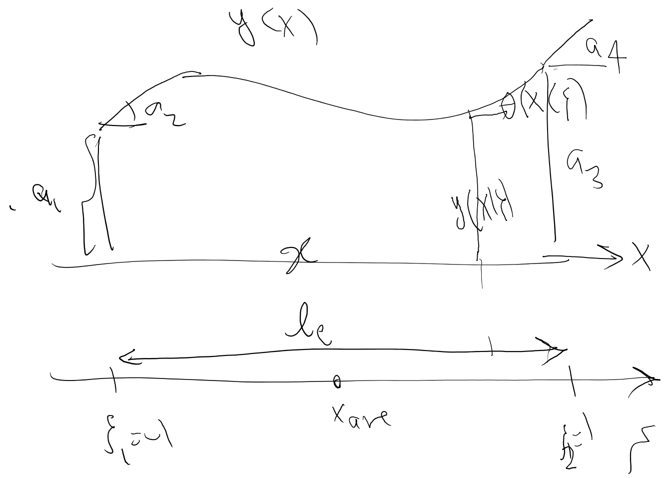
FEM20241111

Monday, November 11, 2024 12:56 PM

From last time:

$$\boxed{y(x(\xi)) = \sum_i a_i N_i(\xi)} \quad (1)$$



$$\theta = \frac{dy}{dX} = \sum a_i \frac{dN_i(\xi)}{dX}$$

$$= \sum a_i \frac{dN_i(\xi)}{d\xi} \cdot \left(\frac{d\xi}{dX}\right)$$

$$X(\xi) = X_{ave} + \frac{l_e}{2}\xi$$

$$= \frac{1}{\underbrace{\left(\frac{dX}{d\xi}\right)}_{J}} \sum a_i \frac{dN_i(\xi)}{d\xi}$$

$$\boxed{J = \frac{dX}{d\xi} = \frac{l_e}{2}}$$

$\frac{d\xi}{dX}$ not known in general

$$= \frac{1}{\frac{l_e}{2}} \left(\frac{dN_1(\xi)}{d\xi} \quad \frac{dN_2}{d\xi} \quad \frac{dN_3}{d\xi} \quad \frac{dN_4}{d\xi} \right) \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

$$N_1^e(\xi) = \frac{1}{4}(2 - 3\xi + \xi^3) \qquad N_2^e(\xi) = \frac{L^e}{8}(1 - \xi - \xi^2 + \xi^3) \qquad \textbf{(421a)}$$

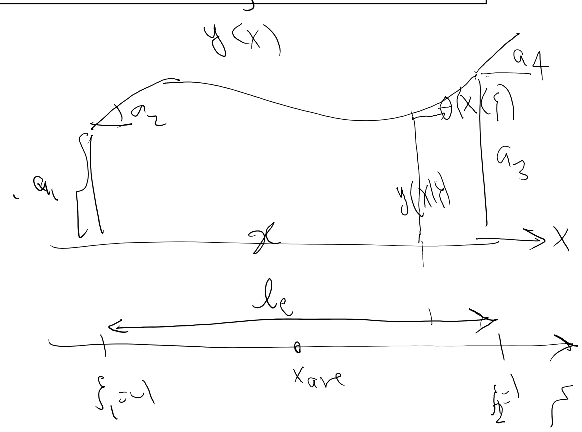
$$N_3^e(\xi) = \frac{1}{4}(2 + 3\xi - \xi^3) \qquad N_4^e(\xi) = \frac{L^e}{8}(-1 - \xi + \xi^2 + \xi^3) \qquad \textbf{(421b)}$$

$$\boxed{\theta(\xi) = \frac{2}{l_e}\left[\underbrace{\frac{1}{4}(-3 + 3\xi^2)}_{\frac{dN_1}{d\xi}}, \ \frac{l_e}{8}(-1 - 2\xi + 3\xi^2), \ \frac{1}{4}(3 - 3\xi^2), \ \underbrace{\frac{l_e}{8}(-1 + 2\xi + 3\xi^2)}_{\frac{dN_4}{d\xi}} \right] \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}} \quad (2)$$

Weak statement: beam

$$\int_{D} \frac{d^2 w}{dx^2} \underbrace{EI}_{D} \frac{d^2 y}{dx^2} dx$$

$D = [0, L]$



$$\mathcal{L}_m = (\quad)''$$

$$\boxed{K = \int B^t D B \, dV} \qquad \boxed{B = \mathcal{L}_m(N)}$$

$$\boxed{K = \int_{\mathcal{D}} B^T D B \, dV} \qquad \boxed{B = \mathcal{L}_m(N)}$$

for a beam problem we have

$$B = \mathcal{L}_m N = \frac{d^2}{dx^2}\overbrace{\left[N_1(\xi)\ N_2(\xi)\ N_3(\xi)\ N_4(\xi)\right]}^{N}$$

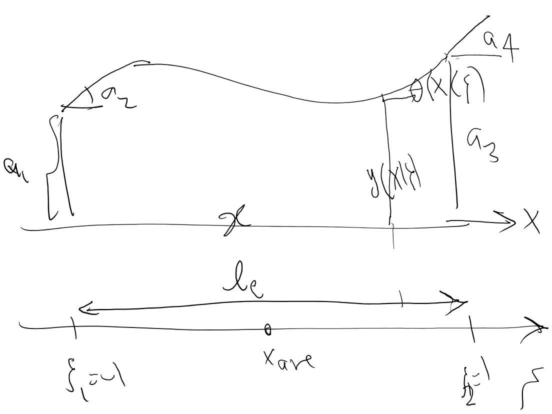
$$B = \frac{d^2 N(\xi)}{dx^2} = \frac{d}{dx}\left(\frac{dN}{dx}\right)(\xi) = \frac{d}{d\xi}\left[\frac{dN}{dx}(\xi)\right]\frac{1}{\frac{dx}{d\xi}} \quad \frac{l_e}{2}$$

$$B = \left(\frac{2}{l_e}\,\frac{2}{l_e}\right)\left[\frac{d\left(\frac{1}{4}(-3+3\xi^2)\right)}{d\xi} \quad \frac{d\,\frac{l_e}{8}(-1-2\xi+3\xi^2)}{d\xi} \quad \frac{d\,\frac{1}{4}(3-3\xi^2)}{d\xi} \quad \frac{d\,\frac{l_e}{8}(-1+2\xi+3\xi^2)}{d\xi}\right]$$

$$B = \frac{4}{l_e^2}\left[\frac{1}{4}(6\xi) \quad \frac{l_e}{8}(-2+6\xi) \quad \frac{1}{4}(-6\xi) \quad \frac{l_e}{8}(2+6\xi)\right]$$

$$\boxed{B(\xi) = \frac{d^2N}{dx^2} = \left[\frac{6\xi}{l_e^2} \quad \frac{-1+3\xi}{l_e} \quad \frac{-6\xi}{l_e^2} \quad \frac{1+3\xi}{l_e}\right]} \qquad (3)$$

$$\frac{d^2y}{dx^2}(\xi) = ?$$



$$y = N_1 a_1 + N_2 a_2 + N_3 a_3 + N_4 a_4$$

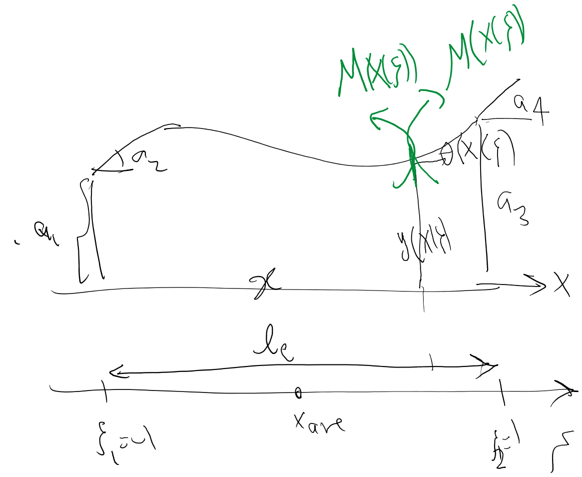
$$= \underbrace{\left[N_1\ N_2\ N_3\ N_4\right](\xi)}_{N(\xi)}\underbrace{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}}_{a}$$

$$\frac{dy}{dx} = \left[\frac{dN_1}{dx} \quad \frac{dN_2}{dx} \quad \frac{dN_3}{dx} \quad \frac{dN_4}{dx}\right]\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\frac{dy}{dx} = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} & \frac{dN_3}{dx} & \frac{dN_4}{dx} \end{bmatrix} \begin{bmatrix} a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

$$\frac{d^2y}{dx^2} = \underbrace{\begin{bmatrix} \frac{d^2N_1}{dx^2} & \frac{d^2N_2}{dx^2} & \frac{d^2N_3}{dx^2} & \frac{d^2N_4}{dx^2} \end{bmatrix}}_{B} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

$$\boxed{\frac{d^2y}{dx^2} = Ba = \sum_{i=1}^{4} B_i a_i \qquad M = EIy''}$$



(3-M)

$$V = \frac{dM}{dX} = \frac{d\,EIy''}{dX} \qquad \text{if } EI \text{ is constant}$$

$$= EI\frac{d^3y}{dx^3}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\underbrace{Ba}_{y''} = \frac{d}{d\xi}(Ba)\frac{1}{\left(\frac{dx}{d\xi}\right)_{\frac{l_e}{2}}}$$

$$= \frac{d}{d\xi}\underbrace{\begin{bmatrix} \frac{6\xi}{l^2} & \frac{-1+3\xi}{l_e} & \frac{-6\xi}{l_e^2} & \frac{1+3\xi}{l_e} \end{bmatrix}}\frac{1}{\frac{l_e}{2}} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

$$\boxed{\frac{d^3y(\xi)}{} = \begin{bmatrix} \frac{12}{} & \frac{6}{} & \frac{-12}{} & \frac{6}{l^2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \ldots \end{bmatrix}}$$

(4)

$$\frac{d^3y}{dx^3}(\xi) = \begin{bmatrix} \frac{12}{l_e^3} & \frac{6}{l_e^2} & \frac{-12}{l_e^3} & \frac{6}{l_e^2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \quad (4)$$

$$V(\xi) = EI \frac{d^3y}{dx^3}$$

Beam Example: Calculation of y, θ, M, V within element

- After the solution of global free dofs, they are transferred to elements.
- Once element dofs are know, we have the Displacement in the entire elements:

$$y(\xi) = N_1^e(\xi)a_1^e + N_2^e(\xi)a_2^e + N_3^e(\xi)a_3^e + N_4^e(\xi)a_4^e.$$

element shape functions are given in (421).

- Rotation: Obtained by differentiating previous equation w.r.t. x & noting that $\frac{\mathrm{d}x}{\mathrm{d}\xi} = \frac{L^e}{2}$:

$$\theta(\xi) = \frac{\mathrm{d}y}{\mathrm{d}x}(\xi) = \frac{\frac{\mathrm{d}y}{\mathrm{d}\xi}(\xi)}{\frac{\mathrm{d}x}{\mathrm{d}\xi}(\xi)} = \frac{2}{L^e}\left\{\frac{\mathrm{d}N_1^e}{\mathrm{d}\xi}(\xi)a_1^e + \frac{\mathrm{d}N_2^e}{\mathrm{d}\xi}(\xi)a_2^e + \frac{\mathrm{d}N_3^e}{\mathrm{d}\xi}(\xi)a_3^e + \frac{\mathrm{d}N_4^e}{\mathrm{d}\xi}(\xi)a_4^e\right\}$$

- Moment is directly obtained by differentiating the above equation:

$$M(\xi) = E(\xi)I(\xi)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}(\xi) = E(\xi)I(\xi)\mathbf{B}^e(\xi)$$
$$= E(\xi)I(\xi)\left\{B_1^e(\xi)a_1^e + B_2^e(\xi)a_2^e + B_3^e(\xi)a_3^e + B_4^e(\xi)a_4^e\right\} \quad \textit{cf.}\ (424)\ \text{for}\ \mathbf{B}^e$$

- Shear force is obtained by differentiating M w.r.t. x. It's a similar process to deriving θ from y with the difference that if EI are not constant we need to take it into account. For constant EI we have:

$$V(\xi) = \frac{\mathrm{d}M}{\mathrm{d}x}(\xi) = \frac{\frac{\mathrm{d}M}{\mathrm{d}\xi}(\xi)}{\frac{\mathrm{d}x}{\mathrm{d}\xi}(\xi)} = \frac{2EI}{L^e}\left\{\frac{\mathrm{d}B_1^e}{\mathrm{d}\xi}(\xi)a_1^e + \frac{\mathrm{d}B_2^e}{\mathrm{d}\xi}(\xi)a_2^e + \frac{\mathrm{d}B_3^e}{\mathrm{d}\xi}(\xi)a_3^e + \frac{\mathrm{d}B_4^e}{\mathrm{d}\xi}(\xi)a_4^e\right\}$$

- To obtain these fields for the entire beam we evaluate these equations for all elements.

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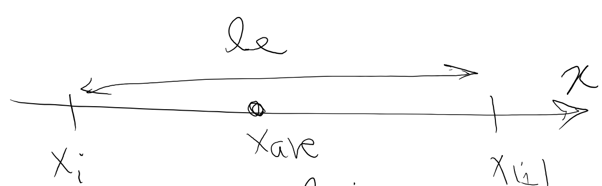
Going back to stiffness calculati

$$\underbrace{\int \frac{d^2w}{dx^2} EI \frac{d^2y}{dx^2} dx}_{\text{LHS of WK}} =$$



$$x(\xi) = x_{ave} + \frac{l_e}{2}\xi$$
$$\Rightarrow J = \left(\frac{l_e}{2}\right)$$

$$K^e = \int_{x_i}^{x_{i+1}} B^t D B\, dx$$

$B = \frac{d^2N}{dx^2}$ $\quad D = EI$

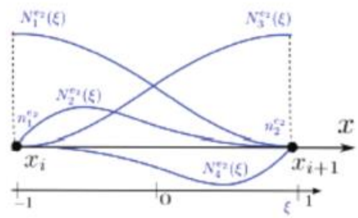
$$dx = \underbrace{\frac{dx}{d\xi}}_{J} d\xi$$

$$K^e = \int_{-1}^{1} \begin{bmatrix} \frac{6\xi}{l^2} \\ \ldots \end{bmatrix} \ldots \frac{dx}{(\ldots)}$$

$$K^e = \int_{-1}^{1} \underbrace{\begin{bmatrix} \frac{6\xi}{\ell^2} \\ \frac{(-1+3\xi)}{\ell} \\ -\frac{6\xi}{\ell^2} \\ \frac{1+3\xi}{\ell} \end{bmatrix}}_{B^t(\xi)} EI(\xi) \begin{bmatrix} \frac{6\xi}{\ell^2} & \cdots & \frac{1+3\xi}{\ell} \end{bmatrix} \left(\frac{\ell_e}{2} d\xi\right)$$

FEM formula for the stiffness of a beam

FEM formulation of beam elements: Stiffness matrix



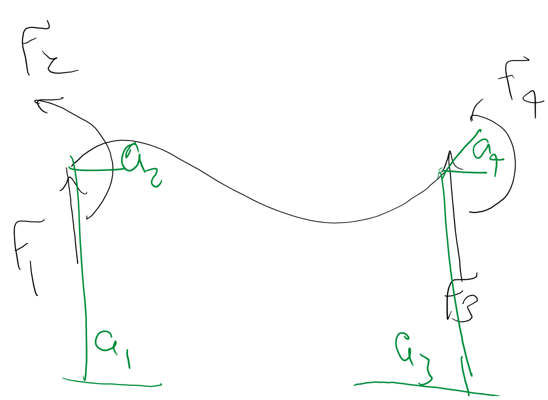
- From (424) and (425) we have,

$$\mathbf{k}^e = \frac{1}{2L^e} \int_{-1}^{1} \begin{bmatrix} \frac{6\xi}{L^e} \\ -1+3\xi \\ -\frac{6\xi}{L^e} \\ 1+3\xi \end{bmatrix} E(\xi)I(\xi) \begin{bmatrix} \frac{6\xi}{L^e} & -1+3\xi & -\frac{6\xi}{L^e} & 1+3\xi \end{bmatrix} d\xi \qquad (426)$$

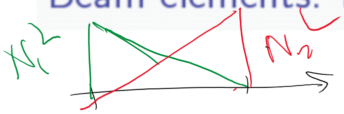
- If E and I are constant, we can take those out of the equation and have:

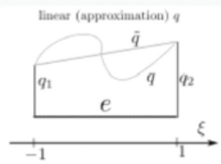
$$\mathbf{k}^e = \frac{EI}{L^{e3}} \begin{bmatrix} 12 & 6L^e & -12 & 6L^e \\ & 4L^{e2} & -6L^e & 2L^{e2} \\ & & 12 & -6L^e \\ \text{sym.} & & & 4L^{e2} \end{bmatrix} \quad \text{for constant } E \text{ and } I \qquad (427)$$

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = K^e \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$



Beam elements: Forces: A. Source term forces





$$q \approx \tilde{q} = \begin{bmatrix} N_1^L & N_2^L \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \qquad f_r = \int N^T q \, dx$$

$$= \int \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} q(\xi) \, J \, d\xi$$

then the source term force is:

$$\mathbf{f}_r^e \approx \int_{-1}^{1} \mathbf{N}^e(\xi)^{\mathrm{T}}.\tilde{q}(\xi)\frac{L^e}{2}\mathrm{d}\xi = \mathbf{r}^e \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \quad \text{where}$$

$$\mathbf{r}^e = \frac{L^e}{2}\int_{-1}^{1} \mathbf{N}^e(\xi)^{\mathrm{T}}.\mathbf{N}_L^e(\xi)\mathrm{d}\xi = \frac{L^e}{2}\int_{-1}^{1} \underbrace{\begin{bmatrix} \frac{1}{4}(2-3\xi+\xi^3) \\ \frac{L^e}{8}(1-\xi-\xi^2+\xi^3) \\ \frac{1}{4}(2+3\xi-\xi^3) \\ \frac{L^e}{8}(-1-\xi+\xi^2+\xi^3) \end{bmatrix}}_{4\times 1} . \underbrace{\begin{bmatrix} \frac{1-\xi}{2} & \frac{1+\xi}{2} \end{bmatrix}}_{1\times 2} \mathrm{d}\xi \quad \Rightarrow$$

$$f_r^e = \int \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix}_{\text{beam}} \underbrace{\begin{bmatrix} N_1^L & N_2^L \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}}_{\tilde{q}} J \, d\xi$$

$$\boxed{\underbrace{\mathbf{f}_r^e \approx \mathbf{r}^e}_{4\times 2} \underbrace{\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}}_{2\times 1} \quad \text{where } \mathbf{r}^e = L^e \begin{bmatrix} \frac{7}{20} & \frac{3}{20} \\ \frac{1}{20}L^e & \frac{1}{30}L^e \\ \frac{3}{20} & \frac{7}{20} \\ -\frac{1}{30}L^e & -\frac{1}{20}L^e \end{bmatrix} \quad \text{exact for linear } q} \qquad (433)$$

q_1 and q_2 are shown in the right figure. For constant $q(x) = \bar{q}$, $q_1 = q_2 = \bar{q}$, from (433) we get:

$$\mathbf{f}_r^e = \mathbf{r}^e \begin{bmatrix} \bar{q} \\ \bar{q} \end{bmatrix} = \begin{bmatrix} \frac{\bar{q}L^e}{2} \\ \frac{\bar{q}L^{e\,2}}{12} \\ \frac{\bar{q}L^e}{2} \\ -\frac{\bar{q}L^{e\,2}}{12} \end{bmatrix} \quad \text{constant } q(x) = \bar{q} \qquad (434)$$

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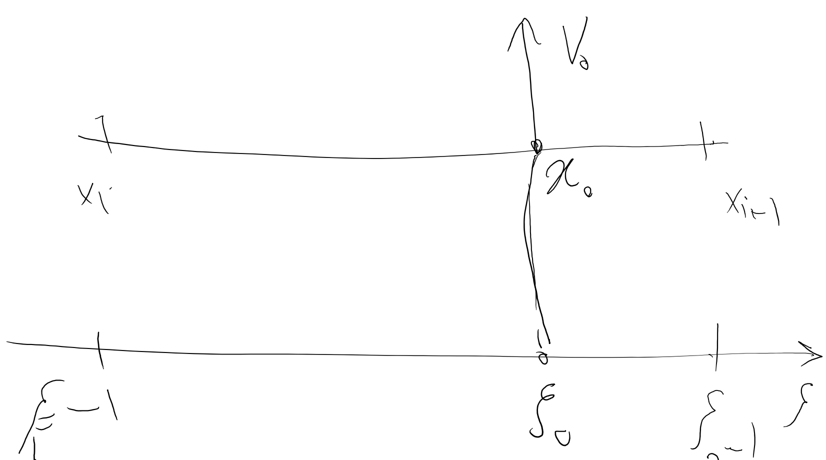
$$f_r = \int_e \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} q(x) \, dx$$

$$q(x) = V_0 \underbrace{\delta(x - x_0)}_{\text{Dirac delta}}$$



$$\boxed{f_r = V_0 \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} (\xi_0)}$$

$$\boxed{x = x_{ave} + \frac{l_e}{2}\xi}$$

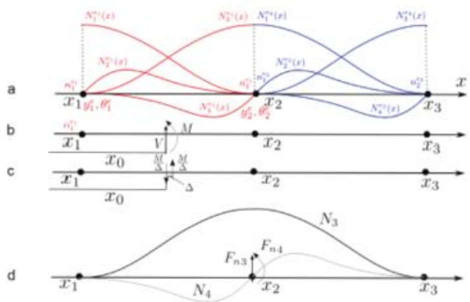
Beam elements: Concentrated load

- Since, for a concentrated vertical load V applied at x_0 the corresponding q is $V\delta(x-x_0)$ the global force F_{ri} is:

$$F_{ri} = \int_0^L N_i(x)\{V\delta(x-x_0)\}\,\mathrm{d}x$$

$$F_{ri} = V N_i(x_0) \qquad (436)$$

- If x_0 is inside an element (for example in figures (b) and (c)) the concentrated force can be added to $\mathbf{f}_r^e$:



$$\mathbf{f}_r^e = \int_0^{L^e} \mathbf{N}^{e\,\mathrm{T}}.q\,\mathrm{d}x = V \begin{bmatrix} N_1^e(\xi_0) \\ N_2^e(\xi_0) \\ N_3^e(\xi_0) \\ N_4^e(\xi_0) \end{bmatrix} \qquad (437)$$

- ξ_0 is the local coordinate corresponding to x_0.
- Off course, if in addition to V at x_0, there is the distributed load q applied on the beam, we add its contribution from the the one of variants on pages 358 to .

$$\frac{dN}{dx} = \frac{dN}{d\xi}\,\frac{1}{\frac{dx}{d\xi}}$$

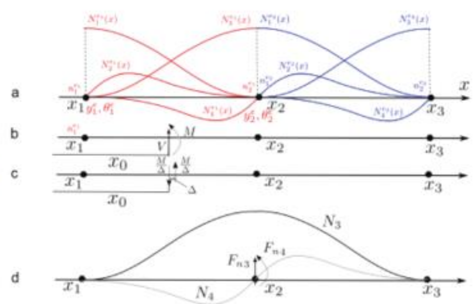
$$J = \frac{l_e}{2}$$

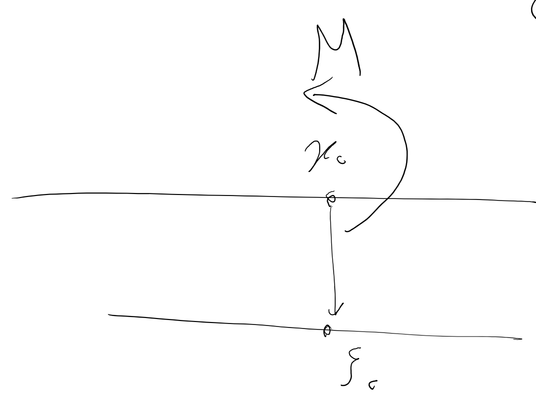
Beam elements: Concentrated load

- Thus, as expected, from (438) we observe that the global force of moment is rotation of shape function at x_0.
- Equations (438) and (436) can be summarized as (figure (b)):

$$F_{ri} = V N_i(x_0) \text{ vertical force } V\ x_0 \qquad (439a)$$

$$F_{ri} = M\frac{\mathrm{d}N_i}{\mathrm{d}x}(x_0) \text{ moment} M\ x_0 \qquad (439b)$$

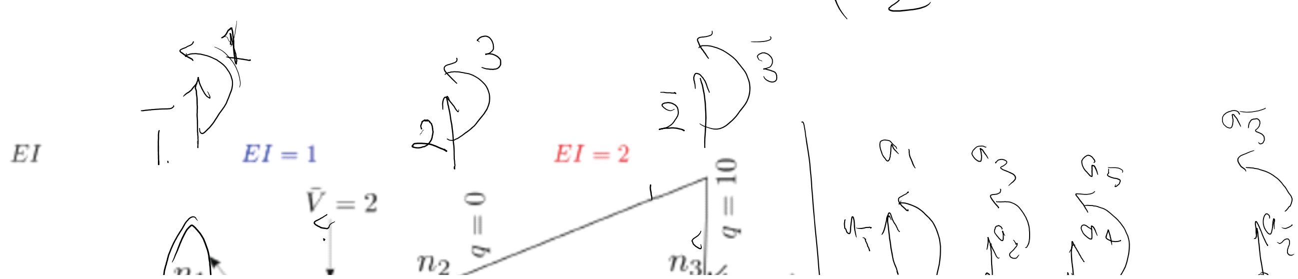


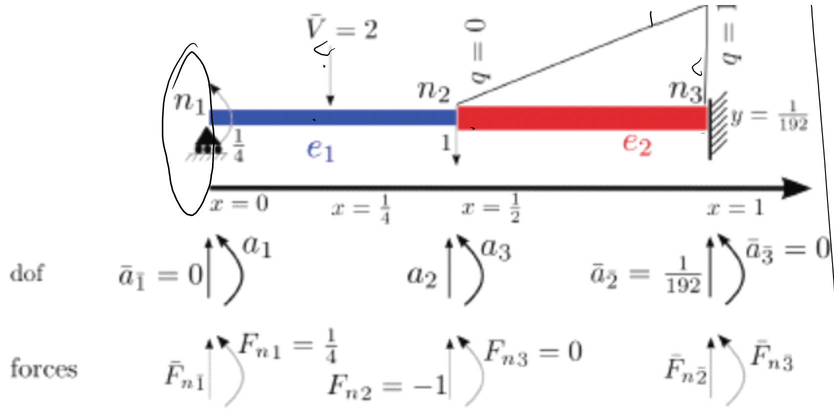


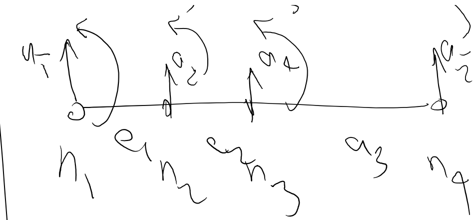
- Clearly, we superpose forces from all concentrated forces.
- For example, in figure (b) $F_{ri} = VN_i(x_0) + M\frac{\mathrm{d}N_i}{\mathrm{d}x}(x_0)$.
- If V and/or M are exerted within an element, *e.g.*, figure (b), their contributions to element source term force is (*cf.* (437)):

$$\mathbf{f}_r^e(V@\,x_0) = V \begin{bmatrix} N_1^e(\xi_0) \\ N_2^e(\xi_0) \\ N_3^e(\xi_0) \\ N_4^e(\xi_0) \end{bmatrix} \qquad \mathbf{f}_r^e(M@\,x_0) = M \begin{bmatrix} \frac{\mathrm{d}N_1^e}{\mathrm{d}x}(\xi_0) \\ \frac{\mathrm{d}N_2^e}{\mathrm{d}x}(\xi_0) \\ \frac{\mathrm{d}N_3^e}{\mathrm{d}x}(\xi_0) \\ \frac{\mathrm{d}N_4^e}{\mathrm{d}x}(\xi_0) \end{bmatrix} \qquad (440)$$

$$= M\,\frac{dN}{d\xi}\cdot\frac{1}{\left(\frac{l_e}{2}\right)}$$







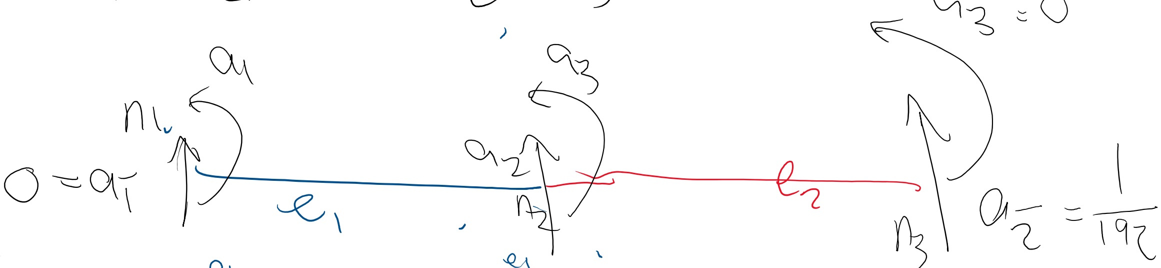
more accurate sln

$n_f = 5$

$n_p = 3$

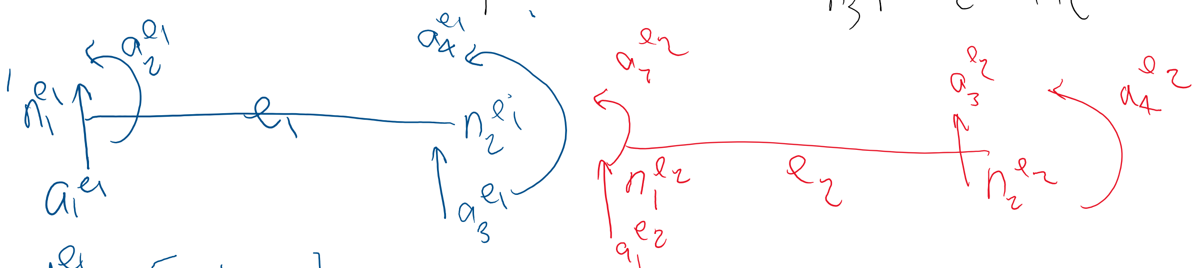
$$F_n = \begin{bmatrix} 1/4 \\ -2 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$F_n = \begin{bmatrix} 1/4 \\ -1 \\ 0 \end{bmatrix} \quad a_p = \begin{bmatrix} 0 \\ 1/192 \\ 0 \end{bmatrix}$$



$n_f = 3$

$n_p = 3$



$LEM^{e_1} = [1, 2]$ $\qquad$ $LEM^{e_2} = [2, 3]$

$M^{e_1} = [\bar{1}, 1, 2, 3]$ $\qquad$ $M^{e_2} = [2, 3, \bar{2}, \bar{3}]$

Beam example: Assembly of global system

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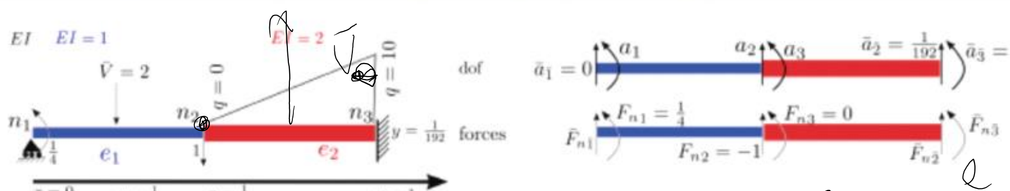

1D elements

$$f^e = f_r^e + f_N^e - f_D^e$$

e	e_1	e_2
$\mathbf{k}^e$	$\mathbf{k}^{e_1} = \frac{1}{(\frac{1}{2})^3} \begin{bmatrix} 12 & 3 & -12 & 3 \\ 3 & 1 & -3 & 0.5 \\ -12 & -3 & 12 & -3 \end{bmatrix}$	$\mathbf{k}^{e_2} = \frac{2}{(\frac{1}{2})^3} \begin{bmatrix} 12 & 3 & -12 & 3 \\ 3 & 1 & -3 & 0.5 \\ -12 & -3 & 12 & -3 \end{bmatrix}$

e	e_1	e_2
$\mathbf{k}^e$	$\mathbf{k}^{e_1} = \frac{1}{(\frac{1}{2})^3}\begin{bmatrix}12 & 3 & -12 & 3\\ 3 & 1 & -3 & 0.5\\ -12 & -3 & 12 & -3\\ 3 & 0.5 & -3 & 1\end{bmatrix}$ $= \begin{array}{c}\\ \bar{1}\\ 1\\ 2\\ 3\end{array}\begin{array}{c}\bar{1}\ 1\quad 2\quad 3\\ \begin{bmatrix}96 & 24 & -96 & 24\\ 24 & 8 & -24 & 4\\ -96 & -24 & 96 & -24\\ 24 & 4 & -24 & 8\end{bmatrix}\end{array}$	$\mathbf{k}^{e_2} = \frac{2}{(\frac{1}{2})^3}\begin{bmatrix}12 & 3 & -12 & 3\\ 3 & 1 & -3 & 0.5\\ -12 & -3 & 12 & -3\\ 3 & 0.5 & -3 & 1\end{bmatrix}$ $= \begin{array}{c}\\ 2\\ 3\\ \bar{2}\\ \bar{3}\end{array}\begin{array}{c}2\ 3\quad \bar{2}\quad \bar{3}\\ \begin{bmatrix}192 & 48 & -192 & 48\\ 48 & 16 & -48 & 8\\ -192 & -48 & 192 & -48\\ 48 & 8 & -48 & 16\end{bmatrix}\end{array}$
$\mathbf{f}_r^e$	(440) (1st eqn) ($\xi = 0$) $\bar{V}\begin{bmatrix}N_1^{e_1}(\xi_0)\\ N_2^{e_1}(\xi_0)\\ N_3^{e_1}(\xi_0)\\ N_4^{e_1}(\xi_0)\end{bmatrix} = -2\begin{bmatrix}\frac{1}{2}\\ \frac{2}{8}\\ \frac{1}{2}\\ -\frac{2}{8}\end{bmatrix} = \begin{array}{c}\bar{1}\\ 1\\ 2\\ 3\end{array}\begin{bmatrix}-1\\ -\frac{1}{8}\\ -1\\ \frac{1}{8}\end{bmatrix}$	equation (433); $\mathbf{r}^e\,[q_1 \quad q_2]^T$ $\frac{1}{2}\begin{bmatrix}\frac{7}{20} & \frac{3}{20}\\ \frac{1}{40} & \frac{1}{60}\\ \frac{3}{20} & \frac{7}{20}\\ -\frac{1}{60} & -\frac{1}{40}\end{bmatrix}\begin{bmatrix}0\\ 10\end{bmatrix} = \begin{array}{c}2\\ 3\\ \bar{2}\\ \bar{3}\end{array}\begin{bmatrix}\frac{3}{4}\\ \frac{1}{12}\\ \frac{7}{4}\\ -\frac{1}{8}\end{bmatrix}$
$\mathbf{f}_D^e$	$\mathbf{k}^{e_1}\mathbf{a}_1^e =$ $\begin{bmatrix}96 & 24 & -96 & 24\\ 24 & 8 & -24 & 4\\ -96 & -24 & 96 & -24\\ 24 & 4 & -24 & 8\end{bmatrix}\begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix} = \begin{array}{c}1\\ \bar{1}\\ 2\\ 3\end{array}\begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}$	$\mathbf{k}^{e_2}\mathbf{a}_2^e =$ $\begin{bmatrix}192 & 48 & -192 & 48\\ 48 & 16 & -48 & 8\\ -192 & -48 & 192 & -48\\ 48 & 8 & -48 & 16\end{bmatrix}\begin{bmatrix}0\\ 0\\ \frac{1}{192}\\ 0\end{bmatrix} = \begin{array}{c}2\\ 3\\ \bar{2}\\ \bar{3}\end{array}\begin{bmatrix}-1\\ -\frac{1}{4}\\ 1\\ -\frac{1}{4}\end{bmatrix}$
$\mathbf{f}_e^e$	$\mathbf{f}_e^{e_1} = \mathbf{f}_r^{e_1} + \mathbf{f}_N^{e_1} - \mathbf{f}_D^{e_1} = \begin{array}{c}\bar{1}\\ 1\\ 2\\ 3\end{array}\begin{bmatrix}-1\\ -\frac{1}{8}\\ -1\\ \frac{1}{8}\end{bmatrix}$	$\mathbf{f}_e^{e_2} = \mathbf{f}_r^{e_2} + \mathbf{f}_N^{e_2} - \mathbf{f}_D^{e_2} = \begin{array}{c}2\\ 3\\ \bar{2}\\ \bar{3}\end{array}\begin{bmatrix}\frac{7}{4}\\ \frac{1}{3}\\ \frac{3}{4}\\ \frac{1}{8}\end{bmatrix}$

$$\mathbf{k}^e = \frac{EI}{L^{e3}}\begin{bmatrix}12 & 6L^e & -12 & 6L^e\\ & 4L^{e2} & -6L^e & 2L^{e2}\\ & & 12 & -6L^e\\ \text{sym.} & & & 4L^{e2}\end{bmatrix} \quad \text{for constant } E \text{ and } I \qquad (427)$$

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$f_D^e = k^e a^e$

$$\mathbf{K} = \begin{bmatrix}8 & -24 & 4\\ & 96+192 & -24+48\\ \text{sym.} & & 8+16\end{bmatrix}$$

$$= \begin{bmatrix}8 & -24 & 4\\ & 288 & 24\\ \text{sym.} & & 24\end{bmatrix}$$

$$\mathbf{F} = \mathbf{F}_n + \mathbf{F}_e$$

$$= \begin{bmatrix}\frac{1}{4}\\ -1\\ 0\end{bmatrix} + \begin{bmatrix}-\frac{1}{8}\\ -1+\frac{7}{4}\\ \frac{1}{8}+\frac{1}{3}\end{bmatrix} = \begin{bmatrix}-\frac{1}{8}\\ -\frac{3}{4}\\ \frac{11}{24}\end{bmatrix} \Rightarrow$$