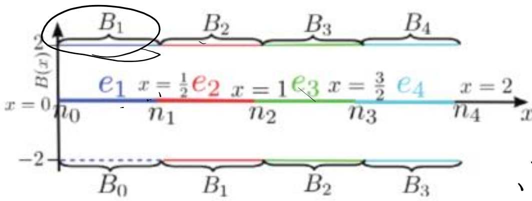
FEM20241028

Monday, October 28, 2024 12:57 PM

From last time

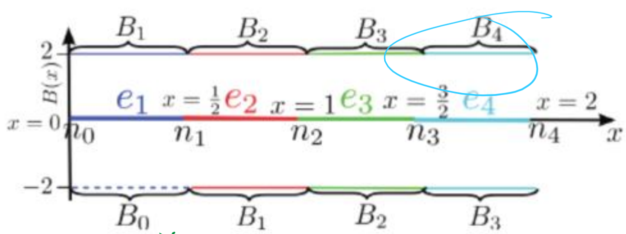
Local approach (element-centered)



$$K^{e_1} = \int_{e_1} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} [B_1\ B_2\ B_3\ B_4]\,dx = \int_0^{.5} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} [2\ 0\ 0\ 0]\,dx = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} 2 & -2 \\ -2 & 2 \end{matrix}$$

$$K^{e_2} = \int_{e_2} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} [B_1\ B_2\ B_3\ B_4]\,dx = \int_{.5}^{1} \begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \end{bmatrix} [-2\ 2\ 0\ 0] = \begin{bmatrix} 2 & -2 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



$$K^{e_3} = \int_1^{1.5} \begin{bmatrix} 0 \\ -2 \\ 2 \\ 0 \end{bmatrix} [0\ -2\ 2\ 0]\,dx = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K^{4} = \int \begin{bmatrix} 0 \\ 0 \\ -2 \\ 2 \end{bmatrix} [0\ 0\ -2\ 2]\,dx = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix}$$

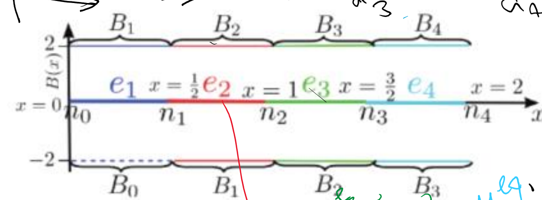
All four elements have local stiffnesses

$$k^e = \frac{A^e E^e}{L^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

for all elements $A^e E^e = 1$, $L^e = \frac{2}{4} = .5$

Local approach (element-centered)

$a_1 \rightarrow$ $a_1 \rightarrow$ $a_2 \rightarrow$ $a_3 \rightarrow$ $a_4 \rightarrow$



n_1

$M^{e_1} = [\bar{1}, 1]$ $M^{e_2} = [1, 2]$ $M^{e_3} = [2, 3]$ $M^{e_4} = [3, 4]$

$$k^{e_1} = k^{e_2} = k^{e_3} = k^{e_4} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$k^{e_1} = \begin{array}{c|c} M^{e_1} & \begin{array}{cc} \bar{1} & 1 \end{array} \\ \hline \begin{array}{c} \bar{1} \\ 1 \end{array} & \begin{bmatrix} 2 & -2 \\ -2 & ? \end{bmatrix} \end{array}$$

$$k^{e_2} = \begin{array}{c|c} M^{e_2} & \begin{array}{cc} 1 & 2 \end{array} \\ \hline \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \end{array}$$

$$k^{e_3} = \begin{array}{c|c} M^{e_3} & \begin{array}{cc} 2 & 3 \end{array} \\ \hline \begin{array}{c} 2 \\ 3 \end{array} & \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \end{array}$$

$$k^{e_4} = \begin{array}{c|c} & \begin{array}{cc} 3 & 4 \end{array} \\ \hline \begin{array}{c} 3 \\ 4 \end{array} & \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \end{array}$$

Assembly

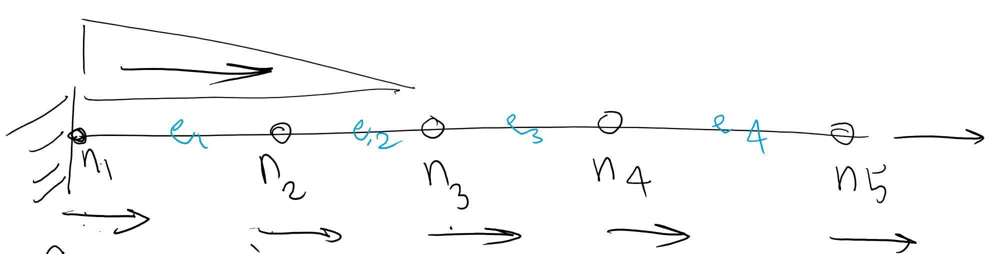
$$K = K_{ff} = \begin{bmatrix} 2+2 & -2 & & \\ -2 & 2+2 & -2 & \\ & -2 & 2+2 & -2 \\ & & -2 & 2 \end{bmatrix}$$

$$K = \begin{bmatrix} 4 & -2 & & \\ -2 & 4 & -2 & \\ & -2 & 4 & -2 \\ & & -2 & 2 \end{bmatrix}$$

Before we solve the same bar example with this local (element-centered) approach, we discuss the concepts of node and dof.

$n_f = 4$

$n_p = 1$



Nodal Element Map / Local Element Map (LEM)

NEM

$NEM_{e_1} = [1, 2]$ $NEM_2 = [2, 3]$ $NEM_{e_4} = [4, 5]$ the nodes the element has

Element ... $M_{e_1} = [\bar{1}, 1]$ $M_{e_2} = [1, 2]$ $M_{e_3} = [3, 4]$ $M_{e_4} = [4, 5]$

nodes element has

Element dof Map (edof) $M_{e_1} = [\bar{1}, 1]$ $M_{e_2} = [1, 2]$ $M_{e_3} = [3, 4]$ $M_{e_4} = [4, 5]$

Node and dof numbers don't match even when there is one dof / node

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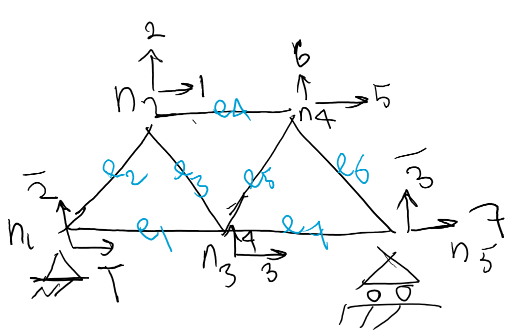
Examples from problems that have more than 1 dof per node:

Ex 1. Truss

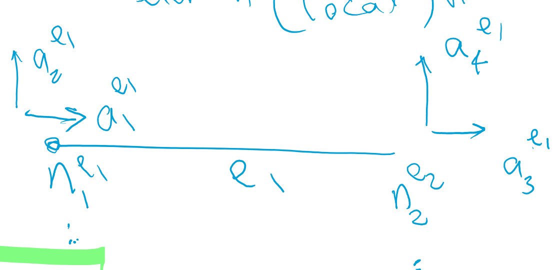


$n_f = 7$

$n_p = 3$

$\frac{ndof}{node} = 2$

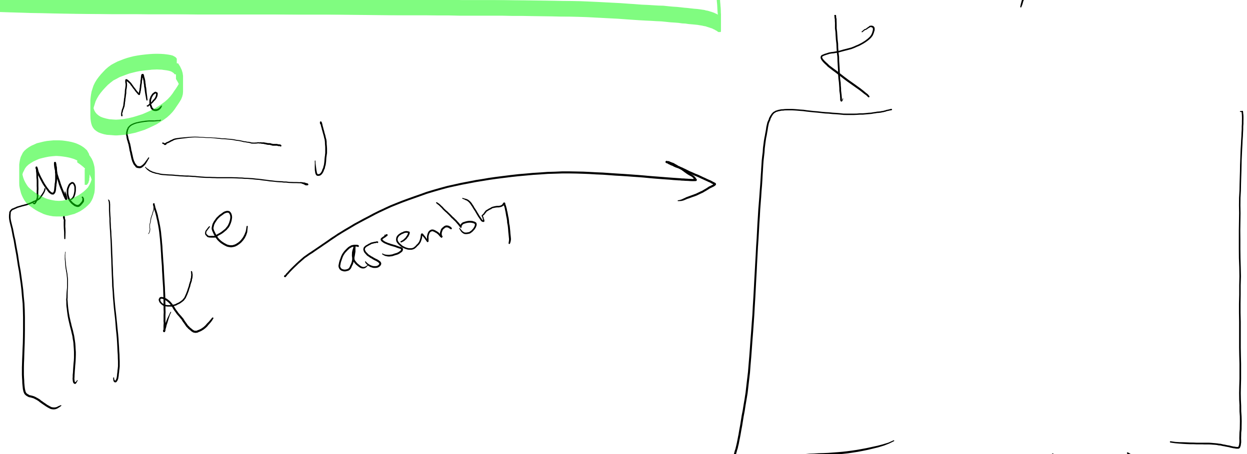
element (local) view



$LEM_{e_1} = [1, 3]$ (nodes)

$\boxed{M_{e_1} = [\bar{1}, \bar{2}, 3, 4] \text{ dof}}$

Needed for assembly



LEN is used in computer implementation to form M (edof)

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plate problem

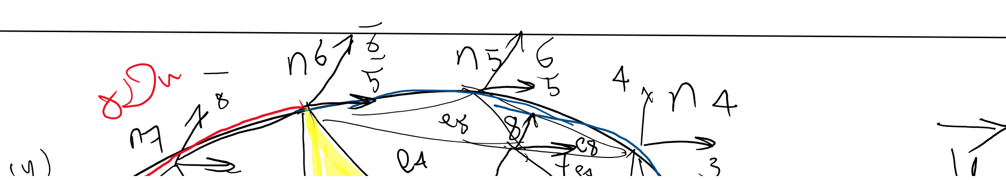
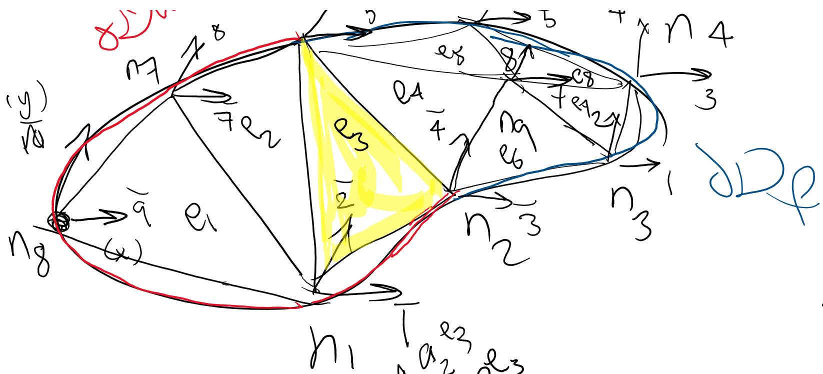


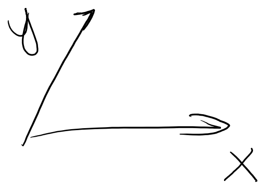
plate problem



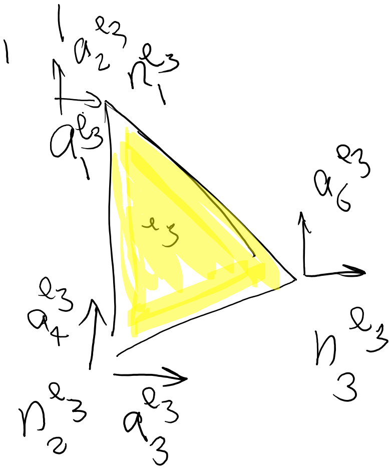
$n_f = 8$

$n_p = 10$

$\vec{u}$



$$\frac{ndof}{node} = 2$$



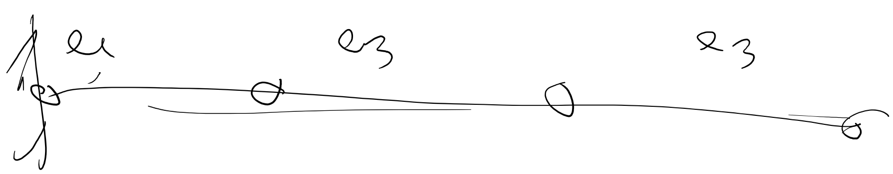
$LEM_{e_3} = [6, 1, 2]$ (nodes)

$M_{e_3} = [\bar{5}, \bar{6}, \bar{1}, \bar{2}, \bar{3}, \bar{4}]$

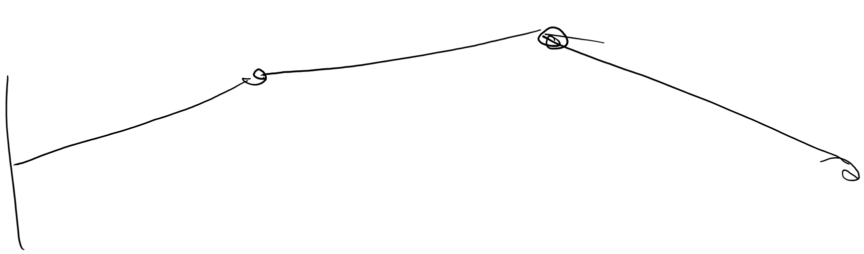
Ex3 Bending problem $(EIy'')'' + q = 0$

$M = 4 \quad m = \frac{M}{2} = 2 \quad C^{m-1} = C^1$

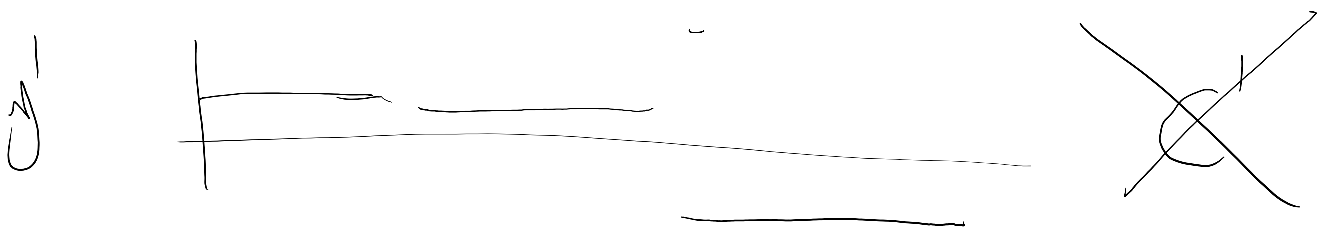
global continuity needed



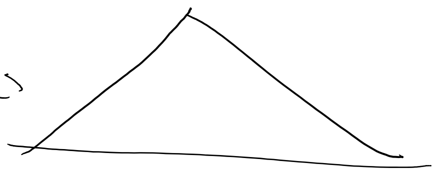
y



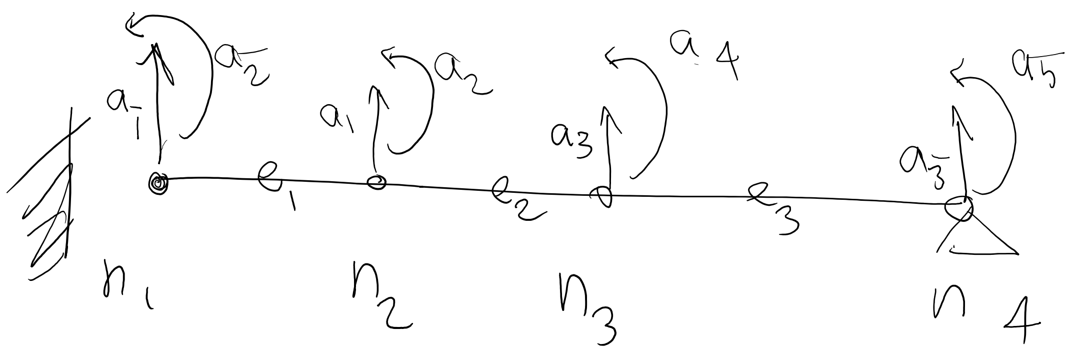
C^0



hat functions

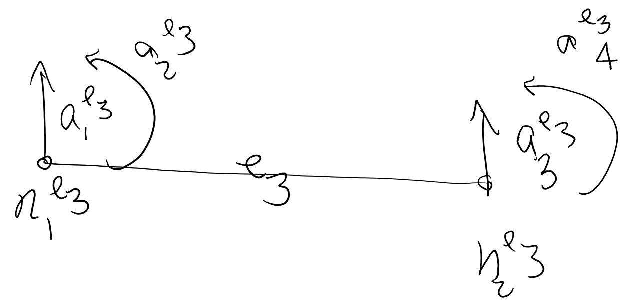


don't work for a beam problem



$\left\{ \begin{array}{l} y \\ \theta = y' \end{array} \right.$ are dofs/node

$\dfrac{\text{dof}}{\text{node}} = 2$

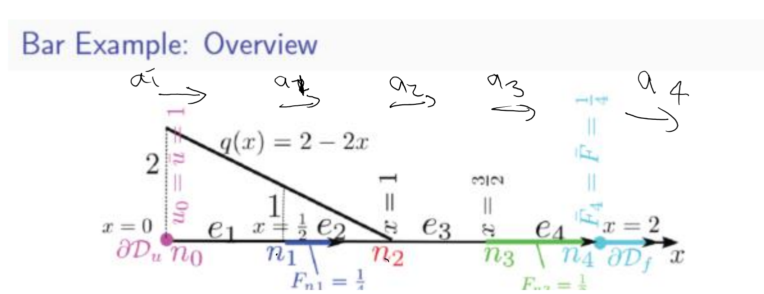


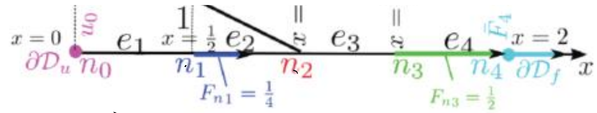
$LEM_{e_3} = [3, 4]$ (nodes)

we need this for assembly ← $M_{e_3} = [3, 4, \bar{3}, 5]$

We solve this with a local approach and cover a few new things along the way:

LEM_{e_1} [0, 1] LEM_{e_2} [1, 2] - -

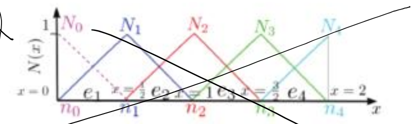




- We discretize the domain shown $\mathcal{D}=[0\ 2]$ to **four** elements e_1, e_2, e_3, e_4.
- The problem has **five** nodes n_0, n_1, n_2, n_3, n_4 at $x=0, \frac{1}{2}, 1, \frac{3}{2}$ and 2 respectively.
- Nodes $\{n_1, n_2, n_3, n_4\}$ are **free** $\Rightarrow n_f = 4$.
- Node n_0 is **prescribed (on $\partial\mathcal{D}_u$)** with the value $\bar{a}_1 = \bar{u} = 1 \Rightarrow n_p = 1$.
- The material and section properties are chosen: $E=1$, $A=1$.

Finite element shape functions:

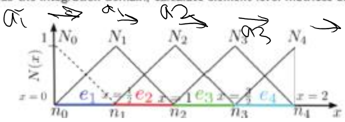
global bad



2. **Local approach** is **element centered**: As we eventually the form of the shape functions change element to element, it is more convenient to first divide the integration domain, calculate element level matrices and vectors, and add them together:

element approach

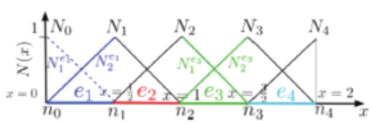
$a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow a_4$



$M_{e_1} = [0,1]$ $\quad M_{e_2} = [1,2]$ $\quad M_{e_3} = [2,3]$ $\quad M_{e_4} = [3,4]$

We already did the stiffness assembly:

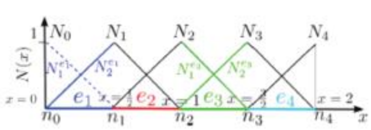
Example: Formation of $\mathbf{K}$ and $\mathbf{F}$



1. Local stiffness matrix: Since $A=1, E=1, L=\frac{1}{2}$ for all elements in the example shown, local stiffness matrix is:
$$\mathbf{k}^{e_i} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}, i=1,2,3,4.$$
2. Assembly to global system: Around the local stiffness matrix for e_3 we have the corresponding dof of the local dof in the global system:
$$\begin{array}{c|cc} & 2 & 3 \\ \hline 2 & 2 & -2 \\ 3 & -2 & 2 \end{array}$$

This means that for example $k_{11}^{e_3}$ will be added to K_{22}, $k_{12}^{e_3}$ to K_{23} and so forth:
$$\mathbf{K}^{e_3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ \text{sym.} & & 2 & 0 \\ & & & 0 \end{bmatrix}$$

Example: Formation of $\mathbf{K}$ and $\mathbf{F}$



Similarly for element e_1 the local stiffness matrix $\mathbf{k}^{e_1}$ and corresponding **free** global dof are:
$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 2 & -2 \\ 1 & -2 & 2 \end{array}$$

Note that 0 in the first row and column means that dof 1 in the element does not correspond to any free global dofs (n_0 is a prescribed node on $\partial\mathcal{D}_u$). In This case, only $k_{22}^{e_1} = 2$ is added to K_{11} from e_1. The corresponding global level matrix from element e_1 is:
$$\mathbf{K}^{e_1} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ \text{sym.} & & 0 & 0 \\ & & & 0 \end{bmatrix}$$

- For example for our sample problem, these maps are,

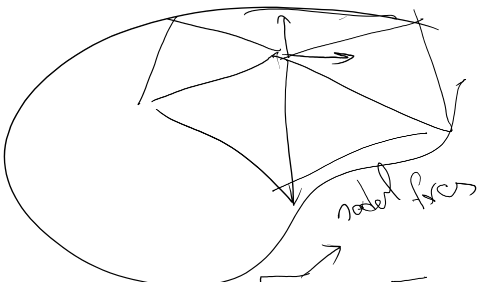
$$\mathbf{M}^{e_1} = [0\ 1]^{\mathrm{T}} \qquad \bar{\mathbf{M}}^{e_1} = [1\ 0]^{\mathrm{T}}$$
$$\mathbf{M}^{e_2} = [1\ 2]^{\mathrm{T}} \qquad \bar{\mathbf{M}}^{e_2} = [0\ 0]^{\mathrm{T}}$$
$$\mathbf{M}^{e_3} = [2\ 3]^{\mathrm{T}} \qquad \bar{\mathbf{M}}^{e_3} = [0\ 0]^{\mathrm{T}}$$
$$\mathbf{M}^{e_4} = [3\ 4]^{\mathrm{T}} \qquad \bar{\mathbf{M}}^{e_4} = [0\ 0]^{\mathrm{T}}$$

- We also combine both maps into $\mathbf{M}^e_t$ that includes both free and prescribed dofs:

$$\mathbf{M}^{e_1}{}_t = [\bar{1}\ 1]^{\mathrm{T}} \qquad \mathbf{M}^{e_2}{}_t = [1\ 2]^{\mathrm{T}} \qquad \mathbf{M}^{e_3}{}_t = [2\ 3]^{\mathrm{T}} \qquad \mathbf{M}^{e_4}{}_t = [3\ 4]^{\mathrm{T}}$$

$$\mathbf{K} = \begin{bmatrix} 2+2 & -2 & 0 & 0 \\ -2 & 2+2 & -2 & 0 \\ 0 & -2 & 2+2 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix}$$

We form the Element level force vector and assemble it to the global force vector



nodal forces

$$Ka = F = F_n + F_e$$

from elements

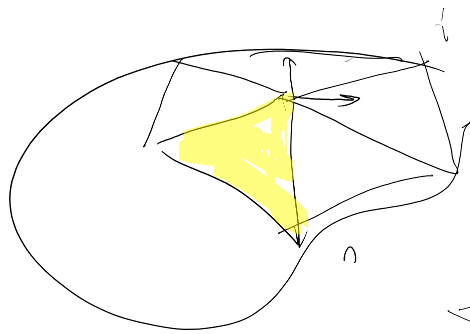
$$F_e = F_r + F_N - F_D$$

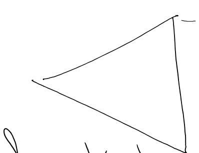
Source term

Neumann BC (2D/3D elements)

Dirichlet BC

$$F_D = K_{fp} a_p$$





k^e → assembled to the global structure

$$f^e = f_r^e + f_N^e - f_D^e$$

Also assembled to the global force F_e

$f_D^e = k^e a^e$ for any element

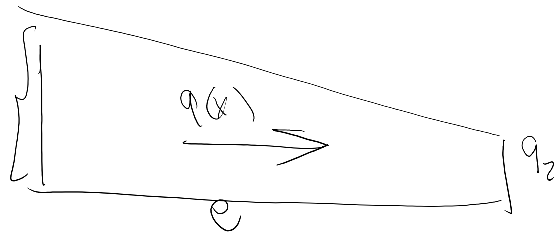
I'll show you

$$f_r^e = \frac{1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

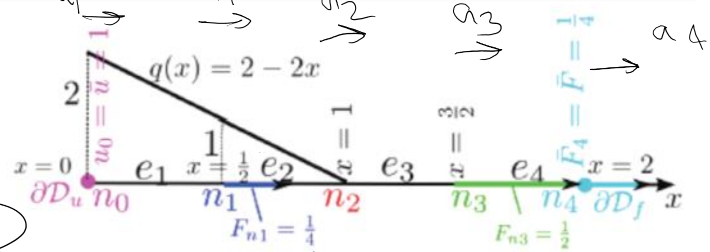
I'll show you

1D bar

$$f_r = \frac{1}{6}\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} q_2 \end{bmatrix}$$

q_1  q_2

a_i a_1 a_2 a_3 a_4



We already have the global stiffness matrix.
Let's calculate the global force vector:

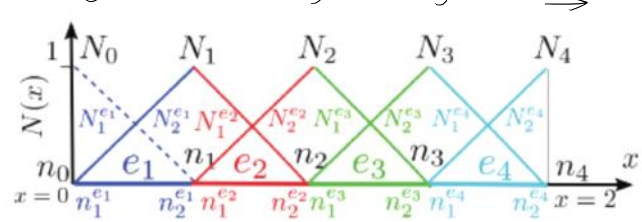
$$Ka = F = \underset{\text{node}}{F_n} + \underset{\text{element}}{F_e} \quad \text{next}$$

$$F_n = \begin{bmatrix} 1/4 \\ 0 \\ 1/2 \\ 1/4 \end{bmatrix}$$

$$F_e = F_r + \underbrace{\cancel{F_N}}_{0 \text{ for 1D elements}} - F_D$$

F_e: calculate f^e for all elements

$a_1 = 1$ a_1 a_2 a_3 a_4



e_1 $$f^{e_1} = f^{e_1}_r + \underbrace{\cancel{f^{e_1}_N}}_{\text{1D}} - f^{e_1}_D$$

$$f^{e_1}_D = K^{e_1} a^{e_1} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} a_1^{e_1} \\ a_2^{e_1} \end{bmatrix}$$

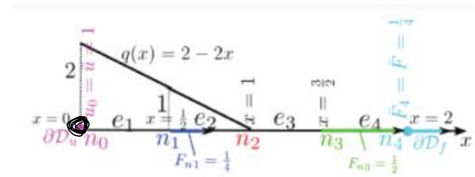
$a_1^{e_1} \rightarrow$ e_1 $\rightarrow a_2^{e_1}$

$M_{e_1} = [\bar{1}, 1]$

$$= \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} a_{\bar{1}} \\ a_1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ ? \end{bmatrix}$$

we use

we use zero instead



$$f_D^{e_1} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

free dof we put zero

distributed forces at the nodes of element

$$f_r^{e_1} = \frac{L_{e_1}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} q_1^{e_1} \\ q_2^{e_1} \end{bmatrix} = \frac{\frac{1}{2}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$f^{e_1} = f_r^{e_1} + \cancel{f_N^{e_1}} - f_D^{e_1} = \frac{1}{12} \begin{bmatrix} 5 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -19/12 \\ 28/12 \end{bmatrix}$$

$$f^{e_1} \quad \begin{array}{c} 1 \\ 1 \end{array} \begin{bmatrix} -19/12 \\ \frac{28}{12} \end{bmatrix}$$

assembly

$$F_e = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} \frac{7}{3} \\ \\ \\ \\ \end{bmatrix}$$

$$M_{e_1} = [1, 1]$$