solution itself. In general, the



- The errors in solution derivative is larger than those in the solution itself. In general, the accuracy of FE solution decreases for solution derivatives (*e.g.*, strains, stresses, *etc.*).
- Approximate solution u^h exhibits jumps in $\frac{\mathrm{d}u^h}{\mathrm{d}x}$ at all interior nodes. This is because the solution is piece-wise constant in $H^1([0\ 2])$.
- Even the exact solution exhibits jumps in $\frac{\mathrm{d}u}{\mathrm{d}x}$ at n_1 and n_3 from the concentrated forces.
- The $H^1([0\ 2])$, rather than $C^1([0\ 2])$, is the right solution space for u and u^h as none of them belong to the latter space.

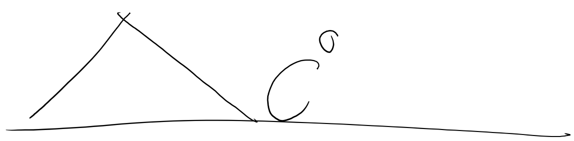
$\|u^h - u_{exact}\| \le C h$ guaranteed

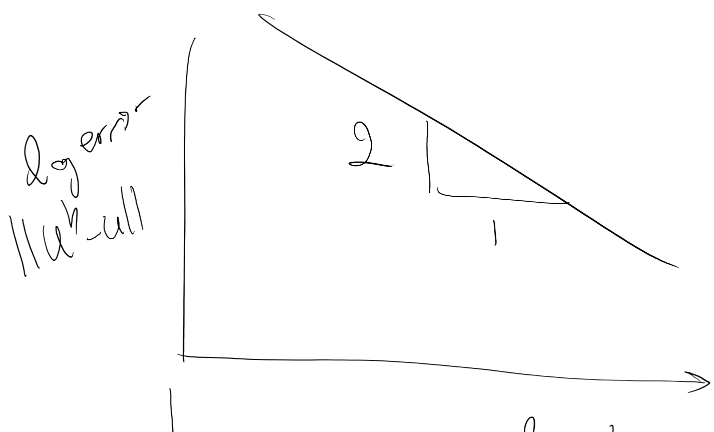
every time we take a derivative the convergence rate decreases by one

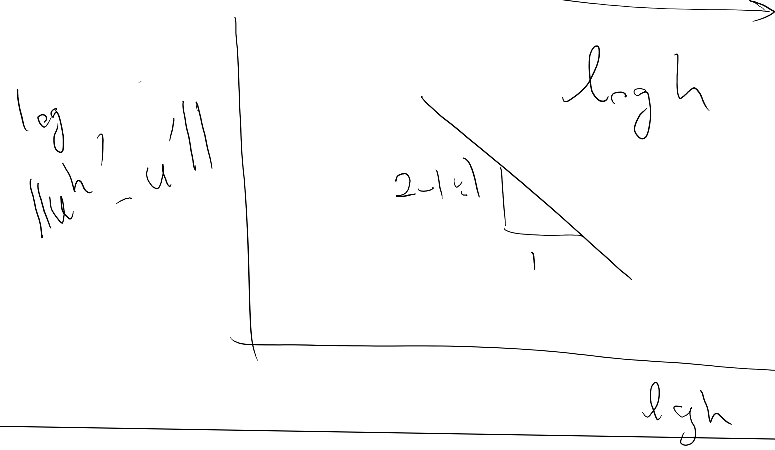
$\|u^{h\prime} - u'_{exact}\| \le C h^p$

u^{exact} C^0 ✓

C^1 ✗





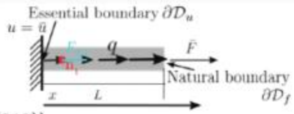


Summary:

Summary: Force vectors

- Force vector is given by:

$$\boxed{\mathbf{F} = \mathbf{F}_r + \mathbf{F}_N + \mathbf{F}_n - \mathbf{F}_D} \qquad (311)$$



- $\mathbf{F}_r$, $\mathbf{F}_N$, $\mathbf{F}_n$ and $\mathbf{F}_D$ are given by (*cf.* (301) and (310))

$$\mathbf{F}_r = \left(\mathbf{N}^{\mathrm{T}}, q\right)_r = \int_{\mathcal{D}} \mathbf{N}^{\mathrm{T}} q \,\mathrm{dv} = \int_0^L \begin{bmatrix} N_1 \\ \vdots \\ N_{n_{\mathrm{f}}} \end{bmatrix} q \,\mathrm{d}x \qquad (312a)$$

$$\mathbf{F}_N = \left(\mathbf{N}^{\mathrm{T}}, \bar{F}\right)_N = \int_{\partial\mathcal{D}_f} \mathbf{N}^{\mathrm{T}} \bar{\mathbf{F}}.\mathbf{N} \,\mathrm{ds} = \left(\begin{bmatrix} N_1 \\ \vdots \\ N_{n_{\mathrm{f}}} \end{bmatrix} \bar{F} \right)_{x=L} \qquad (312b)$$

$$\mathbf{F}_D = \mathcal{A}\left(\mathbf{N}^{\mathrm{T}}, \phi_p\right) = \int_{\mathcal{D}} \frac{\mathrm{d}}{\mathrm{d}x} \mathbf{N}^{\mathrm{T}} EA \frac{\mathrm{d}}{\mathrm{d}x} \phi_p \,\mathrm{dv} \qquad (312c)$$

$$= \left\{ \int_{\mathcal{D}} \mathbf{B}^{\mathrm{T}} EA \bar{\mathbf{B}} \,\mathrm{dv} \right\} \bar{\mathbf{a}} = \left\{ \int_0^L \begin{bmatrix} B_1 \\ \vdots \\ B_{n_{\mathrm{f}}} \end{bmatrix} EA \begin{bmatrix} \bar{B}_{\bar{1}} & \cdots & \bar{B}_{\bar{n_{\mathrm{p}}}} \end{bmatrix} \mathrm{d}x \right\} \begin{bmatrix} \bar{a}_{\bar{1}} \\ \vdots \\ \bar{a}_{\bar{n_{\mathrm{p}}}} \end{bmatrix} = \mathbf{K}_{fp} \bar{\mathbf{a}}$$

$$\mathbf{F}_n = \begin{bmatrix} F_{n\,1} \\ \vdots \\ F_{n\,n_{\mathrm{f}}} \end{bmatrix} \qquad (312d)$$

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Force Essential Boundary Condition

- We have used (309) in (312c) to write,

$$\boxed{\mathbf{F}_D = \mathcal{A}\left(\mathbf{N}^{\mathrm{T}}, \phi_p\right) = \mathbf{K}_{fp} \bar{\mathbf{a}}} \qquad (313)$$

- The prescribed to free stiffness matrix $\mathbf{K}_{fp}$ is an $n_{\mathrm{f}} \times n_{\mathrm{p}}$ matrix given by,

$$\mathbf{K}_{fp} = \int_{\mathcal{D}} \mathbf{B}^{\mathrm{T}} EA \bar{\mathbf{B}} \,\mathrm{dv} = \int_0^L \begin{bmatrix} B_1 \\ \vdots \\ B_{n_{\mathrm{f}}} \end{bmatrix} EA \begin{bmatrix} \bar{B}_{\bar{1}} & \cdots & \bar{B}_{\bar{n_{\mathrm{p}}}} \end{bmatrix} \mathrm{d}x \qquad (314)$$

- From (306) we had,

$$\mathbf{K} = \mathcal{A}\left(\phi^{\mathrm{T}}, \phi\right) = \int_{\mathcal{D}} \mathbf{B}^{\mathrm{T}} EA \mathbf{B} \,\mathrm{dv} = \int_0^L \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{n_{\mathrm{f}}} \end{bmatrix} EA \begin{bmatrix} B_1 & B_2 & \cdots & B_{n_{\mathrm{f}}} \end{bmatrix} \mathrm{d}x$$

where $\mathbf{K}$ was an $n_{\mathrm{f}} \times n_{\mathrm{p}}$ matrix.

- "Prescribed" dofs $\bar{\mathrm{i}}$ do not go into $\mathbf{K}$ because their value $\bar{a}_{\bar{\mathrm{i}}}$ are already known.
- This is opposite to dofs $I = 1, \ldots, n_{\mathrm{f}}$ which correspond to "free" dofs.

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weak statement

$$\int_{\mathcal{D}} \mathcal{L}_m(w) \underbrace{D}_{\text{mat see property}} \mathcal{L}_m(u) \,dv.$$

	$\mathcal{L}_m$	D
bar	$(\ )'$	EA
beam	$(\ )''$	EI
heat conduction	∇	k

---

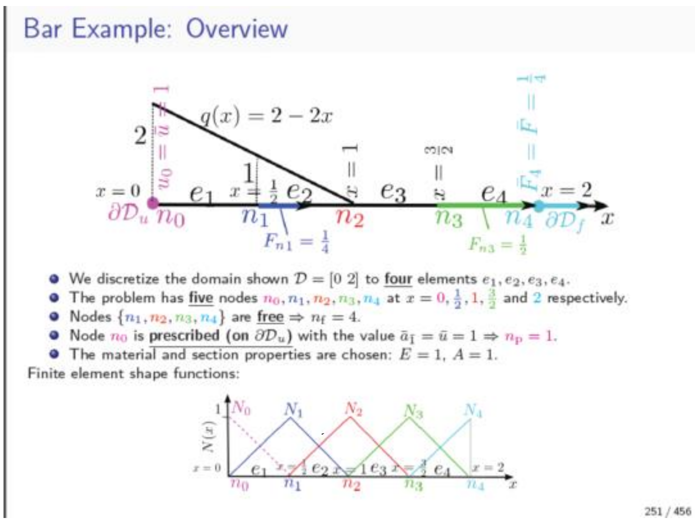
n_f $\quad L \quad [N_1$

conduct v ~

$$u = \Phi_p + \sum_{i=1}^{n_f} a_i N_i \qquad W = N^t = \begin{bmatrix} N_1 \\ \vdots \\ N_{nf} \end{bmatrix}$$

$$K = \int_{\mathcal{D}} \begin{bmatrix} L_m(N_1) \\ \vdots \\ L_m(N_{nf}) \end{bmatrix} D \underbrace{\left[L_m(N_1) \cdots L_m(N_{nf}) \right]}_{B_f} dV = \int_{\mathcal{D}} B_f^t D B_f \, dV$$

What we did can be referred to as the "global" or "node- centered" approach:



the problem with the global approach

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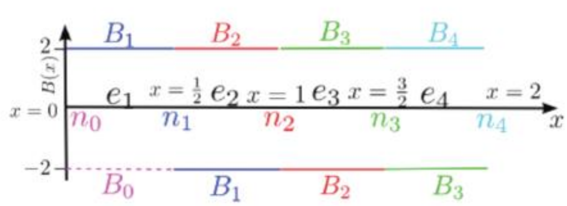
Bar Example: Step 1: Stiffness matrix



$$\mathbf{K} = \int_0^2 \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} EA \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \end{bmatrix} dx = \begin{bmatrix} \int_0^2 B_1 B_1\, dx & \int_0^2 B_1 B_2\, dx & \int_0^2 B_1 B_3\, dx & \int_0^2 B_1 B_4\, dx \\ & \int_0^2 B_2 B_2\, dx & \int_0^2 B_2 B_3\, dx & \int_0^2 B_2 B_4\, dx \\ & & \int_0^2 B_3 B_3\, dx & \int_0^2 B_3 B_4\, dx \\ \text{sym.} & & & \int_0^2 B_4 B_4\, dx \end{bmatrix}$$

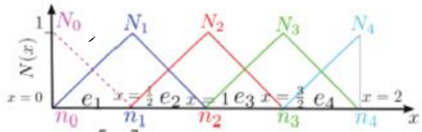
$$K_{11} = \int_0^2 B_1 B_1 \, dx = \int_{e_1} B_1 B_1 \, dx + \int_{e_2} B_1 B_1 \, dx$$

$$.5 \times (2)(2) + .5 \times (-2)(-2) = 2$$

It's much easier that each element does all the integrals (and calculations) that are pertained to it.

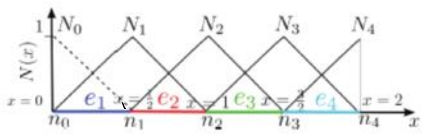
Finite Element Method: Global versus Local approach

1. **Global approach**: This approach is **shape function-centered** and we directly compute each component of the stiffness matrix and force vector by integration of shape functions (or their derivatives) over the entire domain $\mathcal{D}$. The integrals are carried out and summed over all the elements in $\mathcal{D}$.



$$\mathbf{K}=\int_0^2 \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} EA\,[B_1 \quad B_2 \quad B_3 \quad B_4]\,\mathrm{d}x$$

2. **Local approach** is **element centered**: As we eventually the form of the shape functions change element to element, it is more convenient to first divide the integration domain, calculate element level matrices and vectors, and add them together:



For example in the figure,

$$\mathbf{K}=\int_0^2 \mathbf{B}^T EA\mathbf{B}\,\mathrm{d}x=\mathbf{K}^{e1}+\mathbf{K}^{e2}+\mathbf{K}^{e3}+\mathbf{K}^{e4} \quad \text{where} \tag{326}$$

$$\mathbf{K}^{e1}=\int_{e_1}\mathbf{B}^T EA\mathbf{B}\,\mathrm{d}x \quad \mathbf{K}^{e2}=\int_{e_2}\mathbf{B}^T EA\mathbf{B}\,\mathrm{d}x \quad \mathbf{K}^{e3}=\int_{e_3}\mathbf{B}^T EA\mathbf{B}\,\mathrm{d}x \quad \mathbf{K}^{e4}=\int_{e_4}\mathbf{B}^T EA\mathbf{B}\,\mathrm{d}x$$

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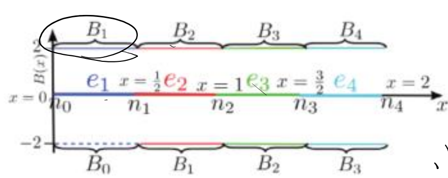
$$K=\int_0^2 \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix}[B_1 \quad B_2 \quad B_3 \quad B_4]\,dx = \underbrace{\int_{e_1} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix}[B_1 \quad B_2 \quad B_3 \quad B_4]\,dx}_{K^{e_1}} + K^{e_2} + K^{e_3} + K^{e_4}$$

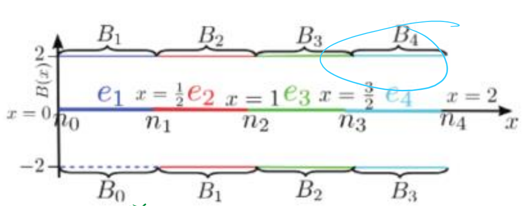
Local approach (element-centered)



$$K^{e_1}=\int_{e_1} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix}[B_1 \quad B_2 \quad B_3 \quad B_4]\,dx = \int_0^{.5} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}[2 \quad 0 \quad 0 \quad 0]\,dx = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

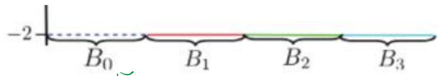
$$\begin{matrix} 2 & -2 \\ -2 & 2 \end{matrix}$$

$$K^{e_2}=\int_{e_2} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix}[B_1 \quad B_2 \quad B_3 \quad B_4]\,dx \quad \int_{.5}^{1} \begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \end{bmatrix}[-2 \quad 2 \quad 0 \quad 0] = \begin{bmatrix} 2 & -2 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



1.5

$[0 \quad 0 \quad 0 \quad 0]$



$$K^{e3} = \int_1^{1.5} \begin{bmatrix} 0 \\ -2 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & -2 & 2 & 0 \end{bmatrix} dx = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K^{4} = \int \begin{bmatrix} 0 \\ 0 \\ -2 \\ 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 & 2 \end{bmatrix} dx = \left[\begin{array}{c|cc} 0 & 0 & \\ \hline 0 & 2 & -2 \\ & -2 & 2 \end{array}\right]$$