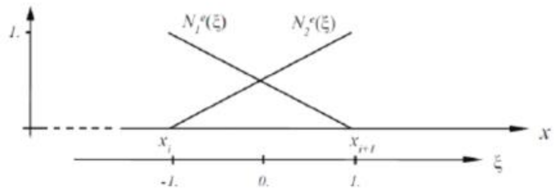
FEM20241104

Sunday, November 3, 2024 11:50 AM

From last time:

Stiffness matrix: Local coordinate system



- Finally, we plug (371) and (372) into (366) to obtain,

$$\mathbf{k}^e = \int_{-1}^{1} \begin{bmatrix} -\frac{1}{L^e} \\ \frac{1}{L^e} \end{bmatrix} E(\xi)A(\xi) \begin{bmatrix} -\frac{1}{L^e} & \frac{1}{L^e} \end{bmatrix} \{\frac{L^e}{2}\mathrm{d}\xi\} \quad \Rightarrow$$

$$\boxed{\mathbf{k}^e = \frac{1}{2L^e}\int_{-1}^{1} E(\xi)A(\xi)\,\mathrm{d}\xi \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}} \qquad (373)$$

FEM stiffness k in $P = k\Delta u$

- If A and E are constant along the bar, we have:

$$\boxed{\mathbf{k}^e = \frac{AE}{L^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{(constant } A \text{ and } E\text{)}} \qquad (374)$$

k

305 / 456

FEM

$u = N_1 u_1 + N_2 u_2$

$\Downarrow$

$\varepsilon = \frac{du}{dx}$

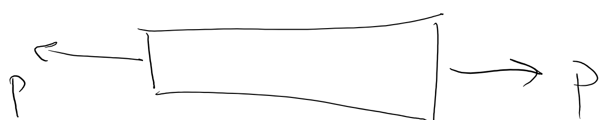
$\Downarrow$

$\sigma = E\varepsilon$

$\Downarrow$

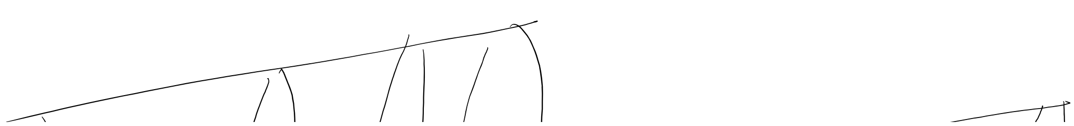
$F = \sigma A$

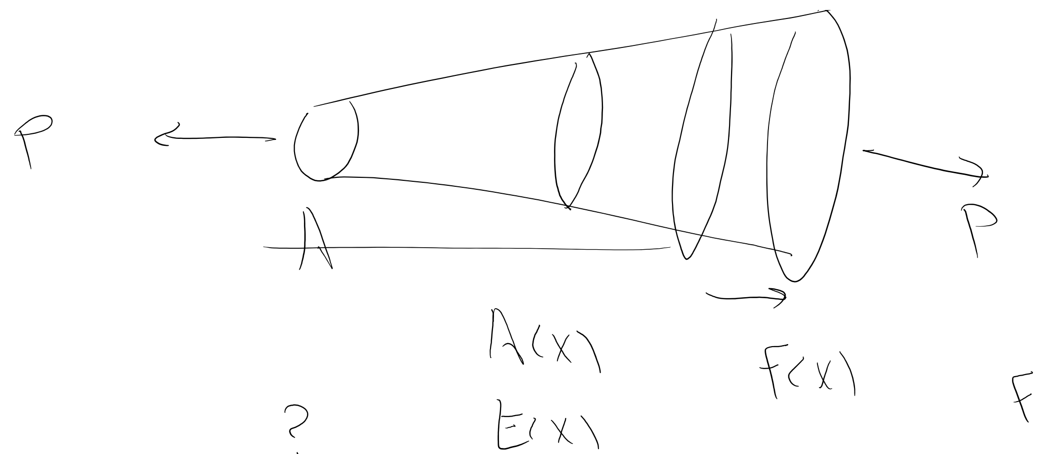
$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \underbrace{k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{= K}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Rightarrow P = F_2 = -F_1 = k\underbrace{(u_2 - u_1)}_{\Delta u}$$

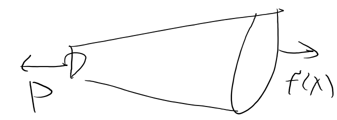


$P = k\Delta u$

A, E constant $\qquad k = \frac{AE}{L}$







$$F(x) = P$$

$$\downarrow$$

?

$$P = (k)\, \Delta u$$

$$\sigma(x) = \frac{F(x)}{A(x)} = \frac{P}{A(x)}$$

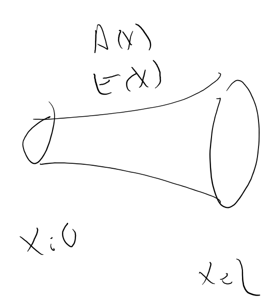
$$\downarrow$$

$$\varepsilon(x) = \frac{\sigma(x)}{E(x)} = \frac{P}{A(x)E(x)}$$

$$\varepsilon = \frac{du}{dx} \Rightarrow \quad u_2 - u_1 = \int_{x=0}^{x=L} \varepsilon(x)\,dx = \int_0^L \frac{P}{A(x)E(x)}\,dx$$

$$\Rightarrow \underbrace{u_2 - u_1}_{\Delta u} = P \int_0^L \frac{dx}{A(x)E(x)} \Longrightarrow$$

$$\frac{P}{\Delta u} = \frac{1}{\int_0^L \frac{dx}{A(x)E(x)}} = k^{exact}$$



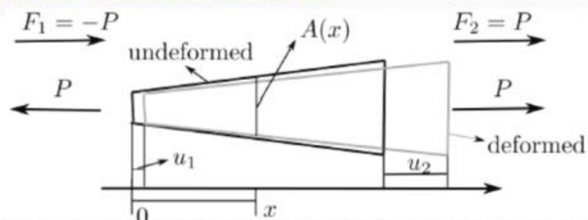
$$\text{FEM} \quad \frac{P}{\Delta u} = \frac{1}{L}\int_0^L A(x)E(x)\,dx = k^{FEM}$$

$$\overline{\overline{K}} = k \begin{bmatrix} 1 & -1 \end{bmatrix}$$

$$\overline{K} = k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Please read:

Stiffness matrix: Direct method



- The engineering concepts of finite element method were first based on directly obtaining relations between deformations and forces without restoring to shape functions (*e.g.*, assumed linear displacement field we used before).
- First, we observe that for static equilibrium $F_1 = -F_2$. We denote $P := F_2$.
- Second, we mechanics equations to obtain u_2 assuming that the displacement is equal to u_1 at the left end ($x = 0$). Let $F(x)$ be the axial force at x:

$$F(x) = P \qquad \text{Axial force} \qquad (375a)$$

$$s(x) = \frac{F(x)}{A(x)} \qquad \text{Stress} \qquad (375b)$$

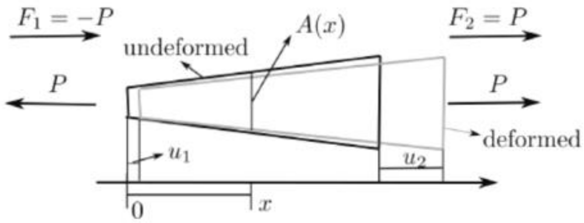
$$\epsilon(x) = \frac{s(x)}{E(x)} = \frac{F(x)}{A(x)E(x)} \qquad \text{Strain} \qquad (375c)$$

$$u(x) = u_1 + \int_0^x \epsilon(y)\,\mathrm{d}y = u_1 + \int_0^L \frac{F(y)}{A(y)E(y)}\,\mathrm{d}y \qquad \text{Displacement} \qquad (375d)$$

$$u_2 - u_1 = \int_0^L \frac{F(x)}{A(x)E(x)}\,\mathrm{d}x = P\int_0^L \frac{1}{A(x)E(x)}\,\mathrm{d}x \qquad \text{Displacement jump} \qquad (375e)$$

307 / 456

Stiffness matrix: Direct method



- Thus, noting that $F_2 = -F_1 = P$ we have,

$$\boxed{\mathbf{k}^e = \frac{1}{\int_0^L \frac{1}{E(x)A(x)}\,\mathrm{d}x}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}} \tag{376}$$

- If A and E are constant along the bar, we have:

$$\boxed{\mathbf{k}^e = \frac{AE}{L^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (\text{constant } A \text{ and } E)} \tag{377}$$

- We observe that (374) and (377) coincide. Why is that?
- Do (373) and (376) match in general? If not, why? Which one is approximate and which one is exact?

308 / 456

Summary of local (element-centered) approach
- K and F are formed at the element level
- Using M (dofMap) they are assembled to the global K and F

- Local stiffness matrix, $\mathbf{k}^e$, is given by (*cf.* (346b) and (340)):

$$\mathbf{k}^e = \int_e \mathbf{B}^{e\mathrm{T}}\mathbf{D}\mathbf{B}^e\,\mathrm{dv}, \quad \text{where} \quad \mathbf{B}^e = L_m(\mathbf{N}^e) \tag{362}$$

- Local force vector, $\mathbf{f}_e^e$, is the assembly of all element level forces:

$$\mathbf{f}_e^e := \mathbf{f}_r^e + \mathbf{f}_N^e - \mathbf{f}_D^e \tag{363a}$$

$$\mathbf{f}_r^e = \int_e \mathbf{N}^{e\mathrm{T}}.\mathbf{r}\,\mathrm{dv} \tag{363b}$$

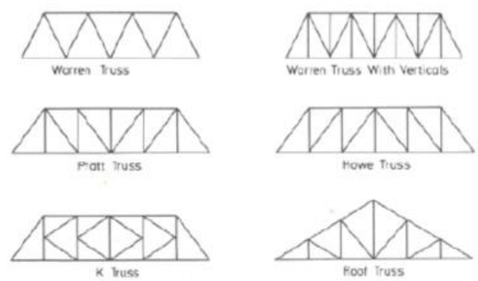
$$\mathbf{f}_N^e = \int_{\partial e_f} \mathbf{N}^{e\mathrm{T}}\bar{\mathbf{F}}.\,\mathrm{ds} \tag{363c}$$

$$\mathbf{f}_D^e = \mathbf{k}^e\mathbf{a}^e \tag{363d}$$

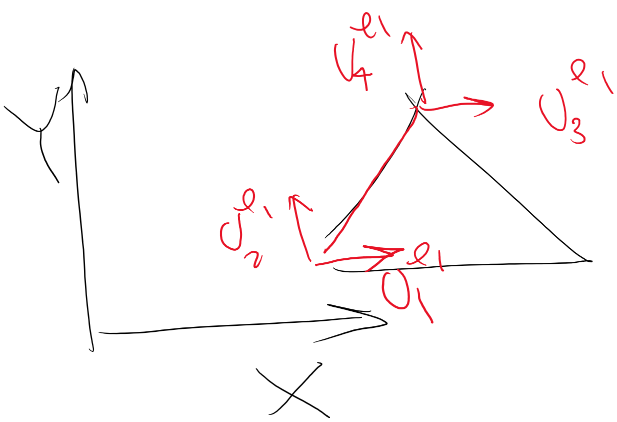
295 / 456

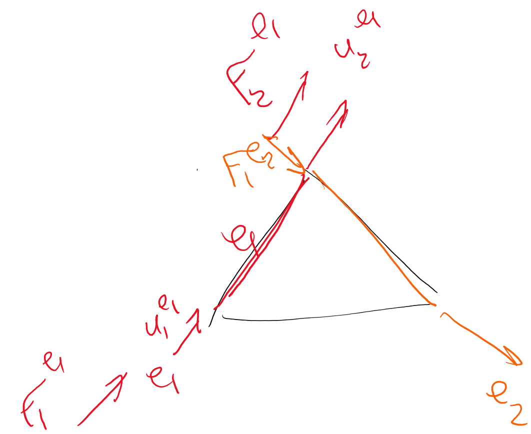
New topic:
Trusses:

- Having a separate coordinate system for the element and the global domain.



Types of simple Plane truss





By all elements having X and Y displacements and forces, they can communicate and we can assemble the global K and F

---

lower case: bar

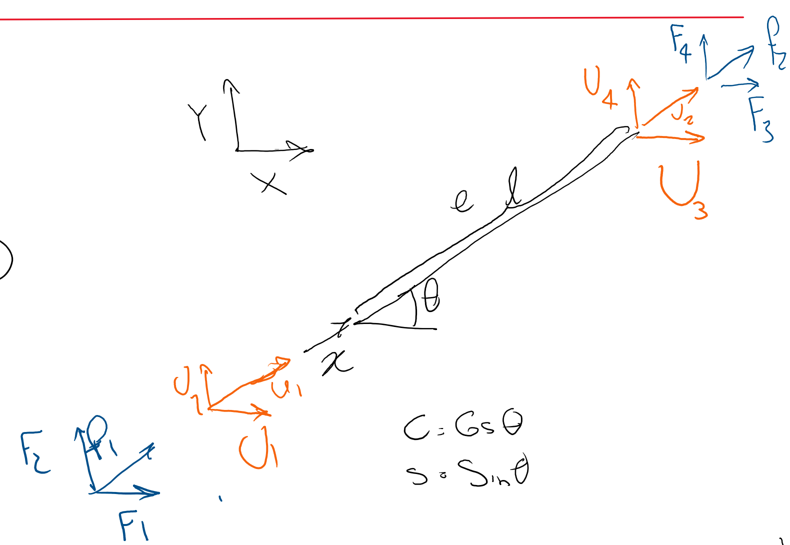
upper case: Truss

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \underset{\text{bar stiffness}}{k^e_{2\times 2}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad (1)$$

$$k^e = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad (2)$$

we want:

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F \end{bmatrix} = K^e_{4\times 4} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ \vdots \end{bmatrix} \qquad (3)$$



$C = \cos\theta$

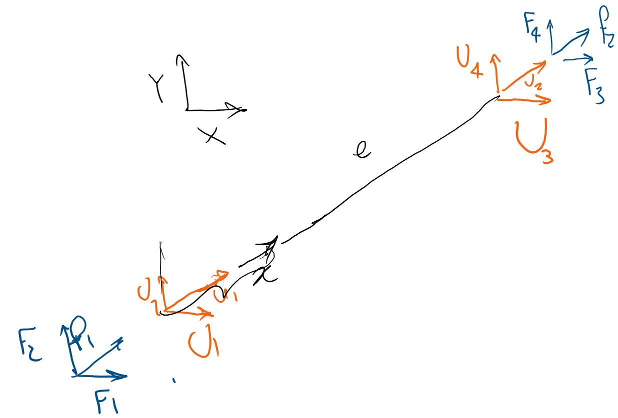
$s = \sin\theta$

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = K_{2\times 4} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}$$

$$\begin{bmatrix} F_3 \\ F_4 \end{bmatrix} \quad \underbrace{4\times4}_{\text{truss stiffness}} \begin{bmatrix} U_3 \\ U_4 \end{bmatrix} \qquad \underbrace{\qquad 4\times4 \begin{bmatrix} U_3 \\ U_4 \end{bmatrix}}_{\text{we'll first get here}}$$

(1)

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = k^e \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (1)$$



$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_{2\times1} = T_U\,{}_{2\times4} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}_{4\times1} \quad (2)$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_{2\times1} = \underbrace{\begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix}_{2\times4}}_{T_U} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}_{4\times1}$$

$U_1 = 1$
$U_2 = 0$
$U_3 = 0$
$U_4 = 0$

(1)
(2)
$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = k^e\, T_U \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} \quad (3)$$

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} \quad (3)$$

$$\boxed{f_2 = -f_1 = \frac{AE}{L}\left(c(U_3 - U_1) + s(U_4 - U_2)\right)}$$

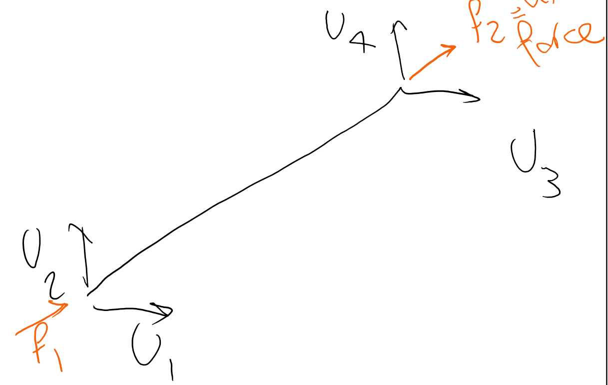
(4)

axial

$U_4 \uparrow \rightarrow f_2 =$ axial force

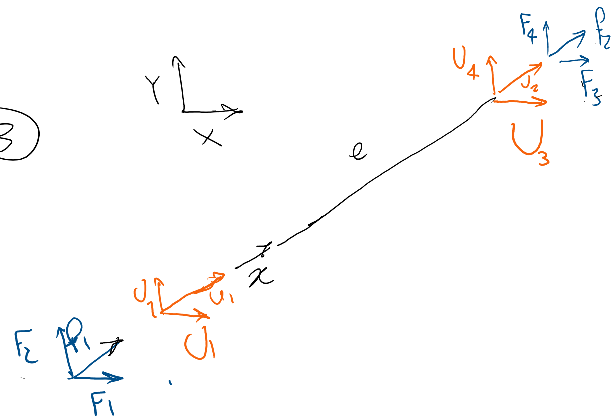
(4) axial force

Calculating axial force from U_1 to U_4



Continue from eqn (3)

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = k^e T_{UU} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} \quad (3)$$



$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \Big(\quad \Big) \;"$$

we need

$$\underset{4\times1}{\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}} = \underbrace{\underset{4\times2}{\begin{bmatrix} c & 0 \\ s & 0 \\ 0 & c \\ 0 & s \end{bmatrix}}}_{T_{Ff} = T_{UU}^{t}} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}_{2\times1}$$

$f_1 = 1$

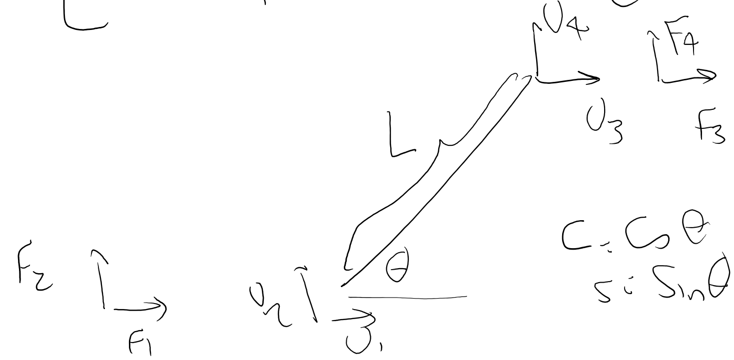
$f_2 = 0$

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \underbrace{T_{Ff\,4\times2}\, k^e_{2\times2}\, T_{UU\,2\times4}}_{K^e} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}$$

$$K_{4\times4}$$

(4)

$$\begin{bmatrix}F_1\\F_2\\F_3\\F_4\end{bmatrix} = \frac{AE}{L}\left[\begin{array}{c|c} Kb_{2x2} & -Kb_{2x2}\\ \hline -Kb_{2x2} & Kb_{2x2}\end{array}\right], \quad k_b = \begin{bmatrix}c^2 & cs\\ cs & s^2\end{bmatrix}$$



$c = \cos\theta$
$s = \sin\theta$

(386)

- Noting that f_2 corresponds to tensile axial force in the bar, which we denote by T we have,

$$\boxed{T = \frac{AE}{L}\left\{c(U_3 - U_1) + s(U_4 - U_2)\right\}} \tag{387}$$

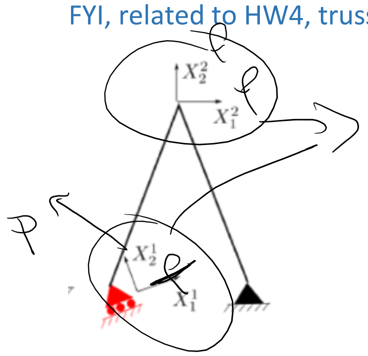
315 / 456

$$\mathbf{K} = \mathbf{T}^{\mathrm{T}}\mathbf{k}\mathbf{T} \;\Rightarrow \mathbf{K} = \frac{AE}{L}\left[\begin{array}{c|c}\mathbf{k}_b & -\mathbf{k}_b\\ -\mathbf{k}_b & \mathbf{k}_b\end{array}\right], \text{ where } \mathbf{k}_b = \begin{bmatrix}c^2 & cs\\ cs & s^2\end{bmatrix} \text{ that is}$$

$$\mathbf{K} = \frac{AE}{L}\begin{bmatrix}c^2 & cs & -c^2 & -cs\\ cs & s^2 & -cs & -s^2\\ -c^2 & -cs & c^2 & cs\\ -cs & -s^2 & cs & s^2\end{bmatrix} \tag{390}$$

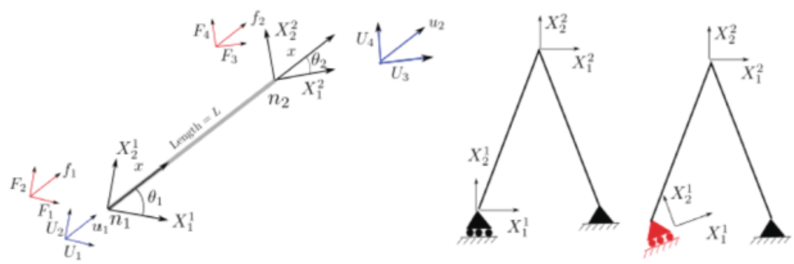
316 / 456

FYI, related to HW4, truss problem



different coordinate systems at ends of the element

Truss element /two different coordinate systems



- As before $\mathbf{T} := \mathbf{T}_{\mathbf{uU}} = \mathbf{T}_{\mathbf{Ff}}$ and in this case is given by,

$$\mathbf{T} = \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ 0 & 0 & c_2 & s_2 \end{bmatrix} \tag{393}$$

- Accordingly, from $\mathbf{K} = \mathbf{T}^{\mathrm{T}}\mathbf{k}\mathbf{T}$ we obtain,

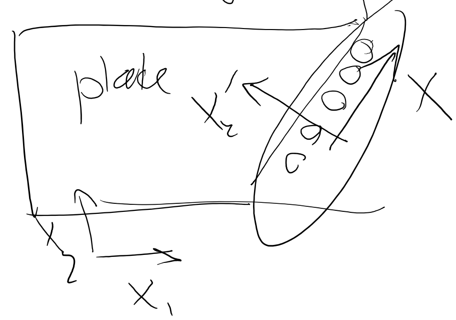
$$\mathbf{K} = \frac{AE}{L}\begin{bmatrix} c_1^2 & c_1 s_1 & -c_1 c_2 & -c_1 s_2 \\ c_1 s_1 & s_1^2 & -c_2 s_1 & -s_1 s_2 \\ -c_1 c_2 & -c_2 s_1 & c_2^2 & c_2 s_2 \\ -c_1 s_2 & -s_1 s_2 & c_2 s_2 & s_2^2 \end{bmatrix} \tag{394}$$

- Finally the axial tensile force in the bar, which is the second line of $\mathbf{k}\mathbf{T}_{\mathbf{uU}} = \mathbf{k}\mathbf{T}$ is (compare to one global coordinate in (387)):

$$\boxed{T = AE/L\,(-c_1 U_1 - s_1 U_2 + c_2 U_3 + s_2 U_4)} \tag{395}$$

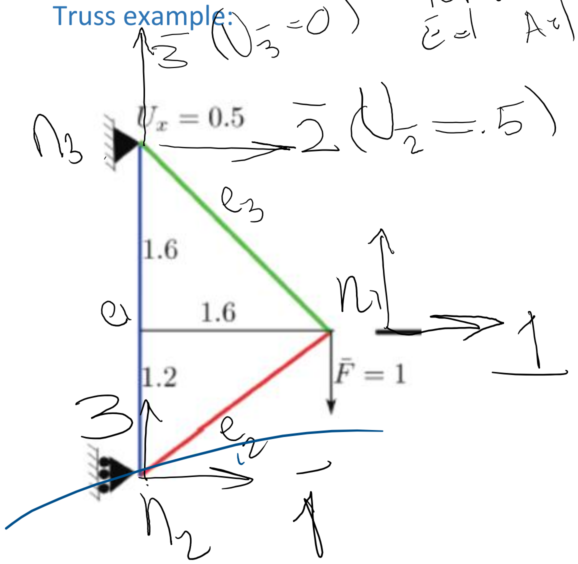
319 / 456

slant mixed BC
same technique is used

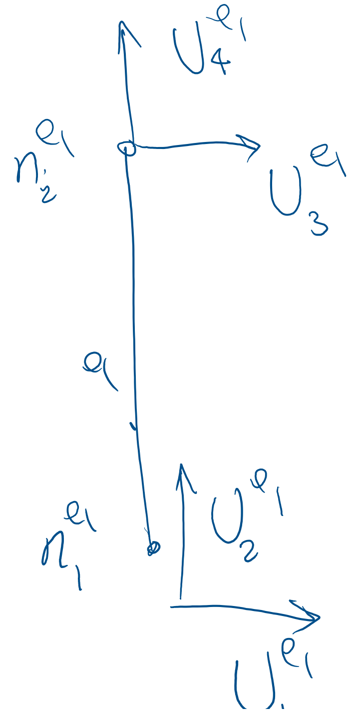


Truss example:

for all 3 bars $E=1$ $A=1$



$np = 3$



$L^{e_1} = 2.8$
$A^{e_1} = 1$
$E^{e_1} = 1$
$\theta^{e_1} = \frac{\pi}{2}$
$c^{e_1} = 0$
$s^{e_1} = 1$

$(a_{\bar{1}} = 0)$ $\quad n_f = 3$ $\quad n_p = 3$

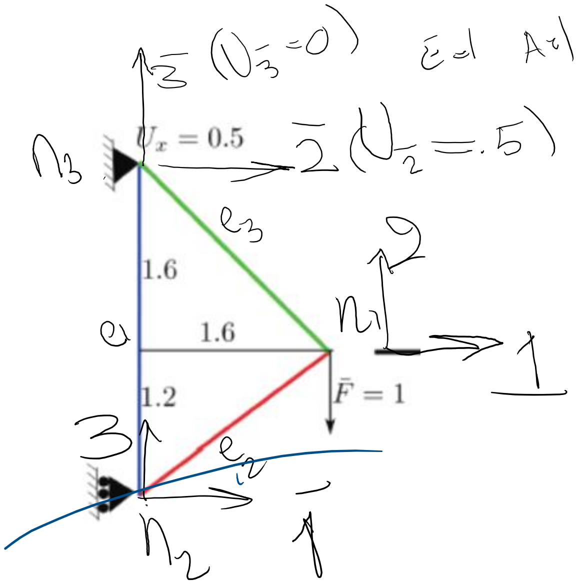
$$a_p = \begin{bmatrix} a_{\bar{1}} \\ a_{\bar{2}} \\ a_{\bar{3}} \end{bmatrix} = \begin{bmatrix} 0 \\ .5 \\ 0 \end{bmatrix}$$

$\vec{U}_1^{e_1}$

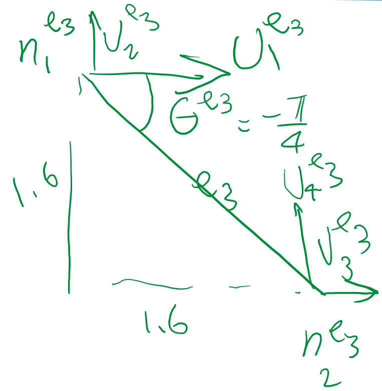
$LEM^{e_1} = [2, 3]$ nodal map

$M^{e_1} = [\bar{1}, 3, \bar{2}, \bar{3}]$ dof map

All the calculations for element 3:



$(a_{\bar{1}} = 0)$



$l^{e_3} = \sqrt{2} \times 1.6$

$A^{e_3} = 1 \quad E^{e_3} = 1$

$c^{e_3} = \cos -\frac{\pi}{4} = +\frac{\sqrt{2}}{2}$

$s^{e_3} = \sin(-\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$

$LEM^{e_2} = [3, 1]$

$M^{e} = [\bar{2}, \bar{3}, 1, 2]$

$$K_{4\times4}^{e_3} = \left(\frac{AE}{l}\right)^{e_3} \left[\begin{array}{c|c} k_b & -k_b \\ \hline -k_b & k_b \end{array}\right] \qquad k_b = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}$$

M^{e_3}

$$K^{e} = \begin{array}{c|cccc} & \bar{2} & \bar{3} & 1 & 2 \\ \hline \bar{2} & .221 & -.221 & -.221 & +.221 \\ \bar{3} & -.221 & .221 & +.221 & -.221 \\ 1 & & & .221 & -.221 \\ 2 & & & -.221 & .221 \end{array}$$

$$K = \begin{bmatrix} .221 & -.221 & \\ -.221 & .221 & \\ & & \end{bmatrix}$$

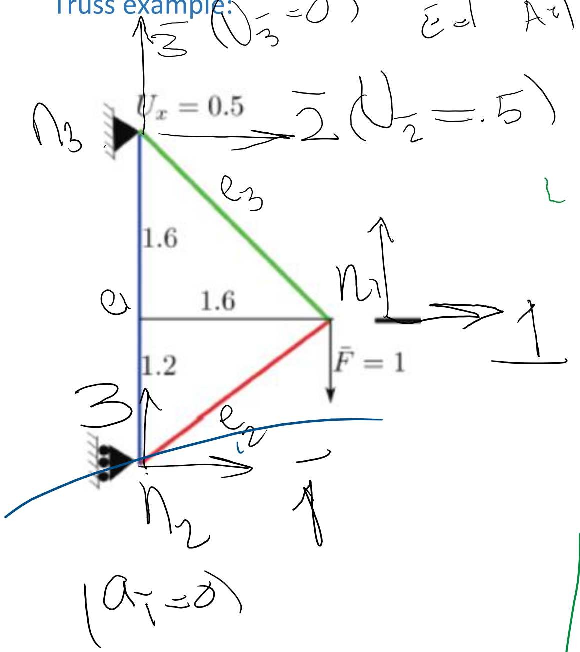
Assemble the element force to global force

$f^{e_3} = f^{e_3} + f^{e_3} - f^{e_3}$

Truss example:

Assemble the element force to global force

Truss example:



$$f^{e_3} = \cancel{f_r^{e_3}} + \cancel{f_N^{e_3}} - f_D^{e_3}$$

0 ignore the weight

1D elements

$$f_D^{e_3} = k^{e_3} a^{e_3} =$$

$$\begin{bmatrix} .221 & -.221 & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} .5 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.1105 \\ -.1105 \\ -.1105 \\ 0.1105 \end{bmatrix}$$

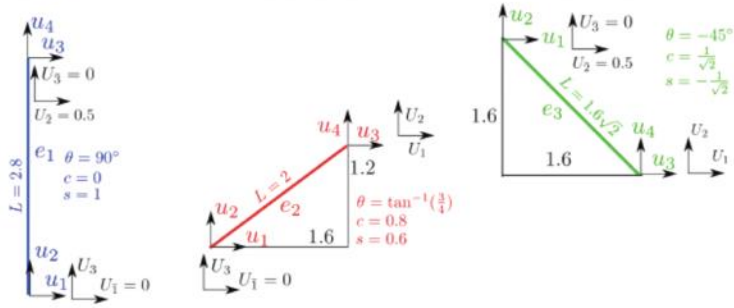
put zero for free dofs

$$M^{e_3} : [\bar{2}, \bar{3}, 1, 2] \qquad a^{e_3} : \begin{bmatrix} U_{\bar{2}} \\ U_{\bar{3}} \\ U_1 \\ U_2 \end{bmatrix} \xrightarrow{M^{e_3}}$$

$$f^{e_3} = \cancel{f_r^{e_3}} + \cancel{f_N^{e_3}} - f_D^{e_3} = \begin{array}{c} \bar{2} \\ \bar{3} \\ 1 \\ 2 \end{array} \begin{bmatrix} \cancel{-0.1105} \\ \cancel{0.1105} \\ 0.1105 \\ -0.1105 \end{bmatrix}$$

$$F^e = \begin{bmatrix} .1105 \\ -.1105 \\ \\ \end{bmatrix}$$

Truss Example



- Table below summarizes parameters for each element:

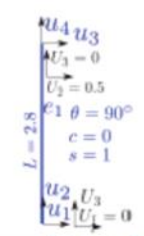
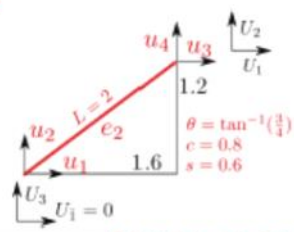
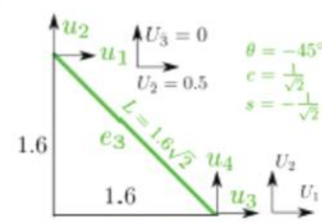
e	L^e	θ	c	s	$\mathbf{M}_t^e$
e_1	2.8	90°	0	1	$[\bar{1}\ \ 3\ \ \bar{2}\ \ \bar{3}]$
e_2	2	$\tan^{-1}(\frac{3}{4})$	0.8	0.6	$[\bar{1}\ \ 3\ \ 1\ \ 2]$
e_3	$1.6\sqrt{2}$	$-45°$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$[\bar{2}\ \ \bar{3}\ \ 1\ \ 2]$

- Local stiffness matrices are given by (390):

$$\mathbf{k}^e = \frac{AE}{L}\left[\begin{array}{c|c} \mathbf{k}_b & -\mathbf{k}_b \\ \hline -\mathbf{k}_b & \mathbf{k}_b \end{array}\right],\ \mathbf{k}_b = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}\ :\ \mathbf{k}^e = \frac{AE}{L}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

- As mentioned for trusses generally $\mathbf{f}_r^e = 0$ (no body force), similar to bars we lump natural BC into nodal forces, and finally $\mathbf{f}_D^e = \mathbf{k}^e\mathbf{a}^e$.

Truss example: Assembly of global system

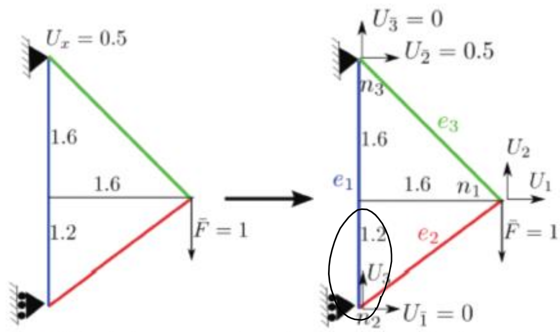
e	e_1	e_2	e_3
			
$\mathbf{k}^e$	$\mathbf{k}^{e_1} = \frac{(1)(1)}{2.8}\begin{bmatrix} 0&0&0&0\\0&1&0&-1\\0&0&0&0\\0&-1&0&1\end{bmatrix}$ $= \begin{array}{c\|cccc} & \bar{1} & 3 & \bar{2} & \bar{3} \\ \hline \bar{1} & 0 & 0 & 0 & 0 \\ 3 & 0 & 0.3571 & 0 & -0.3571 \\ \bar{2} & 0 & 0 & 0 & 0 \\ \bar{3} & 0 & -0.3571 & 0 & 0.3571 \end{array}$	$\mathbf{k}^{e_2} = \frac{(1)(1)}{2}\begin{bmatrix} 0.64&0.48&-0.64&-0.48\\0.48&0.36&-0.48&-0.36\\-0.64&-0.48&0.64&0.48\\-0.48&-0.36&0.48&0.36\end{bmatrix}$ $= \begin{array}{c\|cccc} & \bar{1} & 3 & 1 & 2 \\ \hline \bar{1} & 0.32 & 0.24 & -0.32 & -0.24 \\ 3 & 0.24 & 0.18 & -0.24 & -0.18 \\ 1 & -0.32 & -0.24 & 0.32 & 0.24 \\ 2 & -0.24 & -0.18 & 0.24 & 0.18 \end{array}$	$\mathbf{k}^{e_3} = \frac{(1)(1)}{1.6\sqrt{2}}\begin{bmatrix} 0.5&-0.5&-0.5&0.5\\-0.5&0.5&0.5&-0.5\\-0.5&0.5&0.5&-0.5\\0.5&-0.5&-0.5&0.5\end{bmatrix}$ $= \begin{array}{c\|cccc} & \bar{2} & \bar{3} & 1 & 2 \\ \hline \bar{2} & 0.221 & -0.221 & -0.221 & 0.221 \\ \bar{3} & -0.221 & 0.221 & 0.221 & -0.221 \\ 1 & -0.221 & 0.221 & 0.221 & -0.221 \\ 2 & 0.221 & -0.221 & -0.221 & 0.221 \end{array}$
$\mathbf{f}_D^e$	$\mathbf{k}^{e_1}\mathbf{a}_1^e = \begin{bmatrix} 0&0&0&0\\0&0.3571&0&-0.3571\\0&0&0&0\\0&-0.3571&0&0.3571\end{bmatrix}\begin{bmatrix}0\\0\\0.5\\0\end{bmatrix} = \begin{array}{c\|c} \bar{1} & 0 \\ 3 & 0 \\ \bar{2} & 0 \\ \bar{3} & 0 \end{array}$	$\mathbf{k}^{e_2}\mathbf{a}_2^e = \begin{bmatrix} 0.32&0.24&-0.32&-0.24\\0.24&0.18&-0.24&-0.18\\-0.32&-0.24&0.32&0.24\\-0.24&-0.18&0.24&0.18\end{bmatrix}\begin{bmatrix}0\\0\\0\\0\end{bmatrix} = \begin{array}{c\|c} \bar{1} & 0 \\ 3 & 0 \\ 1 & 0 \\ 2 & 0 \end{array}$	$\mathbf{k}^{e_3}\mathbf{a}_3^e = \begin{bmatrix} 0.221&-0.221&-0.221&0.221\\-0.221&0.221&0.221&-0.221\\-0.221&0.221&0.221&-0.221\\0.221&-0.221&-0.221&0.221\end{bmatrix}\begin{bmatrix}0.5\\0\\0\\0\end{bmatrix} = \begin{array}{c\|c} \bar{2} & 0.1105 \\ \bar{3} & -0.1105 \\ 1 & -0.1105 \\ 2 & 0.1105 \end{array}$
$\mathbf{f}_e^e$	$\mathbf{f}_e^{e_1} = \mathbf{f}_r^{e_1} + \mathbf{f}_N^{e_1} - \mathbf{f}_D^{e_1} = \begin{array}{c\|c} \bar{1} & 0 \\ 3 & 0 \\ \bar{2} & 0 \\ \bar{3} & 0 \end{array}$	$\mathbf{f}_e^{e_2} = \mathbf{f}_r^{e_2} + \mathbf{f}_N^{e_2} - \mathbf{f}_D^{e_2} = \begin{array}{c\|c} \bar{1} & 0 \\ 3 & 0 \\ 1 & 0 \\ 2 & 0 \end{array}$	$\mathbf{f}_e^{e_3} = \mathbf{f}_r^{e_3} + \mathbf{f}_N^{e_3} - \mathbf{f}_D^{e_3} = \begin{array}{c\|c} \bar{2} & -0.1105 \\ \bar{3} & 0.1105 \\ 1 & 0.1105 \\ 2 & -0.1105 \end{array}$

$$\mathbf{K} = \begin{bmatrix} 0.32+0.221 & 0.24-0.221 & -0.24 \\ 0.24-0.221 & 0.18+0.221 & -0.18 \\ -0.24 & -0.18 & 0.3571+0.18 \end{bmatrix} = \begin{bmatrix} 0.5410 & 0.019 & -0.24 \\ 0.019 & 0.401 & -0.18 \\ -0.24 & -0.18 & 0.5371 \end{bmatrix}$$

$$\mathbf{F} = \mathbf{F}_N + \mathbf{F}_e = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.1105 \\ -0.1105 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.1105 \\ -1.1105 \\ 0 \end{bmatrix} \Rightarrow \boxed{\mathbf{U} = \mathbf{K}^{-1}\mathbf{F} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} -0.2123 \\ -3.2980 \\ -1.200 \end{bmatrix}}$$

- Numbers encircled in the computation of essential BC force are displacements corresponding to free dofs. As mentioned before, in reality we do not consider them in computation of this force, but in hand calculation we just put zero for those values.

those values.



$$F_n = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$