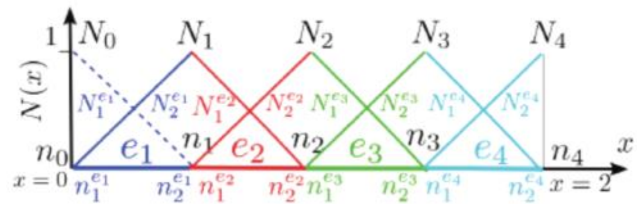
FEM20241030

Wednesday, October 30, 2024 12:51 PM

Continuing the bar problem from last time

Fe: calculate f^e for all elements

$a_1 = 1$ $\quad a_1$ $\quad a_2$ $\quad a_3$ $\quad a_4$



$$\boxed{e_1} \quad f^{e_1} = f_r^{e_1} + \cancel{f_N^{e_1}} - f_D^{e_1}$$

1D

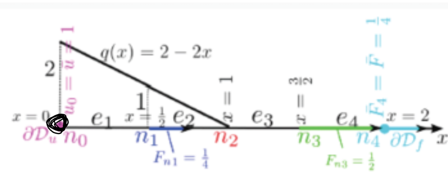
$$f_D^{e_1} = K^{e_1} a^{e_1} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} a_1^{e_1} \\ a_2^{e_1} \end{bmatrix}$$

$a_1^{e_1} \rightarrow$ ——— e_1 ——— $\rightarrow a_2^{e_1}$

$M_{e_1} = [1, 1]$

$$= \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ ? \end{bmatrix}$$

we use zero instead



$$f_D^{e_1} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

free dof we put zero

distributed forces at the nodes of element

$$f_r^{e_1} = \frac{L_{e_1}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} q_1^{e_1} \\ q_2^{e_1} \end{bmatrix} = \frac{\frac{1}{2}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$f^{e_1} = f_r^{e_1} + \cancel{f_N^{e_1}} - f_D^{e_1} = \frac{1}{12}\begin{bmatrix} 5 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -19/12 \\ 28/12 \end{bmatrix}$$

$$f^{e_1} = \begin{array}{c} 1 \\ 1 \end{array} \begin{bmatrix} -19/12 \\ \frac{28}{12} \end{bmatrix}$$

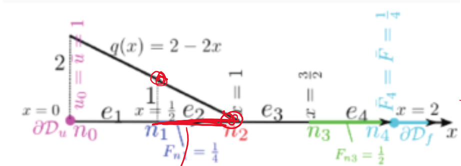
assembly

$$F_e = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} \frac{7}{3} \\ \\ \\ \end{bmatrix}$$

$$M_{e_1} = [1, 1]$$

$$\begin{array}{c} 3 \\ 4 \end{array}\begin{bmatrix} \\ \\ \end{bmatrix}$$

Element 2 force calculations:



$$f^{e_2} = f_r^{e_2} + \cancel{f_N^{e_2}} - f_D^{e_2}$$

1D

e_2

$$f_r^{e_2} = \frac{l_{e_2}}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} q_1^{e_2} \\ q_2^{e_2} \end{bmatrix} = \frac{\frac{1}{2}}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} \\ \frac{1}{12} \end{bmatrix}$$

$f_r^{e_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ clear no node on Dirichlet BC

$$f_r^{e_2} = f_r^{e_2} + \cancel{f_N^{e_2}} - f_D^{e_2} \rightarrow 0 = \begin{array}{c} M^{e_2} \\ 1 \\ 2 \end{array}\begin{bmatrix} \frac{1}{6} \\ \frac{1}{12} \end{bmatrix} \rightarrow \text{will be added to global } F_e$$

$a_1 \rightarrow$ $a_1 \rightarrow$ $a_2 \rightarrow$ $\rightarrow a_3$ $\rightarrow a_4$

n_1 $f_i = \frac{1}{4}$ n_2 n_3 n_4

n_0 e_1 e_2 e_3 e_4

$M^{e_2} = [1, 2]$ $\quad f_3 = \frac{1}{2}$ $\quad F_4 = \frac{1}{4}$

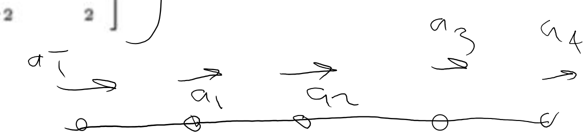
added from e_1

$$F_e = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{bmatrix} \frac{7}{3} + \frac{1}{6} \\ \frac{1}{12} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{30}{12} \\ \frac{1}{12} \\ 0 \\ 0 \end{bmatrix}$$

$$F = F_n + F_e = \begin{bmatrix} \frac{1}{4} \\ 0 \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix} + \begin{bmatrix} \frac{30}{12} \\ \frac{1}{12} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{11}{4} \\ \frac{1}{12} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}$$

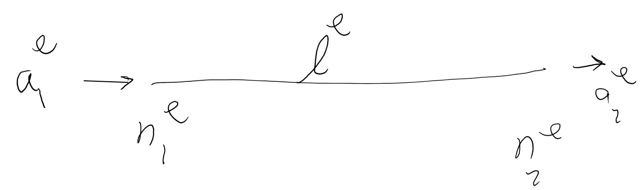
$$Ka = F$$

$$\mathbf{K} = \begin{bmatrix} 2+2 & -2 & 0 & 0 \\ -2 & 2+2 & -2 & 0 \\ 0 & -2 & 2+2 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix} \Rightarrow a = \begin{bmatrix} 43/24 \\ 53/24 \\ 31/12 \\ 65/24 \end{bmatrix}$$

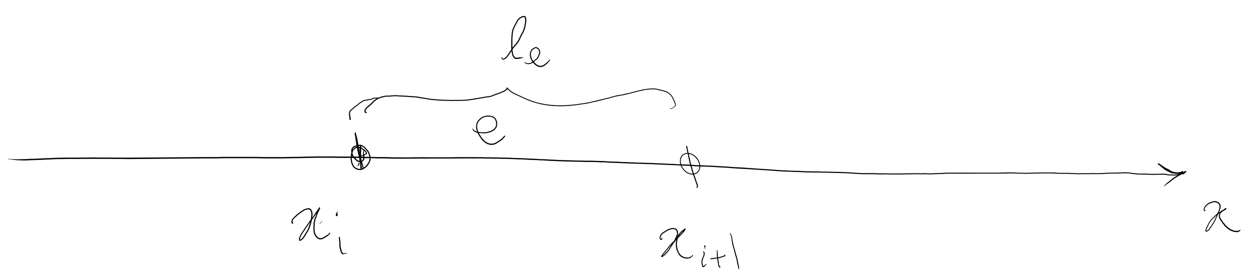


Next thing

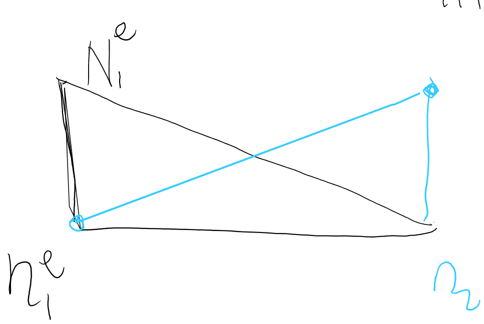
$$k^e = \left(\frac{AE}{L}\right)_e \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$



$$f_f^e = r^e \begin{bmatrix} q^1 \\ q^2 \end{bmatrix} \qquad r^e = \frac{l_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$



$$N_1^e = \frac{x_{i+1} - x}{x_{i+1} - x_i} = \frac{x_{i+1} - x}{l_e}$$



$$N_2^e = \frac{x - x_i}{x_{i+1} - x_i} = \frac{x - x_i}{l_e}$$

$$K^e = \int_e B^{e^T} D B^e \, dx$$

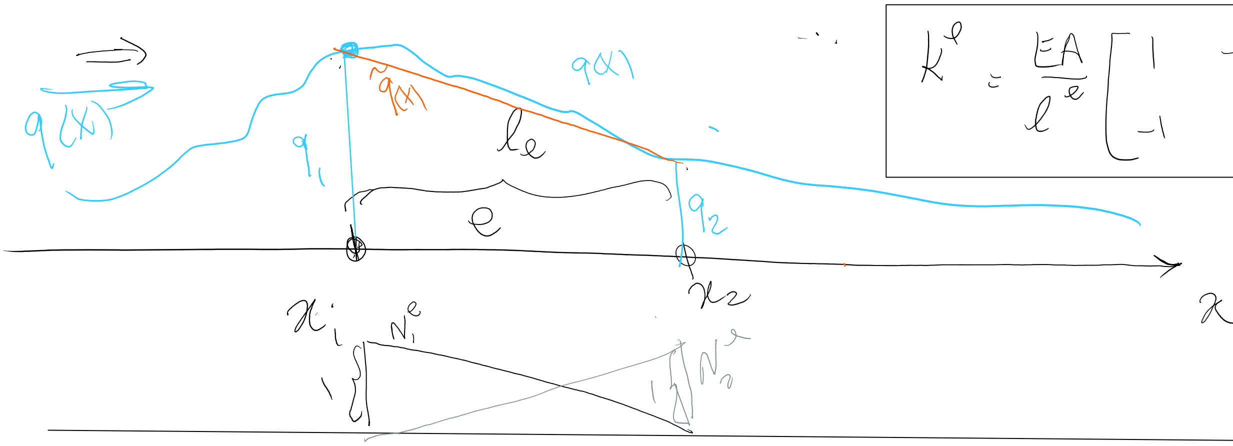
$$N^e = [N_1^e, N_2^e] = \left[\frac{x_{i+1} - x}{l_e}, \frac{x - x_i}{l_e} \right]$$

$$B^e = \mathcal{L}_m (N^e) = [N_1^e, N_2^e]' = \frac{1}{l_e}[-1, 1]$$

$D = EA$ assume EA are constant.

$$K^e = \int_{x_i}^{x_{i+1}} \frac{1}{l_e}\begin{bmatrix} -1 \\ 1 \end{bmatrix} EA \frac{1}{l_e}[-1 \;\; 1]\, dx = \frac{EA}{l_e^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \underbrace{(x_{i+1} - x_i)}_{l_e}$$

$$\boxed{K^e = \frac{EA}{l^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$$



2nd part

$$f_r^e = \int_e N^t q\, dx = \int_{x_i}^{x_{i+1}} \begin{bmatrix} N_1^e \\ N_2^e \end{bmatrix} q(x)\, dx \approx \int \begin{bmatrix} N_1^e \\ N_2^e \end{bmatrix} \tilde{q}(x)\, dx \quad (i)$$

$\tilde{q}(x)$: Linear approximation of $q(x)$

it matches $q(x)$ @ nodes

$$\tilde{q}(x) = q_1 N_1^e(x) + q_2 N_2^e(x) \quad (ii)$$

@ x_i $(N_1^e) = q_1 \times 1 + q_2 \times 0 = q_1$

$q(x) = q_1 N_1(x) + q_2 N_2(x)$ (iii) @ $x_i(n_1^e) = q_1 \times 1 + q_2 \times 0 = q_1$

@ $x_{i+1}(n_2^e) = q_1 \times 0 + q_2 \times 1 = q_2$

Similar to
$u(x) = a_1 N_1^e(x) + a_2 N_2^e(x)$
solution

plug (iii) in (i): $f_1^e \approx \int_{x_i}^{x_{i+1}} \begin{bmatrix} N_1^e \\ N_2^e \end{bmatrix} (N_1^e q_1 + N_2^e q_2)\, dx = \underbrace{\left(\int_{x_i}^{x_{i+1}} \begin{bmatrix} N_1^e \\ N_2^e \end{bmatrix} [N_1^e \; N_2^e]\, dx \right)}_{(r_e)_{2\times 2}} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$

$$\boxed{r_e = \frac{l_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \;\&\; f_1^e = r_e \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}}$$

- We already have a discretization error in FEM

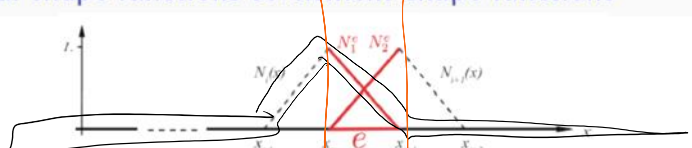
$e = Ch^{p+1}$ here $= Ch^2$

for element of order p ($p=1$ here)
& element size h
as $h \to 0$

- Any other approximation that we make along the way whose corresponding error is of the **same order or smaller** than discretization error is perfectly fine.

$q(x) \to \tilde{q}(x)$

Global shape functions to element shape functions



- Finite Elements are the domain subdivisions that are used for the construction of the shape functions
- Restriction of (global) shape functions to elements form the elements' shape functions (local).
- To distinguish element level and global level quantities, any element level value is decorated by $(.)^e$.
- Local node numbers in the element start from 1 to number of nodes in element n_n^e and are denoted by $n_1^e, \ldots, n_{n_n^e}^e$.
- Similarly local dof start from 1 to the number of dof in element n_{dof}^e.
- For example in the figure both n_n^e and n_{dof}^e are both 2 and the range for local node number and dof is from 1 to 2.
- Element shape functions satisfy the condition,

$$N_i^e(n_j^e) = \delta_{ij} \qquad (325)$$

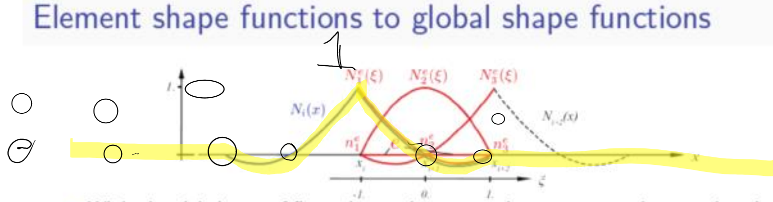
number and dof is from 1 to 2.

- Element shape functions satisfy the condition,

$$N_i^e(n_j^e) = \delta_{ij} \tag{325}$$

- More generally (*e.g.*, beam elements), shape function i has a value 1 at dof i while has a value zero at all other element dofs.

Element shape functions to global shape functions



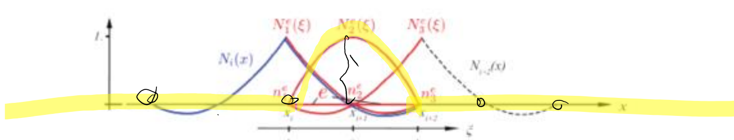
- While the global view of finite element has some advantages in mathematical analysis, we often form the shape functions at the local level and if needed form global shape functions.
- It was this local perspective that first was employed in engineering finite element analysis.
- For example, in the figure the 1D bar element has three nodes with one being internal node and has interpolation order $p = 2$.
- We observe that,

$$N_i^e(n_j^e) = \delta_{ij} \quad \Rightarrow N_I(n_J) = \delta_{IJ}$$

which was the condition we first stipulated for finite elements in global view.

- As an example, we observe that the global shape function $N_i(x)$ is formed from local element shape functions.
- Notice that while local element order is $p = 2$ the global shape functions are still C^0 (piece-wise quadratic in this case).
- Elements can have internal nodes. This generally occurs for higher than linear elements ($p > 1$).

Element shape functions to global shape functions



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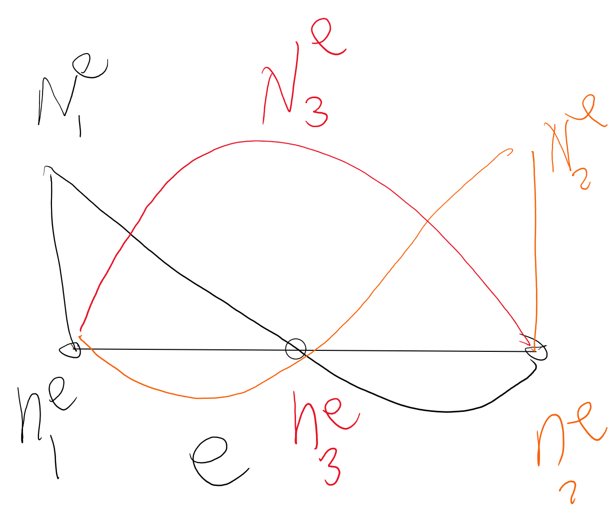
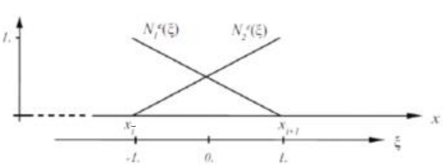
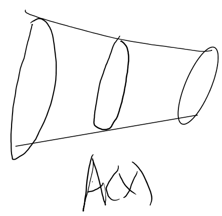


p=2 element

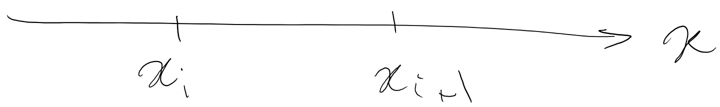
Calculate the element stiffness matrix when E and A are not constant

Local coordinate system



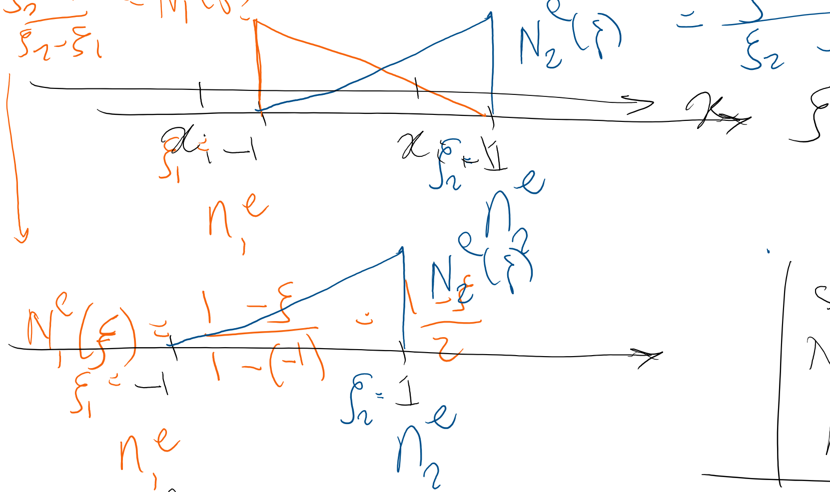


$E(x)$



$$\frac{\xi_2-\xi}{\xi_2-\xi_1} = N_1^e(\xi)$$

$$N_2^e(\xi) = \frac{\xi-\xi_1}{\xi_2-\xi_1} = \frac{\xi-(-1)}{1-(-1)} = \frac{\xi+1}{2}$$

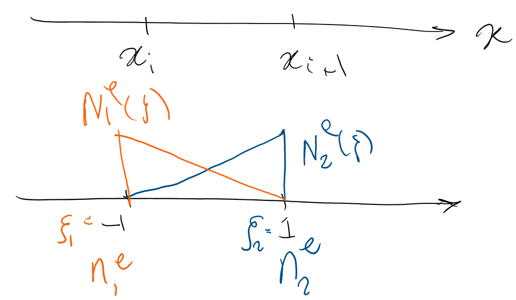


$$N_1^e(\xi) = \frac{1-\xi}{1-(-1)} = \frac{1-\xi}{2}$$

side note we previously wrote N's in terms of x

$$N_1^e(x) = \frac{x_{i+1}-x}{x_{i+1}-x_i} \qquad N_2(x) = \frac{x-x_i}{x_{i+1}-x_i}$$

$$K^e = \int_e B^{e^T} D_e B^e \, dx$$



$$B^e = \mathcal{L}_m(N_e) = \frac{d}{dx}\left[N_1^e \ N_2^e\right]$$

$$\cancel{\frac{d}{d\xi}}$$

$$B^e = \frac{d}{dx}\left[N_1^e(\xi) \quad N_2^e(\xi)\right]$$

weak statement $\int \frac{dw}{dx} EA \frac{du}{dx}$

$$B^e = \underbrace{\frac{d}{d\xi}\left[N_1^e \ N_2^e\right]}_{B^e} \frac{d\xi}{dx}$$

?

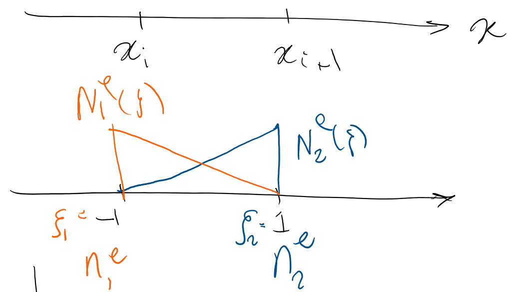
$$\boxed{B^e = B^e_\xi \left(\frac{d\xi}{dx}\right)}$$

B^e ... $\frac{d}{\ }\left[1-\xi \quad 1+\xi\right.$

$$\underbrace{}\; B^e_\xi$$

$$\boxed{\begin{aligned} B^e_\xi &= \frac{d}{d\xi} N = \frac{d}{d\xi}\left[\frac{1-\xi}{2},\ \frac{1+\xi}{2}\right] \\ B^e_\xi &= \left[\frac{-1}{2},\ \frac{1}{2}\right] \end{aligned}}$$

$$K^e = \int_{x_i}^{x_{i+1}} B^{e^t} \underbrace{E^e A^e}_{D^e} B^e \, \underbrace{dx}$$



change of variable

$$K^e = \int_{-1}^{1} B^{e^t} E^e A^e B^e \left[\left(\frac{dx}{d\xi}\right) d\xi\right]$$

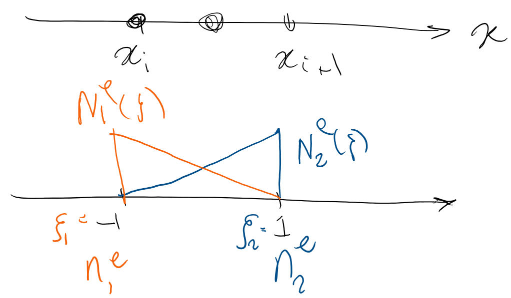
$\hookrightarrow$ Jacobian

All we need is $\frac{dx}{d\xi}$

Approach 1 (dumb)

$$x_{ave} = \frac{x_i + x_{i+1}}{2}$$

$$L_e = x_{i+1} - x_i$$



$$x = A + B\xi$$

@ $\xi_1 = -1 \implies x = x_i$

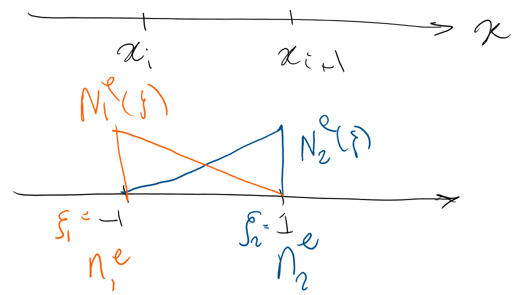
@ $\xi_2 = 1 \implies x = x_{i+1}$

$$\left.\begin{aligned} A - B &= x_i \\ A + B &= x_{i+1} \end{aligned}\right\} \implies \begin{aligned} A &= \frac{x_{i+1} + x_i}{2} = x_{ave} \\ B &= \frac{x_{i+1} - x_i}{2} = \frac{l_e}{2} \end{aligned}$$

$$\boxed{x(\xi) = \frac{x_{i+1} + x_i}{2} + \xi\left(\frac{x_{i+1} - x_i}{2}\right)}$$

$$= x_{ave} + \frac{\xi}{2} L_e$$

2) Smart approach



$$x(\xi) = x_i N_1^e(\xi) + x_{i+1} N_2^e(\xi)$$

$$x_i = x_i \times 1 + x_{i+1} \times 0 \qquad \xi=-1$$

$$x_{i+1} = x_i \times 0 + x_{i+1} \times 1 \qquad \xi=1$$

$$x(\xi) = x_i\left(\frac{1-\xi}{2}\right) + x_{i+1}\left(\frac{1+\xi}{2}\right)$$

$$= \underbrace{\frac{x_i + x_{i+1}}{2}}_{x_{ave}} + \underbrace{\left(\frac{x_{i+1}-x_i}{2}\right)}_{\frac{l_e}{2}}\xi$$

other examples of shape function

$$u(x) = a_1 N_1 + a_2 N_2$$

$$\tilde{q}(x) = \tilde{q}_1 N_1 + \tilde{q}_2 N_2$$

$$x(\xi) = x_1^e N_1 + x_2^e N_2$$

Either way

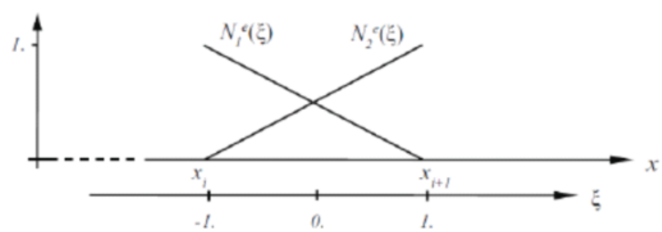
$$\Longrightarrow \boxed{\begin{aligned} x(\xi) &= x_{ave}^e + \frac{l^e}{2}\xi \\ \frac{dx}{d\xi} &= \frac{l^e}{2} \end{aligned}}$$

we had

$$K^e = \int_e B^{e^t} D^e B^e \, dx = \int_{\xi=-1}^{1} B^{e^t} D^e B^e \underbrace{\left(\frac{dx}{d\xi}\right)}_{} d\xi \qquad \frac{l^e}{2}$$

$$B^e = B^e_\xi \frac{d\xi}{dX} = B^e_\xi \left(\frac{1}{\frac{dX}{d\xi}}\right) = \left[-\frac{1}{2} \quad \frac{1}{2}\right] \frac{2}{l^e} = [-1 \quad 1]$$

Stiffness matrix: Local coordinate system



$A(X) \to A(\xi)$

- Finally, we plug (371) and (372) into (366) to obtain,

$$\mathbf{k}^e = \int_{-1}^{1} \begin{bmatrix} -\frac{1}{L^e} \\ \frac{1}{L^e} \end{bmatrix} E(\xi)A(\xi) \left[-\frac{1}{L^e} \quad \frac{1}{L^e}\right] \left\{\frac{L^e}{2} \mathrm{d}\xi\right\} \quad \Rightarrow$$

$$\boxed{\mathbf{k}^e = \frac{1}{2L^e} \int_{-1}^{1} E(\xi)A(\xi) \, \mathrm{d}\xi \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}} \tag{373}$$

- If A and E are constant along the bar, we have:

$$\boxed{\mathbf{k}^e = \frac{AE}{L^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{(constant } A \text{ and } E\text{)}} \tag{374}$$